* This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-68-1415.

THE MEASURABILITY OF A STOCHASTIC PROCESS OF SECOND ORDER AND ITS LINEAR SPACE

Stamatis Cambanis*

Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 832

June, 1972

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1. THE MEASURABILITY OF A STOCHASTIC PROCESS OF SECOND ORDER

Let T be a separable metric space and $\mathcal{B}(T)$ the σ-algebra of Borel sets of T, and let $X_t, t\in T$, be a real stochastic process on the probability space $(\Omega,\mathcal{F},P)$. $X_t, t\in T$, is called measurable if the map $(t,\omega)\to X_t(\omega)$ is $\mathcal{B}(T)\times\mathcal{F}$-measurable. A process Y_t, $t\in T$, on $(\Omega,\mathcal{F},P)$ is called a modification of $X_t, t\in T$, if $P\{X_t=Y_t\}=1$ for all t in T. $X_t, t\in T$, is of second order if $E(X_t^2)<+\infty$ for all t in T, and then its autocorrelation R is defined by $R(t,s)=E(X_tX_s)$ for all t,s in T. It is clear from Fubini's theorem that if a second order process $X_t, t\in T$, has a measurable modification then R is $\mathcal{B}(T)\times\mathcal{B}(T)$-measurable. That the measurability of R is not sufficient for the existence of a measurable modification of $X_t, t\in T$, is demonstrated in Remark 2. It is thus of interest to find a condition which along with the measurability of R would imply the existence of a measurable modification of $X_t, t\in T$. This question is answered in Theorem 1, where in fact necessary and sufficient conditions are given for a second order process to have a measurable modification. A remarkable consequence of these conditions is that the existence of a measurable modification of a second order process is a second order property.

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*This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-68-1415.

The proof of Theorem 1 is based on the necessary and sufficient conditions for a process (not necessarily of second order) to have a measurable modification given in [5], which are expressed as follows (here the terminology of [6] is followed). Let M be the space of all real random variables on $(\Omega,\mathcal{F},P)$ with the topology of convergence in probability, where random variables that are equal a.e. [P] are considered identical. If ξ is a real random variable, its class in M is denoted by $[\xi]$. Then $X_t, t\in T$, has a measurable modification if and only if the map from T to M taking t to $[X_t]$ is measurable and has separable range [5,6]. Moreover, the measurable modification can be taken to be separable and also progressively measurable, the latter if T is an interval and a nondecreasing family $\mathcal{F}_t, t\in T$, of sub-σ-algebras of $\mathcal{F}$ is given.

For a second order process $X_t, t\in T$, we denote by H(X) the closure in $L_2(\Omega,\mathcal{F},P)$ of the linear space of the random variables $\{X_t, t\in T\}$ and we call it the linear space of the process. We also denote by $\mathcal{R}(K)$ the reproducing kernel Hilbert space of a real, symmetric, nonnegative definite function K on T×T. It is well known that $\mathcal{R}(R)$ consists of all functions f on T of the form $f(t)=E(\xi X_t)$, $t\in T$, for some $\xi\in H(X)$, and that the map $\xi\mapsto E(\xi X_t)$ defines an inner product preserving isomorphism between H(X) and $\mathcal{R}(R)$ [16, p.302].

THEOREM 1. Let $X_t, t\in T$, be a real, second order process with autocorrelation R. The following are equivalent.

(i) $X_t, t\in T$, has a measurable modification.

(ii) R is $\mathcal{B}(T)\times\mathcal{B}(T)$-measurable and H(X) (or $\mathcal{R}(R)$) is separable.

PROOF. (a) We first show that (ii) implies (i). It suffices to verify the conditions of [5,6]; the construction of a measurable modification is the same as in [5] or in [6].

Since convergence in $L_2(\Omega,F,P)$ implies convergence in probability, the separability of $H(X)$ as a subset of $L_2(\Omega,F,P)$ implies its separability as a subset of M. Thus its subset $\{[X_t],\ t\in T\}$ is separable in M. To complete the proof it suffices to show that the map X: $T \to M$ defined by $X(t)=[X_t]$ is measurable. The metric ρ on M defined by $\rho(\xi,\eta)=E\left(\frac{|\xi-\eta|}{1+|\xi-\eta|}\right)$, $\xi,\eta\in M$, metrizes the topology of convergence in probability. Thus for the measurability of X it suffices to show that $X^{-1}(B)\in F$ for every set B in M of the form $B=\{Y\in M: \rho(Y,Y_0)\le r\}$, where $Y_0\in M$ and $r > 0$. Since $X^{-1}(B)=\{t\in T: \rho([X_t], Y_0) \le r\}$, it suffices to prove that the real function $\rho([X_t], Y_0)$ on T is $\mathcal{B}(T)$-measurable for all $Y_0\in M$.

Let $\{\xi_n\}_{n=1}^{\infty}$ be a complete orthonormal sequence in $H(X)$ (which exists because $H(X)$ is separable). Then for all $t\in T$ we have

$$X_t = \sum_{n=1}^{\infty} a_n(t)\xi_n$$

in $L_2(\Omega,F,P)$, where $a_n(t) = E(\xi_n X_t)$. Thus $a_n \in R(R)$, and in fact $\{a_n\}_{n=1}^{\infty}$ is a complete orthonormal sequence in $R(R)$. If for every $t\in T$ we let $X_t^{(N)} = \sum_{n=1}^{N} a_n(t)\xi_n$, then $X_t^{(N)}$ converges to X_t in $L_2(\Omega,F,P)$ and thus in probability. Let $Y_t = X_t - Y_0$ and $Y_t^{(N)}= X_t^{(N)}- Y_0$ for all $t\in T$. Then $Y_t^{(N)}$ converges to Y_t in probability, i.e., $\rho([Y_t^{(N)}], [Y_t]) \to o$ as $N \to \infty$. Dropping the index t for simplicity we have

$$\left|\frac{|Y^{(N)}|}{1+|Y^{(N)}|} - \frac{|Y|}{1+|Y|}\right| = \frac{\left||Y^{(N)}|-|Y|\right|}{(1+|Y^{(N)}|)(1+|Y|)} \le \frac{|Y^{(N)}-Y|}{1+|Y^{(N)}-Y|}.$$

Thus

$$E\left|\frac{|Y^{(N)}|}{1+|Y^{(N)}|} - \frac{|Y|}{1+|Y|}\right| \le \rho([Y^{(N)}];[Y])\underset{N\to\infty}{\to} 0.$$

It follows that for all $t \in T$,

$$\rho([X_t], Y_0) = \lim_{N \to \infty} E\left[\frac{|X_t^{(N)} - Y_0|}{1+|X_t^{(N)} - Y_0|}\right].$$

Note that every function in $\mathcal{R}(R)$ is either a finite linear combination of the functions $\{R(t,.), t \in T\}$ or a pointwise limit on T of such functions. Hence, since R is $\mathcal{B}(T)\times\mathcal{B}(T)$ - measurable, $R(t,.)$ is $\mathcal{B}(T)$ - measurable for all $t \in T$, and it follows that every f in $\mathcal{R}(R)$ is $\mathcal{B}(T)$ - measurable. Consequently $Y_t^{(N)}(\omega)$ is $\mathcal{B}(T)\times\mathcal{F}$ - measurable. By Fubini's theorem $E\left(\frac{|X_t^{(N)}-Y_0|}{1+|X_t^{(N)}-Y_0|}\right)$ is $\mathcal{B}(T)$ - measurable, and thus so is $\rho([X_t], Y_0)$, which completes the proof.

(b) We now show that (i) implies (ii). The measurability of R follows from Fubini's theorem and (i). In order to prove the separability of H(X) we first assume that R is uniformly bounded on T:

$$R(t,t) \leq C<+\infty \quad \text{for all t in T.}$$

We will show that this implies the uniform integrability of the family of random variables $\{X_t, t \in T\}$. Indeed we have for all $a>o$,

$$\begin{aligned}\int_{|X_t|>a} |X_t|\, dP &= \int_\Omega I_{\{|X_t|>a\}} |X_t|\, dP \\ &\leq [P\{|X_t|>a\}.\ R(t,t)]^{\frac{1}{2}} \\ &\leq \frac{R(t,t)}{a}.\end{aligned}$$

Thus

$$\lim_{a \to \infty} \sup_{t \in T} \int_{|X_t|>a} |X_t|\, dP \leq \lim_{a \to \infty} \frac{C}{a} = 0$$

and $\{X_t, t \in T\}$ is uniformly integrable.

Now (i) implies that $\{[X_t], t \in T\}$ is separable in M. Thus there exists a countable subset M' of $\{[X_t], t \in T\}$ such that for every t in T, $[X_t]$ is the limit in probability of a sequence in M', and hence also in $L_2(\Omega, F, P)$, since M' is uniformly integrable [14, p. 57]. It follows that H(X) equals the $L_2(\Omega, F, P,)$ closure of the linear span of M' and, since M' is countable, H(X) is separable.

We now consider the general case and define for N = 1, 2,...,

$$T_N = \{t \in T: R(t,t) \leq N\}.$$

Since R is measurable, $T_N \in \mathcal{B}(T)$ and by (i) $\{X_t, t \in T_N\}$ has a measurable modification. It follows by what has been proven that the $L_2(\Omega, F, P)$ closure of the linear span of the random variables $\{X_t, t \in T_N\}$, $H_N(X)$, is separable. Since X_t is of second order, R is finite valued and thus $T_N \uparrow T$. It follows that H(X) is the $L_2(\Omega, F, P)$ closure of $\cup_{N=1}^{\infty} H_N(X)$ and thus H(X) is separable. □

Thus a $\mathcal{B}(T) \times \mathcal{B}(T)$ - measurable, symmetric, nonnegative definite, real function R on T×T is the autocorrelation of a measurable process if and only if $\mathcal{R}(R)$ is separable.

REMARK 1. The mean m and the covariance C of a real second order process X_t, $t \in T$, are defined by $m(t) = E(X_t)$ and $C(t,s) = E([X_t - m(t)][X_s - m(s)])$ for all t, s, in T. Then $R(t,s) = m(t)m(s) + C(t,s)$. In connection with (ii) of Theorem 1 it should be noted that

> R is $\mathcal{B}(T) \times \mathcal{B}(T)$ - measurable if and only if m is $\mathcal{B}(T)$ - measurable and C is $\mathcal{B}(T) \times \mathcal{B}(T)$ - measurable.

The "if" part is obvious. The "only if" part is shown as follows. We have $m(t) = E(X_t I_\Omega)$ for all t in T, where I is the indicator function. Denote by ξ the projection of $I_\Omega \in L_2(\Omega, \mathcal{F}, P)$ onto the subspace H(X). Then $m(t) = E(X_t \xi)$ for all t in T and $\xi \in H(X)$, and thus $m \in \mathcal{R}(R)$. Since R is $\mathcal{B}(T)\times\mathcal{B}(T)$ - measurable, m is $\mathcal{B}(T)$ - measurable (see part (a) of the proof of Theorem 1) and $C(t,s) = R(t,s) - m(t)m(s)$ is $\mathcal{B}(T)\times\mathcal{B}(T)$ - measurable.

REMARK 2. Let $T = [0,1]$ and $R(t,s) = 1$ for $t = s$ in T and $R(t,s) = o$ for $t \neq s$ in T. Since R is symmetric and nonnegative definite, there exists a probability space $(\Omega, \mathcal{F}, P)$ and a real process X_t, $t \in T$, on it with autocorrelation R. R is clearly $\mathcal{B}(T)\times\mathcal{B}(T)$ - measurable, but since the values of X_t are orthogonal in $L_2(\Omega, \mathcal{F}, P)$, $E(X_t X_s) = 0$ for $t \neq s$ in T, H(X) is not separable and by Theorem 1, $X_t, t \in T$, does not have a measurable modification. This can be also shown without using Theorem 1. Indeed, assume that $X_t, t \in T$, has a measurable modification $Y_t, t \in T$. Then

$$E\left(\int_o^1 Y_t^2 \, dt\right) = \int_o^1 R(t,t)\,dt = 1 < +\infty$$

implies that $\int_o^1 Y_t^2 \, dt < +\infty$ a.e.[P]. If $\{\phi_n\}_{n=1}^\infty$ is a complete orthonormal set in $L_2(T) = L_2(T, \mathcal{B}(T), \text{Leb})$ then

$$Y_t = \sum_{n=1}^\infty \xi_n \, \phi_n(t)$$

in $L_2(T)$ a.e. [P], where $\xi_n = \int_o^1 Y_t \phi_n(t)\,dt$ a.e. [P]. Then

$$E(\xi_n^2) = \int_o^1\int_o^1 R(t,s)\,\phi_n(t)\,\phi_n(s)\,dt\,ds = 0$$

i.e., $\xi_n = 0$ a.e. [P], and thus $\int_o^1 Y_t^2 \, dt = \sum_{n=1}^\infty \xi_n^2 = 0$ a.e. [P] which contradicts $E(\int_o^1 Y_t^2 \, dt) = 1$. It follows that $X_t, t \in T$, does not have a measurable modification.

REMARK 3. For Gaussian processes it can be easily shown that (ii) implies (i) without relying on the results of [5]; this is done in [15, p. 44].

COROLLARY 1. Let R be a symmetric, nonnegative definite, real function on T×T. If $\mathcal{R}(R)$ is separable the following are equivalent.

(i) R(t,.) is $\mathcal{B}(T)$ - measurable for all t in T.

(ii) R is $\mathcal{B}(T)\times\mathcal{B}(T)$ - measurable.

PROOF. It suffices to show that (i) implies (ii). Since R symmetric, nonnegative definite and real, there exists a probability space $(\Omega,\mathcal{F},P)$ and a real process $X_t, t\in T$, on it with autocorrelation R. It is clear from part (a) of the proof of Theorem 1 that the separability of $\mathcal{R}(R)$ and (i) imply the existence of a measurable modification of $X_t, t\in T$, and thus (ii). This result can be shown in a simpler way without using an associated process. Indeed, if $\{a_n\}_{n=1}^{\infty}$ is a complete orthonormal set in $\mathcal{R}(R)$, then it is easily seen that $R(t,s) = \sum_{n=1}^{\infty} a_n(t)\, a_n(s)$ for all t,s in T. Now (i) implies as in part (a) of the proof of Theorem 1 that every a_n is $\mathcal{B}(T)$ - measurable and thus (ii) holds. □

COROLLARY 2. A second order process X_t, $t\in T$, which satisfies any of the following conditions has a measurable modification (in(iii) also progressively measurable).

(i) X_t, $t\in T$, is weakly continuous on T.

(ii) T is an arbitrary interval and X_t, $t\in T$, has orthogonal increments.

(iii) T is an abitrary interval and X_t, $t\in T$, is a martingale.

PROOF. (i) Since T is separable and X_t weakly continuous on T, H(X) is separable [16, p. 272]. By the weak continuity of X_t, R(t,.)

is continuous, hence $\mathcal{B}(T)$ - measurable, for all t in T. The conclusion follows from Corollary 1 and Theorem 1.

(ii) It is known that H(X) is separable [8, p. 110]. Also, that X_t has left and right $L_2(\Omega,\mathcal{F},P)$ limits on T and that except on a countable subset of T, $X_{t-} = X_t = X_{t+}$. This implies the measurability of R and the result follows from Theorem 1.

(iii) Define the function F by $F(t) = E(X_t^2)$ for all t in T. By the martingale property, with respect to the nondecreasing family $\mathcal{F}_t$, $t\in T$, of sub-σ-algebras of $\mathcal{F}$, we have for all $s\le t$ in T, $E(X_t X_s) = E[E(X_t X_s/\mathcal{F}_s)] = E[X_s E(X_t/\mathcal{F}_s)] = E(X_s^2)$ and thus

$$E(\{X_t-X_s\}^2) = F(t)-F(s).$$

It follows from this relationship, as in [8, p. 110] and in (ii), that H(X) is separable and R is $\mathcal{B}(T)\times\mathcal{B}(T)$ - measurable. □

REMARK 4. Let X_t, $t\in T$, T an arbitrary interval, be a real separable process of second order with autocorrelation R. If X_t is mean square differentiable on T and $\frac{\partial R(t,s)}{\partial t}$, $\frac{\partial^2 R(t,s)}{\partial t \partial s}$ are locally Lebesgue integrable in t and in t,s respectively,then with probability one the paths of X_t, $t\in T$, are absolutely continuous on every compact subinterval of T. This is shown in [10, pp. 186-187] with the additional assumption that the mean square derivative X_t' of X_t has a measurable modification, which is always satisfied because of Theorem 1. Indeed, since X_t is mean square differentiable on T, it is mean square continuous on T. Thus H(X) is separable and the continuity of R implies the measurability of $\frac{\partial^2 R(t,s)}{\partial t \partial s}$. Since $\frac{\partial^2 R(t,s)}{\partial t \partial s}$ is the autocorrelation of X_t' and since $H(X') \subseteq H(X)$, the conclusion follows from Theorem 1.

We conclude this section with a property which is useful in connection with problems involving conditional probabilities; such as for instance the existence of jointly measurable densities (see [9, pp. 616-617]) and properties related to metric transitivity (see [17, Ch. IV. 8]). A σ-algebra is called separable if it is generated by a countable class of sets. A sub-σ-algebra F' of F is said to coincide mod 0 with the σ-algebra F if for every set E in F there is a set E' in F' such that $P(E \Delta E') = 0$. Let $F(X)$ be the sub-σ-algebra of F generated by the random variables $\{X_t, t \in T\}$. It is known that if X_t is continuous in probability on T, $F(X)$ coincides mod 0 with a separable σ-algebra. Corollary 3 generalizes this result (and in fact, as it is clear from [6], it is valid for any process with values in a compact metric space).

COROLLARY 3. If a real process X_t, $t \in T$, has a measurable modification, then $F(X)$ coincides mod 0 with a separable σ-algebra.

PROOF. Since X_t, $t \in T$, has a measurable modification, $\{[X_t], t \in T\}$ is a separable subset of M. Thus there exists a countable subset $M' = \{[X_t], t \in S\}$ of $\{[X_t], t \in T\}$ (S is a countable subset of T) such that for every t in T, $[X_t]$ is the limit in probability of a sequence from M', and thus X_t is the a.e.[P] limit of a sequence from $\{X_t, t \in S\}$. If F' is the sub-σ-algebra of F generated by the random variables $\{X_t, t \in S\}$, then $F' \subseteq F(X)$, F' is separable and $F(X)$ coincides with F' mod 0. □

2. ON THE SEPARABILITY OF THE LINEAR SPACE OF A SECOND ORDER PROCESS

The linear space H(X) of a second order process X_t, $t \in T$, plays an important role in the structure of such processes and in a variety of problems in statistical inference. If H(X) is separable then X_t admits series representations and also integral representations (Theorem 2) that can be effectively used in problems such as linear mean square estimation. Also the separability of H(X) is the only condition needed for a second order process to have the Hida-Cramér representation (see for instance [11]). It is thus of interest that a measurable second order process has a separable linear space. H(X) is known to be separable when the process X_t, $t \in T$, is weakly continuous [16, p. 272], has orthogonal increments [8, p. 110], or is a martingale (Corollary 2.(iii)). In Theorem 2 necessary and sufficient conditions are given for H(X) to be separable in terms of integral representations of X_t.

Before stating the theorem we mention a few basic facts about random measures, that can be found for instance in [7, 16]. Let $(V, \mathcal{V})$ be a measurable space. A random measure Z on $(V, \mathcal{V})$ is a countably additive map from $\mathcal{V}$ to $L_2(\Omega, \mathcal{F}, P)$; i.e., whenever A is the disjoint union of the sets $A_n \in \mathcal{V}$, $Z(A) = \sum_{n=1}^{\infty} Z(A_n)$ in $L_2(\Omega, \mathcal{F}, P)$. (Here we consider the case where Z is defined on the entire σ-algebra $\mathcal{V}$). To each random measure Z on $\mathcal{V}$ there corresponds a finite signed measure μ on $\mathcal{V} \times \mathcal{V}$ by $\mu(A \times B) = E[Z(A)Z(B)]$, $A, B \in \mathcal{V}$. μ is symmetric and nonnegative definite on the measurable ractangles of $\mathcal{V} \times \mathcal{V}$. A random measure Z is called orthogonal if $\mu(A \times B) = 0$ whenever A and B are disjoint, and to each orthogonal random measure there corresponds a finite nonnegative measure ν on $\mathcal{V}$ by $\nu(A) = E[Z^2(A)]$, $A \in \mathcal{V}$. Let H(Z) be the closure in $L_2(\Omega, \mathcal{F}, P)$ of the linear span of $\{Z(A), A \in \mathcal{V}\}$, and let $\Lambda_2(\mu)$ be the Hilbert space of real, $\mathcal{V}$-measurable functions on V with inner product $\langle f,g \rangle_{\Lambda_2(\mu)} = \int_\Lambda \int_\Lambda f(u)g(v)d\mu(u,v)$ (of course $\Lambda_2(\mu)$ consists of equivalence classes of functions, two functions f and g considered identical if $\langle f-g, f-g \rangle_{\Lambda_2(\mu)} = 0$). There is an inner product preserving isomorphism between $\Lambda_2(\mu)$ and H(Z), denoted by $\leftrightarrow$, such that $I_A \leftrightarrow Z(A)$, $A \in \mathcal{V}$, and integration of functions in $\Lambda_2(\mu)$ with respect to Z is defined by $\xi = \int_V f(u)dZ(u)$, where $f \leftrightarrow \xi$. If Z is orthogonal, there is an inner product preserving isomorphism between $L_2(\nu) = L_2(V,\mathcal{V},\nu)$ and $H(Z)$, denoted again by $\leftrightarrow$, such that $I_A \leftrightarrow Z(A)$, $A \in \mathcal{V}$, and integration of functions in $L_2(\nu)$ with respect to Z is defined by $\xi = \int_V f(u)dZ(u)$, where $f \leftrightarrow \xi$.

THEOREM 2. Let X_t, $t \in T$ be a second order process.

(i) If H(X) is separable then for every finite measure space $(V,\mathcal{V},\nu)$ such that $L_2(\nu) = L_2(V,\mathcal{V},\nu)$ is separable and infinite dimensional, X_t has a representation

$$X_t = \int_V f(t,u)\, dZ(u) \quad \text{for all } t \text{ in } T$$

where Z is an orthogonal measure on $\mathcal{V}$ with corresponding measure ν and $f(t,.) \in L_2(\nu)$ for all t in T. Conversely, if X_t has such a representation, H(X) is separable.

(ii) If H(X) is separable, then for every measurable space $(V,\mathcal{V})$ and every finite signed measure μ on $\mathcal{V}\times\mathcal{V}$ which is symmetric and non-negative difinite on the measurable rectangles of $\mathcal{V}\times\mathcal{V}$, and such that $\Lambda_2(\mu)$ is separable and infinite dimensional, X_t has a representation

$$X_t = \int_V f(t,u)\, dZ(u) \quad \text{for all } t \text{ in } T$$

where Z is a random measure on $\mathcal{V}$ with corresponding measure μ and $f(t,.) \in \Lambda_2(\mu)$ for all t in T. Conversely, if X_t has such a representation, H(X) is separable.

PROOF. (i) being a particular case of (ii), we will prove only (ii). We start with the second claim. If X_t has such a representation then $X_t \in H(Z)$ for all t in T, hence $H(X) \subseteq H(Z)$ and the conclusion follows from the isomorphism between H(Z) and $\Lambda_2(\mu)$ and the separability of the latter. We now prove the first claim. Assume that H(X) is separable and let $\{\xi_n\}_{n=1}^{\infty}$ be a complete orthonormal set. Then for all t in T

$$X_t = \sum_{n=1}^{\infty} a_n(t)\xi_n$$

in $L_2(\Omega,F,P)$, where $a_n(t) = E(X_t\xi_n)$. Let $\{f_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $\Lambda_2(\mu)$. Since μ is finite, $I_A \in \Lambda_2(\mu)$ for all $A \in V$. Then

$$I_A = \sum_{n=1}^{\infty} \lambda_n(A) f_n$$

in $\Lambda_2(\mu)$, where

$$\lambda_n(A) = \langle I_A, f_n \rangle = \int_A\int_V f_n(v)\,d\mu(u,v).$$

Throughout the proof we will write $\langle .,. \rangle$ for $\langle .,. \rangle_{\Lambda_2(\mu)}$. Thus for all n, λ_n is a finite signed measure on (V,V). We also have

$$\sum_{n=1}^{\infty} \lambda_n^2(A) = \langle I_A, I_A \rangle = \mu(A\times A) < +\infty.$$

Hence

$$Z(A) = \sum_{n=1}^{\infty} \lambda_n(A)\xi_n$$

defines a function from V to $L_2(\Omega,F,P)$ (the convergence being in $L_2(\Omega,F,P)$). We will show that Z is a random measure with corresponding measure μ. The latter is clear since for all $A,B \in V$ we have

$$E[Z(A)Z(B)] = \sum_{n=1}^{\infty} \lambda_n(A)\lambda_n(B) = \langle I_A, I_B \rangle = \mu(A\times B).$$

For the countable additivity of Z let $A = \cup_{k=1}^{\infty} A_k$, where $\{A_k\}_{k=1}^{\infty}$ is a disjoint sequence of sets in $\mathcal{V}$. Then

$$\begin{aligned} E[\{Z(A)-\sum_{k=1}^{K} Z(A_k)\}^2] &= \sum_{n=1}^{\infty} \{\lambda_n(A)-\sum_{k=1}^{K} \lambda_n(A_k)\}^2 \\ &= \sum_{n=1}^{\infty} \{\sum_{k=K}^{\infty} \lambda_n(A_k)\}^2 \\ &= \sum_{n=1}^{\infty} \lambda_n^2(\bigcup_{k=K}^{\infty} A_k) \\ &= \mu(\bigcup_{k=K}^{\infty} A_k \times \bigcup_{k=K}^{\infty} A_u) \xrightarrow[K\to\infty]{} o \end{aligned}$$

since $\bigcup_{k=K}^{\infty} A_k \downarrow \emptyset$ as $K \to \infty$. Thus $Z(A) = \sum_{k=1}^{\infty} Z(A_n)$.

We now show that for every g in $\Lambda_2(\mu)$,

$$\int_V g dZ = \sum_{n=1}^{\infty} \langle g, f_n \rangle \xi_n$$

in $L_2(\Omega,\mathcal{F},P)$. This is true for indicator functions by definition of Z, and therefore also for simple functions. Since H(Z) is defined as the $L_2(\Omega,\mathcal{F},P)$ closure of the linear space of $\{Z(A),\ A\epsilon\mathcal{V}\}$, it follows by the isomorphism between $\Lambda_2(\mu)$ and H(Z) that the linear span of $\{I_A,\ A\epsilon\mathcal{V}\}$ is dense in $\Lambda_2(\mu)$. Thus every g in $\Lambda_2(\mu)$ is the $\Lambda_2(\mu)$ - limit of a sequence of simple functions $\{g_k\}_{k=1}^{\infty}$. Thus

$$\begin{aligned} \int_V g dZ &= \lim_{k\to\infty} \int_V g_k dZ \\ &= \lim_{k\to\infty} \sum_{n=1}^{\infty} \langle g_k, f_n \rangle \xi_n \end{aligned}$$

where $\lim_{k\to\infty}$ is in $L_2(\Omega,\mathcal{F},P)$ and the result follows from

$$E[\{\sum_{n=1}^{\infty} \langle g, f_n \rangle \xi_n - \sum_{n=1}^{\infty} \langle g_k, f_n \rangle \xi_n\}^2]$$

$$= E[\{\sum_{n=1}^{\infty} \langle g-g_k, f_n\rangle \ \xi_n\}^2]$$

$$= \sum_{n=1}^{\infty} \langle g-g_n, f_n\rangle^2$$

$$= \langle g-g_k, g-g_k\rangle \xrightarrow[k\to\infty]{} o.$$

In particular we have $\int_V f_n dZ = \xi_n$ which implies that $H(Z) = H(X)$.

Now since $\sum_{n=1}^{\infty} a_n^2(t) = R(t,t) <+\infty$ for all t in T, we can define $f(t,.)$ in $\Lambda_2(\mu)$ for all t in T by

$$f(t,u) = \sum_{n=1}^{\infty} a_n(t) f_n(u)$$

where the convergence is in $\Lambda_2(\mu)$. It follows from the property of the integral just proven that for all t in T we have the following equality in $L_2(\Omega,F,P)$,

$$\int_V f(t,u)dZ(u) = \sum_{n=1}^{\infty} a_n(t)\xi_n = X_t$$

which concludes the proof. □

REMARK 5. We assume throughout this remark that $H(X)$ is separable. Then it is clear that the first claim in (i) and (ii) is valid provided the dimensionality of $L_2(\nu)$ and $\Lambda_2(\mu)$ is no less than the dimensionality of the integers. Also, one can take $(V,\mathcal{V}) = (T, \mathcal{B}(T))$ or as V any interval and $\mathcal{V}$ its Borel sets; in the latter case ν may be taken the Lebesgue measure or one absolutely continuous to it, and μ may be taken absolutely continuous to the Lebesgue measure on $V\times V$. If a series (respectively, integral) representation of X_t is known then one can obtain integral (respectively, series) representations of X_t as indicated in the proof of Theorem 2. These representations will be explicitly obtained if one can find complete orthonormal sets in the spaces

$L_2(\nu)$ and $\Lambda_2(\mu)$. If V is an interval and V its Borel sets, complete orthonormal sets in $L_2(\nu)$ are given in [13] (see also [2]), and complete sets in $\Lambda_2(\mu)$ are given in [3] (In [3] the case where V is the entire real line is treated and the case where V is an interval can be treated similarly). If neither an integral nor a series representation of X_t is available, the problem arises how to obtain explicitly such a representation (in terms of the process X_t $t \in T$, and its autocorrelation R). This problem is solved in [4] for weakly continuous processes X_t $t \in T$, and T an arbitrary interval.

REMARK 6. Theorem 2 may also be stated in terms of integral representation of the autocorrelation R, which for (i) and (ii) are respectively

$$R(t,s) = \int_V f(t,u)\, f(s,u)\, d\nu(u)$$

$$R(t,s) = \int_V \int_V f(t,u)\, f(s,v)\, d\mu(u,v) \qquad \text{for all } t,s \text{ in } T.$$

REMARK 7. In [12] a second order process X_t, $t \in R^1 = (-\infty, +\infty)$ is called oscillatory if it has a representation

$$X_t = \int_{-\infty}^{\infty} e^{itu}\, a_t(u)\, dZ(u) \quad \text{for all } t \text{ in } R^1$$

where Z is an orthogonal random measure on $(R^1, \mathcal{B}(R^1))$ with corresponding measure ν and $a_t(.) \in L_2(\nu)$ for all t in T (this is a generalization of a concept introduced by Priestley). If X_t, $t \in R^1$, is oscillatory then H(X) is separable, since $L_2(R^1, \mathcal{B}(R^1), \nu)$ is separable. Conversely, if H(X) is separable it follows by Theorem 1. (i) that for any finite measure ν on $(R^1, \mathcal{B}(R^1))$ we have $X_t = \int_{-\infty}^{\infty} f(t,u)\, dZ(u)$ for all t in T, where Z is an orthogonal random measure on $(R^1, \mathcal{B}(R^1))$ with corresponding measure ν and $f(t,.) \in L_2(\nu)$ for all t in $\tilde{T}$. If we define $a_t(u) = e^{-itu}f(t,u)$, it becomes clear that X_t, $t \in R^1$, is oscillatory. Thus a second order process is oscillatory if and only if its linear space is separable.

REMARK 8. Some simple sufficient conditions for H(X) to be separable are as follows. If X_t, $t \in T$, is a linear operation on a second order process Y_s, $s \in S$, with separable linear space, then $H(X) \subseteq H(Y)$ and the separability of H(X) follows from that of H(Y). Also, because of the isomorphism between H(X) and $\mathcal{R}(R)$, H(X) is separable if there is a symmetric, nonnegative definite function K on T×T such that $\mathcal{R}(R) \subseteq \mathcal{R}(K)$ and $\mathcal{R}(K)$ is separable. A sufficient condition for $\mathcal{R}(R) \subseteq \mathcal{R}(K)$ is that K-R be nonnegative definite [1, p. 354].

REFERENCES

1. Aronszajn, N.: Theory of reproducing kernels. Trans. Amer. Math. Soc. 68, 337-404 (1950).

2. Cambanis, S.: Bases in L_2 spaces with applications to stochastic processes with orthogonal increments. Proc. Amer. Math. Soc. 29, 284-290 (1971).

3. Cambanis, S., Liu, B.: On harmonizable stochastic processes. Information and Control 17, 183-202 (1970).

4. Cambanis, S., Masry, E.: On the representation of weakly continuous processes. Information Sci. 3, 277-290 (1971).

5. Chung, K.L., Doob, J.L.: Fields, optionality and measurability. Amer. J. Math. 87, 397-424 (1965).

6. Cohn, D.L.: Measurable choice of limit points and the existence of separable and measurable processes. Z. Wahrscheinlichkeitstheorie verw. Geb. 22, 161-165 (1972).

7. Cramér, H.: A contribution to the theory of stochastic processes. Proc. Second Berkeley Symp. Math. Statist. Prob., 329-339. Berkeley: Univ. California Press 1951.

8. Cramér, H., Leadbetter, M.R.: Stationary and Related Stochastic Processes. New York: Wiley 1967.

9. Doob, J.L.: Stochastic Processes. New York: Wiley 1953.

10. Gikhman, I.I., Skorokhod, A.V.: Introduction to the Theory of Random Process. Philadelphia: Saunders 1969.

11. Kallianpur, G., Mandrekar, V.: Multiplicity and representation theory of purely non-deterministic stochastic processes. Theor. Probability Appl. 10, 553-581 (1965).

12. Mandrekar, V.: A characterization of oscillatory processes and their prediction. Proc. Amer. Math. Soc. 32, 280-284 (1972).

13. Masry, E., Liu, B., Steiglitz, K.: Series espansion of wide-sense stationary random processes. IEEE Trans. Information Theory 14, 792-796 (1968).

14. Neveu, J.: Mathematical Foundations of the Calculus of Probability. San Francisco: Holden-Day 1965.

15. Neveu, J.: Processus Aléatoires Gaussiens. Montréal: Presses Univ. Montréal 1968.

16. Parzen, E.: Time Series Analysis Papers. San Francisco: Holden-Day 1967.

17. Rasanov, Yu. A.: Stationary Random Processes. San Francisco: Holden-Day 1967.

Stamatis Cambanis
Department of Statistics
University of North Carolina
Chapel Hill, N.C. 27514