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GENERAL MOMENT FUNCTIONS AND A DENSITY VERSION OF THE CENTRAL LIMIT THEOREM*

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§ 1. INTRODUCTION

Let $X_1, X_2, \ldots,$ *ad inf.* be an infinite sequence of iid random variables such that $EX_i = 0$ and $EX_i^2 = 1$. It is well known that

$$\frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} = Z_n \text{ , say,}$$

tends, in distribution, as $n \to \infty$ to the normal distribution with p.d.f.

$$\phi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} .$$

This is the famous Central Limit Theorem. In some situations it is desirable to be able to claim that the p.d.f. of Z_n tends to $\phi(x)$, a conclusion that is, in general, false. An important theorem on this subject is the following one to be found in Gnedenko-Kolmogorov (1968, p. 224):

THEOREM A: Let $\{X_n\}$ *be a sequence of iid random variables with* $EX_n = 0$ *and* $EX_n^2 = 1$. *Let* $Z_n = (X_1 + X_2 + \ldots + X_n)/\sqrt{n}$. *If, for some integer* $m \geq 1$, Z_m *possesses a density that is integrable in the r-th power* $(1 < r \leq 2)$, *then for all large* n , *the d.f.* F_n, *say, of* Z_n *is absolutely continuous with a p.d.f.* f_n *such that*

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$$f_n(x) \to \phi(x) \quad \text{as} \quad n \to \infty ,$$

uniformly in x, $-\infty < x < +\infty$.

A rather different theorem directed to the same question was given by Smith (1953):

THEOREM B: Let $\{X_n\}$ *be a sequence of iid random variables with* $EX_n = 0$, $EX_n^2 = 1$ *and* Z_n *as before. Suppose that* $E|X_n|^r < \infty$ *for some integer* $r \geq 2$ *and suppose that for some constant* $\alpha > 0$ *and large real* t, $Ee^{itX_n} = O(|t|^{-\alpha})$. *Then* Z_n *has a density* f_n *for all large* n *and*

$$|x|^{\gamma} \, |f_n(x) - \phi(x)| \to 0 , \text{ as } n \to \infty ,$$

uniformly in x, $-\infty < x < +\infty$, *for all* γ *such that* $0 \leq \gamma \leq r$.

Be it noted that Theorem A imposes a weaker condition on the distribution of the $\{X_n\}$ than does Theorem B. On the other hand, granted the condition in Theorem B on the characteristic function, the conclusion is more informative and, possibly, more useful. In particular, Theorem B has enabled Cox and Smith (1954) to prove some quite general renewal theorems.

The purpose of this note is to prove a Theorem which embraces the ideas of both the above theorems and even allows one to introduce quite general kinds of moments of the $\{X_n\}$ rather than the moments of integral order in Theorem B.

DEFINITION: *A function* $M(x)$ *of* x, *defined for* $x \geq 0$ *will be said to belong to the class* $\mathcal{M}$ *of*

i) $M(x)$ *is non-decreasing* ,

ii) $\frac{M(x)}{x}$ *is non-increasing* ,

iii) $M(0) > 0$.

In Smith (1969) some properties of this class of functions are investigated in detail. Typical examples of such functions may be ones which (as $x \to \infty$) asymptotically equal

a) $x^{\delta} \log x$ with $0 < \delta < 1$,

b) a constant.

If $M(x) \in \mathcal{M}$, and $\nu \geq 0$ is a real number, we shall write

$$\mathcal{M}_{\nu} = E|X_n|^{\nu} M(|X_n|)$$

when this moment exists. We shall write

$$\lambda_{\nu} = E|X_n|^{\nu} \; ; \; \mu_{\nu} = EX_n^{\nu}$$

for the familiar absolute and ordinary moments respectively. Our theorem is then:

THEOREM I. Let $\{X_n\}$ *be a sequence of iid random variables with an absolutely continuous d.f.,* $EX_n = 0$ *and* $EX_n^2 = 1$. *Suppose* $M(x) \in \mathcal{M}$. *Then if*

i) $\mathcal{M}_{\nu} < \infty$ *for some* $\nu \geq 2$ *(so that* $\lambda_{\nu} < \infty$*)* ,

ii) for some $m \geq 1$ *the sum* $X_1+\ldots+X_m$ *has a p.d.f. which is integrable in the r-th power* $(1 < r \leq 2)$,

it follows that $Z_n = n^{-\frac{1}{2}}(X_1+\ldots+X_n)$ *has a p.d.f. such that*

$$|x|^{\gamma} M(|x|)\,|f_n(x) - \phi(x)| \to 0$$

as $n \to \infty$, *uniformly in* x , $-\infty < x < \infty$ *for any* γ, $0 \leq \gamma \leq \nu$.

The usefulness of this theorem lies in the fact that it enables us to write, when suitable conditions for the theorem apply,

$$f_n(x) = \phi(x) + \frac{r_n(x)}{1 + |x|^{\nu} M(|x|)}$$

where $r_n(x) \to 0$ as $n \to \infty$, uniformly in x, $-\infty < x < +\infty$. This result can be a useful tool in theoretical investigations, and it is fortunate that it makes full use of whatever knowledge is available concerning the existence of quite general moments of the random variables.

It might be noted that condition I(ii) can easily be shown to be equivalent to the following alternative condition:-

I^*(ii) *If* $\omega(t) = Ee^{itX_n}$ *is the characteristic function of the* $\{X_n\}$ $-\infty < t < \infty$, *then for some* $p > 0$, $\{\omega(t)\}^p$ *is in* $L_1(-\infty, +\infty)$.

In presenting Theorem I we have supposed the $\{X_n\}$ have a p.d.f. This is not assumed in Theorems A and B; their conditions merely ensure that Z_n has a p.d.f. for all sufficiently large n . We have chosen to assume a p.d.f. for the $\{X_n\}$ to simplify in one or two places an already complicated proof. Condition I(ii) will, as in Theorem A, ensure that Z_n has a p.d.f. for all large n; we leave it to the reader to note the places in our proof where, at a slight increase in complications, we can avoid our "simplifying" assumption. Be it noted that considerable ingenuity must be expended to produce an example in which the $\{X_n\}$ do *not* have an absolutely continuous distribution and Z_n, for some n, does.

§2. NOTATION AND PRELIMINARY LEMMAS

Throughout this paper $\{X_n\}$ will be a sequence of iid random variables with a p.d.f. $f(x)$, and c.f. $\omega(t) = Ee^{itX_n}$, $-\infty < t < +\infty$. We shall set

$$Z_n = \frac{X_1+X_2+\ldots+X_n}{\sqrt{n}}$$

and write $\Omega_n(t) = Ee^{itZ_n}$. Thus

$$\Omega_n(t) = \{\omega(t/\sqrt{n})\}^n , \tag{2.1}$$

as is well known.

If r is a positive real we write $\mu_r = EX_n^r$ for the familiar ordinary moment (when it exists) and $\lambda_r = E|X_n|^r$ for the absolute moment.

For each integer n and real x we define

(2.2) $$\alpha_n(t,x) = \int_{|u|\le|x|\sqrt{n}} e^{itu} f(u)\, du ,$$

(2.3) $$\beta_n(t,x) = \omega(t) - \alpha_n(t,x) ,$$

(2.4) $$A_n(t,x) = \{\alpha_n(t/\sqrt{n},x)\}^n ,$$

(2.5) $$B_n(t,x) = \Omega_n(t) - A_n(t,x)$$

$$= \sum_{j=1}^{n} \binom{n}{j} \{\alpha_n(t/\sqrt{n}, x)\}^{n-j} \{\beta_n(t/\sqrt{n}, x)\}^j .$$

Whenever the following inversion integrals are absolutely convergent we set†

(2.6) $$f_n(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itu}\, \Omega_n(t)\, dt ,$$

(2.7) $$a_n(u,x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itu}\, A_n(t,x)\, dt ,$$

(2.8) $$b_n(u,x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itu}\, B_n(t,x)\, dt ,$$

for each integer n and real valued u and x. If r is a positive integer and the following derivatives with respect to t are known to exist, we denote them thus:

$$\alpha_n^{(r)}(t,x) = \frac{\partial^r}{\partial t^r} \alpha_n(t,x) ,$$

$$A_n^{(r)}(t,x) = \frac{\partial^r}{\partial t^r} A_n(t,x) ,$$

---

† These absolutely convergent integrals provide the *continuous* p.d.f's that we shall use, and which are the subjects of our theorem.

and so on.

We also write $p(x) \equiv P\{|X_n| > x\}$ and $q(x) = 1 - p(x)$ for real $x > 0$. We write "$h(x) \in L^{(r)}$" to mean

$$\int_{-\infty}^{+\infty} |h(x)|^r \, dx < \infty .$$

Finally, $\phi(x)$ and $N(t)$ will denote the p.d.f. and the c.f., respectively, of the standard normal distribution.

We are now ready to prove some basic lemmas.

LEMMA 2.1 *For any integer* $r \le n$,

$$n^{\frac{1}{2}r} A_n^{(r)}(t\sqrt{n}, x) = \sum_{j=1}^{r} \sum\nolimits_j^r \; n^{[j]} C_r(r_1, r_2, \ldots, r_j) \alpha_n^{(r_1)}(t,x) \alpha_n^{(r_2)}(t,x) \ldots$$

$$\ldots \alpha_n^{(r_j)}(t,x) [\alpha_n(t,x)]^{n-j}$$

where the summation $\sum_j^r$ *extends over the j-part partitions* $(r_1, r_2, \ldots, r_j)$ *of the integer* r *and the coefficients* $C_r(r_1, r_2, \ldots, r_j)$ *depend only on* $r_1, r_2, \ldots, r_j$ *and are independent of* n.

Note that $n^{[j]} \equiv n(n-1)\ldots(n-j+1)$ and that $r_1, r_2, \ldots, r_j$ may not be all distinct.

PROOF: The lemma follows from the r-fold differentiation with respect to t of (see (2.4)) the equation $A_n(t\sqrt{n}, x) = \{\alpha_n(t, x)\}^n$.

An easy combinatorial argument shows that

$$\frac{r!}{r_1!\, r_2! \ldots r_j!} \times \frac{n!}{j_1!\, j_2! \ldots j_s! (n-j)!}$$

$$= n^{[j]} C_r(r_1, r_2, \ldots, r_j) ,$$

from which the claim made for the coefficients C_r is seen to be valid.

LEMMA 2.2 *For each fixed $x \neq 0$, and all real t, if $\mu_2 = 1$ and $\mu_1 = 0$, $A_n(t,x) \to \exp(-\frac{1}{2}t^2)$, as $n \to \infty$. Moreover, this convergence is uniform with respect to x in $|x| \geq 1$.*

PROOF: By the Central Limit Theorem, $\Omega_n(t) \to \exp(-\frac{1}{2}t^2)$ as $n \to \infty$. Thus, from (2.5), we need to show $B_n(t,x) \to 0$, uniformly for $|x| \geq 1$.

Since $\mu_2 < \infty$

$$nx^2 \, p(|x|\sqrt{n}) \to 0 \, , \; n \to \infty \, , \tag{2.9}$$

uniformly in $|x| \geq 1$. Also

$$|B_n(t,x)| \leq \sum_{j=1}^{n} \binom{n}{j} \left|\alpha_n\left(t/\sqrt{n}\, , \, x\right)\right|^{n-j} \left|\beta_n\left(t/\sqrt{n},x\right)\right|^j$$

$$\leq \sum_{j=1}^{n} \binom{n}{j} \{p(|x|\sqrt{n}\,)\}^j$$

$$\leq \exp\{np(|x|\sqrt{n})\} - 1$$

The right hand member of this inequality tends to zero uniformly in $|x| \geq 1$, by (2.9). Thus the lemma is essentially proved (clearly the set $|x| \geq 1$ can be replaced by $|x| \geq \delta$ for any fixed arbitrarily small $\delta > 0$).

LEMMA 2.3 *If $\mu_2 = 1$ and $\mu_1 = 0$ then for each fixed integer k*

$$A_n^{(k)}(t,x) \to N^{(k)}(t) \, , \quad n \to \infty$$

for all real t and real fixed $x \neq 0$.

PROOF: Fix $x \neq 0$ and real. Since

$$\alpha_n(t,x) = \int_{|u| \leq |x|\sqrt{n}} e^{itu} \, f(u) \, du \, ,$$

it is evident that $\alpha_n(t,x)$ is an entire function in the complex t-plane. Furthermore

$$\alpha_n^{(1)}(0,x) = i \int_{|u|\le|x|\sqrt{n}} u\, f(u)\, du$$

$$= -i \int_{|u|>|x|\sqrt{n}} u\, f(u)\, du \quad ,$$

since $\mu_1 = 0$. Therefore

$$|\alpha_n^{(1)}(0,x)| \le \int_{|u|>|x|\sqrt{n}} u\, f(u)\, du$$

$$\le \frac{1}{|x|\sqrt{n}} \quad ,$$

if we appeal to the fact that $\mu_2 = 1$.

One can also obtain the equation

$$\alpha_n^{(2)}(t,x) = -\int_{|u|\le|x|\sqrt{n}} u^2\, e^{itu}\, f(u)\, du \ ,$$

so that, for any $K > 0$, provided $|\mathcal{I}t| \le K/|x|\sqrt{n}$,

$$|\alpha_n^{(2)}(t,x)| \le e^K \ .$$

But

$$\alpha_n(t,x) = \alpha_n(0,x) - t\alpha_n^{(1)}(0,x) - \int_0^t (z-t)\, \alpha_n^{(2)}(z,x)\, dz \ ,$$

where the contour integral can be taken along a straight line joining 0 to t. Thus, provided $|\mathcal{I}t| \le K/|x|$,

$$\left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right| \le 1 + \frac{|t|}{n|x|} + \frac{|t|^2 e^K}{2n^2} \ .$$

It follows that in the intersection C_{KL}, say, of the circle $|t| \le L$, for some $L > 0$, and the strip $|It| \le K/|x|$, we have

$$|A_n(t,x)| \le \left(1 + \frac{L}{n|x|} + \frac{L^2 e^K}{2n^2}\right)^n ,$$

$$\le D(|x|) \text{ , say,}$$

where

$$D(|x|) = \exp\left(\frac{L}{|x|} + \frac{1}{2} L^2 e^K\right) .$$

Therefore, for each $x \ne 0$, $\{A_n(t,x)\}$ is a sequence of entire functions such that:

(i) $|A_n(t,x)| \le D(|x|)$ for all $t \in C_{KL}$ and all n;

(ii) For all real t $\lambda A_n(t,x) \to e^{-\frac{1}{2}t^2}$, $n \to \infty$.

Hence, by Vitali's convergence theorem,

$$A_n(t,x) \to e^{-\frac{1}{2}t^2} \text{ as } n \to \infty$$

uniformly with respect to t *in* C_{KL} . By a familiar theorem on uniformly convergent sequences of analytic functions (see, e.g. Titchmarsh (1939) §2.81) it follows that $A_n^{(k)}(t,x) \to N^{(k)}(t)$ uniformly with respect to t in C_{KL} . Since L can be chosen arbitrarily large it is clear that the lemma is proved.

LEMMA 2.4 *Suppose $F(x)$ to be absolutely continuous; fix $\delta > 0$, and let $n_0 > 0$ be a fixed integer. Then if*

$$\lambda =_{def} \sup_{\substack{|x| \ge 1 \\ |t| > \delta \\ n \ge n_0}} |\alpha_n(t,x)|$$

it follows that $0 \le \lambda < 1$.

PROOF: If the lemma is false there must be sequences of reals $\{t_n\}$ such that $t_n > t_0$, and $\{y_n\}$ such that $y_n \to \infty$, with the property that

$$\int_{-y_n}^{+y_n} e^{it_n u} f(u)\, du \to 1 , \quad n \to \infty .$$

But, since

$$\int_{|u|>y_n} f(u)\, du \to 0$$

this implies $\omega(t_n) \to 1$. The Riemann-Lebesgue lemma insists that $\{t_n\}$ be a bounded sequence. Thus $\{t_n\}$ must have a finite limit point t^*, say, and the continuity of $\omega(t)$ then requires $\omega(t^*) = 1$. But we must then have $t^* \geq \delta$ and are forced to the contradiction that $F(x)$ is lattice. Thus the lemma is proved.

Let us write $r_0 = mr(r-1)^{-1}$. It is well known that condition I(ii) in section 1 implies that, for all $n > r_0$, $|\omega(t)|^n \epsilon L^{(1)}$ (and hence, since $|\omega(t)| \leq 1$ for all real t, $|\omega(t)|^n \epsilon L^{(2)}$ also). Thus the integral in (2.6) is absolutely convergent so the p.d.f. $f_n(u)$ exists, as claimed, and belongs to $L^{(1)} \cap L^{(2)}$. Since the probability measure defining $\omega(t)$ dominates the defective one defining $\alpha_n(t,x)$ (for fixed $x \neq 0$), the existence of $f_n(u)$ implies that of the defective p.d.f. $a_n(u,x)$ and $0 \leq a_n(u,x) \leq f_n(u)$ almost everywhere in u. Thus

$$\int_{-\infty}^{+\infty} [a_n(u,x)]^2\, du \leq \int_{-\infty}^{+\infty} |f_n(t)|^2\, dt .$$

and so, by Plancharel's theorem,

$$\text{(2.10)} \qquad \int_{\infty}^{\infty} |\alpha_n(t,x)|^{2n} \leq \int_{\infty}^{\infty} |\omega(t)|^{2n}\, dt .$$

This inequality enables us to prove:

LEMMA 2.5 *If* $\omega(t) \in L^{(n)}$ *for all* $n > r_0$ *and if* $\mu_1 = 0$, $\mu_2 < \infty$, *then*

$$\int_{-\infty}^{+\infty} |\alpha_n(t,x)|^n \, dt \le \frac{C}{\sqrt{n}}$$

for all large n *, where* C *is a finite constant independent of* $x \neq 0$.

PROOF: Choose a small $\delta > 0$, then there will be a finite $\eta > 0$ such that

$$|\omega(t)| \le e^{-\eta t^2}$$

for all $|t| \le \delta$. This is an easy consequence of a familiar expansion of $\omega(t)$ about $t = 0$, when $\mu_1 = 0$ and $\mu_2 < \infty$. Thus

$$\int_{-\delta}^{+\delta} |\omega(t)|^n \, dt \le \int_{-\delta}^{+\delta} e^{-n\eta t^2} \, dt$$

$$\le \frac{C_1}{\sqrt{n}} , \tag{2.11}$$

for some finite C_1.

Let

$$\lambda_1 = \sup_{|t| \ge \delta} |\omega(t)| ,$$

then $0 \le \lambda_1 < 1$ since $|\omega(t)|^{2n}$ is the c.f. of an absolutely continuous distribution.

Thus, if we fix $n_0 > r_0$ and let k be any integer, $k > n_0$,

$$\int_{|t|>\delta} |\omega(t)|^k dt \le \lambda_1^{(k-n_0)} \int_{|t|>\delta} |\omega(t)|^{n_0} \, dt$$

$$\le C_2 \lambda_1^k \tag{2.12}$$

for some finite $C_2 \ge 0$. From (2.12) and (2.11) it is immediate that

$$\int_{-\infty}^{+\infty} |\omega(t)|^k dt = O(\frac{1}{\sqrt{k}})$$

and then (2.10) shows the existence of a finite C_3, independent of $x \neq 0$, such that

$$\int_{-\infty}^{+\infty} |\alpha_k(t,x)|^{2k} dt \leq \frac{C_3}{\sqrt{k}}$$

Since $\alpha_{2k}(t,x) = \alpha_k(t, x\sqrt{2})$ we have

$$\int_{-\infty}^{+\infty} |\alpha_{2k}(t,x)|^{2k} dt \leq \frac{C_3}{\sqrt{k}}$$

for all $x \neq 0$, i.e. the lemma is proved for n even. The case of n odd, say $n = 2k+1$, follows easily from the observations:

(i) $$|\alpha_{2k+1}(t,x)|^{(2k+1)} \leq |\alpha_{2k+1}(t,x)|^{2k} ;$$

(ii) $$\alpha_{2k+1}(t,x) = \alpha_{2k}(t,\tilde{x}) ,$$

where $$\tilde{x} = x\sqrt{\frac{2k+1}{2k}} .$$

LEMMA 2.6 *For some sufficiently small* $\delta > 0$ *and all large* n *,*

$$|\alpha_n(t,x)|^2 \leq e^{-\frac{1}{4}t^2}$$

in the range $|t| \leq \delta$ *.*

PROOF: Consider the equation

(2.13) $$|\alpha_n(t,x)|^2 = \iint_K e^{it(u-v)} f(u)f(v)\, du\, dv,$$

where K is the (u,v)-set where $|u| \leq |x|\sqrt{n}$ and $|v| \leq |x|\sqrt{n}$. Plainly

(2.14) $$|\alpha_n(0,x)|^2 \leq 1 .$$

If we differentiate (2.13) once with respect to t we obtain

$$\frac{d}{dt}|\alpha_n(t,x)|^2\Big|_{t=0} = 0 , \tag{2.15}$$

and a second differentiation yields

$$\frac{d^2}{dt^2}|\alpha_n(t,x)|^2 = -\iint_K e^{it(u-v)} (u-v)^2 f(u) f(v)\, du\, dv .$$

Thus

$$\frac{d^2}{dt^2}|\alpha_n(t,x)|^2 - \frac{d^2}{dt^2}|\omega(t)|^2$$

$$= \iint_{\sim K} e^{it(u-v)}(u-v)^2 f(u) f(v)\, du\, dv .$$

But the right-hand double integral will be made small as $n \to \infty$ uniformly in t. Further, since $|\omega(t)|^2$ is the characteristic function of a random variable with zero expectation and variance 2, its second derivative will be arbitrarily near 2 for all sufficiently small $|t|$. Thus, for all sufficiently small $|t|$ and all large n,

$$\frac{d^2}{dt^2}|\alpha_n(t,x)|^2 \le -1 .$$

If we combine this inequality with (2.14) and (2.15) we find

$$|\alpha_n(t,x)|^2 \le 1 - \frac{t^2}{2} \le e^{-\frac{1}{2}t^2}$$

Thus the lemma is proved.

LEMMA 2.7 *Given* $\delta > 0$ *there exists* $\lambda(\delta), 0 < \lambda < 1$, *such that for all sufficiently large* m *and* n,

$$\int_{|t|>\delta} |\alpha_n(t,x)|^m\, dt < A\lambda^m ,$$

where A *is independent of* x, $|x| \ge 1$.

PROOF: Fix an integer q, say, such that

$$\int_{-\infty}^{+\infty} |\alpha_q(t,x)|^q \, dt < A_1 \text{ , say,}$$

for all $|x| \geq 1$. Lemma 2.5 assures us that we can find such a finite A_1 and such a q. Let n_0, λ, be as presented in Lemma 2.4. Then, for $m > q$, $n > \max(q, n_0)$

$$\int_{|t|>\delta} |\alpha_n(t,x)|^m \, dt < \lambda^{m-q} \int_{-\infty}^{+\infty} |\alpha_n(t,x)|^q \, dt \quad .$$

Let $|x^*|\sqrt{q} = |x|\sqrt{n}$ and note that $|x| > 1$ implies $|x^*| > 1$. Then

$$\int_{-\infty}^{+\infty} |\alpha_n(t,x)|^q \, dt = \int_{-\infty}^{+\infty} |\alpha_q(t,x^*)|^q \, dt < A_1 \; .$$

Thus the lemma is proved, with $A = A_1 \lambda^{-q}$.

LEMMA 2.8 *If* $\mu_1 = 0$ *and* $\mu_2 = 1$, *then*

$$\sqrt{n}\, \alpha_n^{(1)} \left(\frac{t}{\sqrt{n}}, x\right) \to -t \quad ,$$

$$\alpha_n^{(2)} \left(\frac{t}{\sqrt{n}}, x\right) \to -1 \quad ,$$

uniformly in $|x| \geq 1$, *as* $n \to \infty$.

PROOF: It is known (Smith(1953), for example) that $\sqrt{n}\, \omega^{(1)}(t/\sqrt{n}) \to -t$ and that $\omega^{(2)}(t/\sqrt{n}) \to -1$. Thus we need to show $\sqrt{n}\, \beta_n^{(1)}(\frac{t}{\sqrt{n}}, x) \to 0$ and $\beta_n^{(2)}(\frac{t}{\sqrt{n}}, x) \to 0$ uniformly. Now

$$\sqrt{n}\, \beta_n^{(1)} \left(\frac{t}{\sqrt{n}}, x\right) = i\sqrt{n} \int_{|u|>|x|\sqrt{n}} e^{itu/\sqrt{n}} \, u \, f(u) \, du$$

so that, for $|x| \geq 1$,

$$|\sqrt{n}\ \beta_n^{(1)}\ (\frac{t}{\sqrt{n}}\ ,x)|\ \leq \int_{|u|>\sqrt{n}} u^2\ f(u)\ du$$

The right-hand member of this inequality tends to zero as $n \to \infty$. A similar argument deals with $\beta_n^{(2)}(\frac{t}{\sqrt{n}}\ ,x)$.

§3 PROOF OF THEOREM I .

It is well-known that condition I(ii) implies that $|\omega(t)|^n \in L^{(1)}$ for all $n > r_0$ and we have seen that this implies the existence of the defective p.d.f. $a_n(u,x)$ for fixed $x \neq 0$ and that $a_n(u,x) \in L^{(1)} \cap L^{(2)}$ This implies that the Fourier transform $A_n(t,x) \in L^{(1)} \cap L^{(2)}$ also (for $n > r_0$). From (2.5) we can then deduce that $B_n(t,x)$ also belongs to $L^{(1)} \cap L^{(2)}$ for $n > r_0$.

In proving the theorem we may plainly assume, with no loss of generality, that $M(|x|) \geq 1$ for all x and that $M(|x|)/|x|$ is non-increasing for $|x| \geq 1$. In view of Theorem A we need only prove that

$$|x|^\gamma\ M(|x|)\,|f_n(x)\ -\ \phi(x)|\ \to 0$$

as $n \to \infty$ uniformly in $|x| \geq 1$. Indeed we need prove this result only for the case $\gamma = \nu$. Thus our proof will be done if we can show

$$\lim_{n\to\infty}\ |x|^\nu\ M(|x|)\ b_n(x,x)\ =\ 0 \tag{3.1}$$

$$\lim_{n\to\infty}\ |x|^\nu\ M(|x|)\,|a_n(x,x)\ -\ \phi(x)|\ =\ 0\ , \tag{3.2}$$

uniformly in $|x| \geq 1$.

To begin, we note that for $|x| \geq 1$,

$$n^{\frac{1}{2}\nu}\ |x|^\nu M(|x|\sqrt{n})\ p(|x|\sqrt{n}\ )$$

$$\leq \int_{|u|\geq\sqrt{n}} |u|^\nu\ M(|u|)\ f(u)\ du$$

$$= Q(\sqrt{n})\ ,\ \text{say.}$$

But, since $M_\nu < \infty$, $Q(\sqrt{n}) \to 0$ as $n \to \infty$. Thus, for any fixed $K > 0$ we can find $n_0(K)$ such that

$$p(|x|\sqrt{n}) \leq \frac{Q(\sqrt{n})}{n^{\frac{1}{2}\nu}M(\sqrt{n})} < e^{-K} \tag{3.3}$$

for all $n \geq n_0$, $|x| \geq 1$.

For ease let us write

$$\sum{}' \quad \text{for} \quad \sum_{j=1}^{j=[\frac{1}{2}n]} ,$$

$$\sum{}'' \quad \text{for} \quad \sum_{j=1+[\frac{1}{2}n]}^{j=n-2k} ,$$

$$\sum{}''' \quad \text{for} \quad \sum_{j=n-2k}^{j=n} ,$$

where k is some fixed large integer, $n > 2k$, and $[y]$ means "the integer part of y." Then (2.5) and (2.8) show

$$|b_n(x,x)| \leq \frac{1}{2\pi} \{ \sum{}' + \sum{}'' + \sum{}''' \} \binom{n}{j} \int_{-\infty}^{+\infty} \left|\alpha_n(\frac{t}{\sqrt{n}},x)\right|^{n-j} \left|\beta_n(\frac{t}{\sqrt{n}}, x)\right|^j dt .$$

We deal with the three sums separately. First we note that

$$|x|^\nu M(|x|) \sum{}' \binom{n}{j} \int_{-\infty}^{+\infty} \left|\alpha_n(\frac{t}{\sqrt{n}},x)\right|^{n-j} \left|\beta_n(\frac{t}{\sqrt{n}},x)\right|^j dt$$

$$= F_n^{(1)}(x) \text{ , say,}$$

$$\leq C_1 |x|^\nu M(|x|) \sum{}' \binom{n}{j} \{p(|x|\sqrt{n})\}^j ,$$

(since $|\beta_n(\frac{t}{\sqrt{n}},x)| \leq p(|x|\sqrt{n}))$ if we can show there is a finite constant C_1 such that

$$\int_{-\infty}^{+\infty} |\alpha_n(\frac{t}{\sqrt{n}},x)|^{n-j}\, dt \le C_1 \quad ,$$

for $1 \le j \le [\frac{1}{2}n]$. But this is ensured by Lemma 2.5. Thus

$$F_n^{(1)}(x) \le C_1[\{1+|x|^\nu M(|x|)\ p(|x|\sqrt{n})\}^n - 1]$$

$$\le C_1[\exp\{n|x|^\nu M(|x|)\ p(|x|\sqrt{n})\} - 1]$$

when we bear in mind that $|x| \ge 1$ and $M(|x|) \ge 1$. If we now appeal to (3.3) we see that this last expression tends to zero as $n \to \infty$. uniformly in $|x| \ge 1$. Thus

$$F_n^{(1)}(x) \to 0 \quad \text{as } n \to \infty$$

uniformly in $|x| \ge 1$.

Next, let

$$F_n^{(2)}(x) = |x|^\nu M(|x|) \sum{}'' \binom{n}{j} \int_{-\infty}^{+\infty} \left|\alpha_n(\frac{t}{\sqrt{n}},x)\right|^{n-j} \left|\beta_n(\frac{t}{\sqrt{n}},x)\right|^j dt \ .$$

Then Lemma 2.5 and (3.3) show that

$$F_n^{(2)}(x) \le C\sqrt{n}|x|^\nu M(|x|)\ p(|x|\sqrt{n})\ e^{-K[\frac{1}{2}n]} \sum{}'' \binom{n}{j}$$

$$\le C\{\sqrt{n}|x|^\nu M(|x|)\ p(|x|\sqrt{n})\} e^{-K[\frac{1}{2}n]}\ 2^n$$

and, provided K is chosen large enough, this last expression also tends to zero uniformly in $|x| \ge 1$.

Since $\beta_n(t,x) = \omega(t) - \alpha_n(t,x)$ we can appeal to Lemma 2.5 to see that a constant C_2, say, exists such that

$$\int_{-\infty}^{+\infty} |\beta_n(t,x)|^{2k}\, dt < \frac{C_2}{\sqrt{n}}$$

for all n and all $x \neq 0$, provided k is large enough. Thus, if we suppose $n > 6k - 1$ and $n - 2k + 1 \le j \le n$,

$$\binom{n}{j} \int_{-\infty}^{+\infty} |\alpha_n(\tfrac{t}{\sqrt{n}},x)|^{n-j} |\beta_n(\tfrac{t}{\sqrt{n}},x)|^j \, dt$$

$$\le C_3 \, n^{n-j+\frac{1}{2}} \{p(|x|\sqrt{n})\}^{j-2k}$$

$$\le C_3 \, n^{2k-\frac{1}{2}} \{p(|x|\sqrt{n})\}^{n-4k+1} ,$$

for some finite constant C_3. Therefore, in an obvious extension of our notation,

$$F_n^{(3)}(x) \le C_3 |x|^{\nu} M(|x|) \sum{}''' \{np(|x|\sqrt{n})\}^{2k}$$

$$= C_3(2k+1)|x|^{\nu} M(|x|)\{np(|x|\sqrt{n})\}^{2k}$$

from which it follows easily that $F_n^{(3)}(x) \to 0$ as $n \to \infty$ uniformly in $|x| \ge 1$. Combining the results on $F_n^{(1)}(x)$, $F_n^{(2)}(x)$, $F_n^{(3)}(x)$ we see that (3.1) is proved.

To deal with (3.2) we have to break the argument down into two cases.

CASE 1: $\nu > 2$

Let us define the integer $\kappa_\nu = [\nu] + 1$. Let us also set $L(|x|) = M(|x|)/|x|$, so that $L(|x|)$ decreases as $|x|$ increases. With a nodding reference to Lemma 2.1 we introduce the following notations.

$$A_{2,n}^{(\kappa_\nu+1)}(t,x) = n^{-\frac{1}{2}(\kappa_\nu-1)} \alpha_n^{(\kappa_\nu+1)}(\tfrac{t}{\sqrt{n}},x)\{\alpha_n(\tfrac{t}{\sqrt{n}},x)\}^{n-1} ,$$

$$A_{3,n}^{(\kappa_\nu+1)}(t,x) = C_{(\kappa_\nu+1)}(1,\kappa_\nu)\, n^{-\frac{1}{2}(\kappa_\nu+1)}\, n^{[2]}$$

$$\times\, \alpha_n^{(\kappa_\nu)}\left(\frac{t}{\sqrt{n}},x\right) \alpha_n^{(1)}\left(\frac{t}{\sqrt{n}},x\right) \left[\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right]^{n-2} ,$$

$$A_{1,n}^{(\kappa_\nu+1)}(t,x) = A_n^{(\kappa_\nu+1)}(t,x) - A_{2,n}^{(\kappa_\nu+1)}(t,x) - A_{3,n}^{(\kappa_\nu+1)}(t,x) .$$

We then write

$$D_{j,n}(x) = \int_{-\infty}^{+\infty} e^{-itx} A_{j,n}^{(\kappa_\nu+1)}(t,x)\, dt , \quad j = 2,3,$$

and

$$D_{1,n}(x) = \int_{-\infty}^{+\infty} e^{-itx}\{A_{1,n}^{(\kappa_\nu+1)}(t,x) - N^{(\kappa_\nu+1)}(t)\}\, dt .$$

We note that, when the "pseudo-density" $a_n(u,x)$ exists (x fixed),

$$A_n(t,x) = \frac{1}{2\pi} \int e^{itu}\, a_n(u,x)\, dn ,$$

where the integral is over a finite range. Thus we may differentiate with respect to t under the integral sign to establish that

$$A_n^{(\kappa_\nu+1)}(t,x)$$

is the Fourier Transform of

$$(iu)^{(\kappa_\nu+1)}\, a_n(u,x)$$

Thus, so long as the integral is absolutely convergent,

$$(ix)^{(\kappa_\nu+1)} a_n(x,x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-itx} A_n^{(\kappa_\nu+1)}(t,x)\, dt .$$

This, and similar considerations for the normal distribution, enable us to write

$$|x|^{\nu} M(|x|)|a_n(x,x) - \phi(x)|$$

$$= \frac{1}{2\pi} \frac{L(|x|)}{|x|^{\kappa_\nu - \nu}} \left| D_{1,n}(x) + D_{2,n}(x) + D_{3,n}(x) \right|$$

For each $x \neq 0$ and every integer $r > 2$, we deduce from dominated convergence that

$$n^{-\frac{1}{2}(r-2)} \alpha_n^{(r)}\left(\frac{t}{\sqrt{n}},x\right) \to 0, \quad n \to \infty \tag{3.4}$$

A mean value theorem gives us that

$$\alpha_n^{(1)}\left(\frac{t}{\sqrt{n}},x\right) = \alpha_n^{(1)}(0,x) + \frac{t}{\sqrt{n}} \ddot{\alpha}_n^{(2)}\left(\frac{\xi t}{\sqrt{n}},x\right), \tag{3.5}$$

for some $0 < \xi < 1$. But $\sqrt{n}\,\alpha_n^{(1)}(0,x)$ converges to zero as $n \to \infty$ uniformly in $|x| \geq 1$. Thus there is a constant $c_1 > 0$ such that

$$\sqrt{n}\left|\alpha_n^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right| \leq c_1 + |t| \tag{3.6}$$

for all n, $|x| \geq 1$. From (3.4) and (3.6) we find that for each x with $|x| \geq 1$,

$$\lim_{n\to\infty} A_{j,n}^{(\kappa_\nu+1)}(t,x) = 0, \quad j = 2,3. \tag{3.7}$$

Next we prove:

LEMMA 3.1 *For all* t, *as* $n \to \infty$,

$$A_{1,n}^{(\kappa_\nu+1)}(t,x) \to N^{(\kappa_\nu+1)}(t),$$

uniformly in $|x| \geq 1$.

PROOF: If we refer to the definition of $A_{1,n}^{(\kappa_\nu+1)}(t,x)$ and Lemma 2.1 we see it is the sum of terms like

$$(3.8)\qquad n^{-\frac{1}{2}(\kappa_\nu+1)} n^{[j]}\, C_{\kappa_\nu+1}(k_1,\ldots,k_j)\alpha_n^{(k_1)}\left(\frac{t}{\sqrt{n}},x\right)\cdots$$

$$\cdots\alpha_n^{(k_j)}\left(\frac{t}{\sqrt{n}},x\right)\left[\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right]^{n-j},$$

with $k_1 + k_2 + \ldots + k_j = \kappa_\nu + 1$. Suppose that ℓ of the numbers $k_1, k_2, \ldots, k_j$ are 1 and the rest are at least 2.

Then

$$\kappa_\nu + 1 \geq \ell + 2(j-\ell)$$

so that

$$(3.9)\qquad j - \tfrac{1}{2}(\kappa_\nu + 1) \leq \tfrac{1}{2}\ell$$

with equality if and only if $k_r \leq 2$ for all $r = 1,2,\ldots,j$.

a) Suppose '<' holds in (3.9).

Then the absolute value of (3.8) is

$$\leq \text{Constant} \cdot n^{j-\frac{1}{2}(\kappa_\nu+\ell+1)}\left|\sqrt{n}\,\alpha_n^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right|^{\ell}$$

$$\leq \text{Constant} \cdot n^{j-\frac{1}{2}(\kappa_\nu+\ell+1)}(c_1 + |t|)^{\ell}$$

by (3.6). This last expression tends to zero as $n \to \infty$, uniformly in $|x| \geq 1$.

b) Suppose '=' holds in (3.9). In this case (3.8) has the special form

$$n^{[j]} n^{-\frac{1}{2}(\kappa_\nu+\ell+1)}\left[\sqrt{n}\,\alpha_n^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right]^{\ell}$$

$$\times\left[\alpha_n^{(2)}\left(\frac{t}{\sqrt{n}},x\right)\right]^{j-\ell}\times\left[\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right]^{n-j}$$

and thus, as $n \to \infty$, converges to

$$(-t)^{\ell} \times (-1)^{j-\ell} \times e^{-\frac{1}{2}t^2} ,$$

uniformly in $|x| \geq 1$, by Lemmas 2.2 and 2.8.

Thus each term of the finite sum representing $A_{1,n}^{(\kappa_\nu+1)}(t,x)$ converges to a finite limit as $n \to \infty$, uniformly in $|x| \geq 1$. In view of Lemma 2.3 and the limits (3.7), we have proved Lemma 3.1.

Let us set, for some large $R > 0$,

$$I_1(n,x) = \int_{-R}^{+R} |A_{1,n}^{(\kappa_\nu+1)}(t,x) - N^{(\kappa_\nu+1)}(t)|\, dt ,$$

$$I_2 \quad = \int_{|t|>R} |N^{(\kappa_\nu+1)}(t)|\, dt ,$$

and for some small $\delta > 0$ and large n,

$$I_3(n,x) = \int_{R\leq|t|\leq\delta\sqrt{n}} |A_{1,n}^{(\kappa_\nu+1)}(t,x)|\, dt ,$$

$$I_4(n,x) = \int_{|t|>\delta\sqrt{n}} |A_{1,n}^{(\kappa_\nu+1)}(t,x)|\, dt.$$

Then we have

$$|D_{1,n}(x)| \leq I_1(n,x) + I_2 + I_3(n,x) + I_4(n,x) \quad .$$

An examination of the proof of Lemma 3.1 in conjunction with (3.6) and the easily proved fact that $|\alpha_n^{(2)}(\frac{t}{\sqrt{n}},x)| \leq 1$, will show that there is a polynomial $P(|t|)$ in $|t|$, of degree not exceeding $\kappa_\nu + 1$ and with non-negative coefficients, such that

$$|A_{1,n}^{(\kappa_\nu+1)}(t,x)| \leq P(|t|)\,|\alpha_n(\frac{t}{\sqrt{n}},x)|^{n-\kappa_\nu-1} \quad . \tag{3.10}$$

In the range $|t| \leq R$, (3.10) shows that the convergence of Lemma 3.1 is bounded convergence, the bound being uniform in $|x| \geq 1$. Thus

(3.11) $$I_1(n,x) \to 0 \quad \text{as } n \to \infty,$$

uniformly in $|x| \geq 1$. Furthermore, given any small $\varepsilon > 0$ we can choose R so large that

(3.12) $$I_2 < \varepsilon$$

Next we observe, from Lemma 2.6 and (3.10), that

$$\left|A_{1,n}^{(\kappa_\nu+1)}(t,x)\right| \leq P(|t|)e^{-\frac{t^2(n-\kappa_\nu-1)}{4n}} .$$

Thus

$$I_3(n,x) \leq \int_{R\leq|t|} P(|t|)\, e^{-\frac{t^2(n-\kappa_\nu-1)}{4n}}\, dt$$

and the right-hand member, by monotone converges, goes to

$$\int_{R\leq|t|} P(|t|)\, e^{-\frac{1}{4}t^2}\, dt \quad , \text{as } n \to \infty .$$

This can be made as small as we please by choice of sufficiently large R. Thus for large fixed R and all large n

(3.13) $$I_3(n,x) < \varepsilon \quad , \quad \text{uniformly in } |x| \geq 1.$$

We must now deal with $I_4(n,x)$. We note first that

$$|\alpha_n^{(r)}(\tfrac{t}{\sqrt{n}},x)| \leq \lambda_r \quad \text{for } r=0,1,\ldots,\kappa_\nu-1.$$

Thus, referring to (3.8) and Lemma 2.1, we see that

$$(3.14)\quad |A_{1,n}^{(\kappa_\nu+1)}(t,x)| \le \frac{1}{n^{\frac{1}{2}(\kappa_\nu+1)}} \sum_{j=2}^{\kappa_\nu+1} \sum_j^{\kappa_\nu+1} n^j C_{\kappa_\nu+1}(k_1,\ldots,k_j)\lambda_{k_1}\lambda_{k_2}\cdots\lambda_{k_j} \times \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-\kappa_\nu-1} .$$

If p and q are any integers, we have from Lemma 2.7 that

$$n^p \int_{|t|>\delta\sqrt{n}} \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-q} dt < n^{p+\frac{1}{2}} A\lambda^{n-q} .$$

Since $0 < \lambda < 1$, the right-hand term tends to zero as $n \to \infty$. Reference to (3.14) then shows that

$$(3.15)\qquad I_4(n,x) \to 0\ , \quad \text{as } n \to \infty\ ,$$

uniformly in x, $|x| \ge 1$.

Combining (3.11), (3.12), (3.13), and (3.15) shows that $|D_{1n}(x)| < 4\varepsilon$ for all sufficiently large n, uniformly in $|x| \ge 1$. Thus,

$$D_{1n}(x) \to 0\ , \quad \text{as } n \to \infty\ ,$$

uniformly in $|x| \ge 1$. It then follows, *a fortiori* that

$$(3.16)\qquad \frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} D_{1n}(x) \to 0\ , \quad \text{as } n \to \infty\ ,$$

uniformly in $|x| \ge 1$.

For fixed x in $|x| \ge 1$, let I_{nx} be the set of reals u such that $1 \le |u| \le |x|\sqrt{n}$. Then, for $|x| \ge 1$, we have an obvious argument,

$$|\alpha_n^{(\kappa_\nu+1)}(t,x)| \le \int_{-1}^{+1} |u|^{\kappa_\nu+1} f(u)\,du + \int_{I_{nx}} |u|^{\kappa_\nu+1} f(u)\,du$$

$$\le 1 + \int_{I_{nx}} |u|^{\kappa_\nu-\nu} \frac{M(|u|)}{L(|u|)} |u|^\nu f(u)\,du$$

$$\le 1 + \frac{|x|^{\kappa_\nu-\nu} n^{\frac{1}{2}(\kappa_\nu-\nu)}}{L(|x|\sqrt{n})} \cdot M_\nu \,. \tag{3.17}$$

Thus, for $|x| \ge 1$,

$$|A_{2n}^{(\kappa_\nu+1)}(t,x)| \le n^{-\frac{1}{2}(\kappa_\nu-1)} + \frac{|x|^{\kappa_\nu-\nu} n^{\frac{1}{2}(x_\nu-\nu)} \cdot M_\nu}{L(|x|\sqrt{n})\, n^{\frac{1}{2}(\kappa_\nu-1)}}$$

$$= n^{-\frac{1}{2}(\kappa_\nu-1)} + \frac{|x|^{\kappa_\nu+1-\nu} \cdot M_\nu}{M(|x|\sqrt{n})\, n^{\frac{1}{2}(\kappa_\nu-2)}}$$

and we note that since $\kappa_\nu > \nu > 2$, the right-hand side tends to zero as $n \to \infty$, for fixed x.

We use (3.17) in the following steps, in which we suppose $|x| \ge 1$,

$$\frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} \left|D_{2n}(x)\right| \le \frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} \cdot n^{-\frac{1}{2}(\kappa_\nu-1)} \int_{-\infty}^{+\infty} \left|\alpha_n^{(\kappa_\nu+1)}\left(\frac{t}{\sqrt{n}},x\right)\right| \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-1} dt$$

$$\le \frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} \cdot n^{-\frac{1}{2}(\kappa_\nu-1)} \left\{1 + \frac{|x|^{\kappa_\nu-\nu} n^{\frac{1}{2}(\kappa_\nu-\nu)}}{L(|x|\sqrt{n})} M_\nu \right\}$$

$$\times \int_{-\infty}^{+\infty} \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-1} dt$$

$$\le \left\{\frac{L(1)}{n^{\frac{1}{2}(\kappa_\nu-1)}} + M_\nu \cdot \frac{L(|x|)}{L(|x|\sqrt{n})} n^{-\frac{1}{2}(\nu-1)}\right\} \times \int_{-\infty}^{+\infty} \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-1} dt$$

$$\leq \left\{\frac{L(1)}{n^{\frac{1}{2}(\kappa_\nu-1)}} + \frac{M_\nu}{n^{\frac{1}{2}(\nu-2)}}\right\} \int_{-\infty}^{+\infty} \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-1} dt$$

since M is non-decreasing, and so

$$\frac{L(|x|)}{\sqrt{n}L(|x|\sqrt{n})} = \frac{M(|x|)}{M(|x|\sqrt{n})} \leq 1 .$$

But

$$\frac{L(1)}{n^{\frac{1}{2}(\kappa_\nu-1)}} + \frac{M_\nu}{n^{\frac{1}{2}(\nu-2)}} \to 0$$

as $n \to \infty$. Further, Lemma 2.5 shows (after an easy modification of a kind we have already demonstrated) that

$$\int_{-\infty}^{+\infty} \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-1} dt$$

is bounded uniformly in $|x| \geq 1$. Thus

(3.18)
$$\frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} D_{2n}(x) \to 0 , \quad n \to \infty ,$$

uniformly in $|x| \geq 1$.

To complete our discussion of the case $\nu > 2$ we must deal with $D_{3n}(n)$. We note that, for $|x| \geq 1$,

$$\frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} \left|D_{3n}(x)\right|$$

$$\leq \frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} n^{-\frac{1}{2}(\kappa_\nu-3)} \int_{-\infty}^{+\infty} \left|\alpha_n^{(\kappa_\nu)}\left(\frac{t}{\sqrt{n}},x\right)\right| \left|\alpha_n^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right| \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-2} dt$$

$$\leq \frac{L(|x|)}{|x|^{\kappa_\nu-\nu}} \cdot n^{-\frac{1}{2}(\kappa_\nu-3)}|x|^{\kappa_\nu-\nu}\, n^{\frac{1}{2}(\kappa_\nu-\nu)}\lambda_\nu$$

$$\times \int_{-\infty}^{+\infty}\left|\alpha_n^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right|\left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-2} dt$$

$$\leq L(1)\lambda_\nu n^{-\frac{1}{2}(\nu-3)}\{J_1(n) + J_2(n) + J_3(n)\}\text{ , say,}$$

where, for $R>0$,

$$J_1(n) = \int_{|t|\leq R}\left|\alpha^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right|\left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-2} dt$$

$$J_2(n) = \int_{R\leq|t|\leq\delta\sqrt{n}}\left|\alpha^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right|\left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-2} dt$$

$$J_3(n) = \int_{|t|>\delta\sqrt{n}}\left|\alpha^{(1)}\left(\frac{t}{\sqrt{n}},x\right)\right|\left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-2} dt\ .$$

By (3.6):

$$n^{-\frac{1}{2}(\nu-3)}J_1(n) \leq n^{-\frac{1}{2}(\nu-3)}\, n^{-\frac{1}{2}}\int_{|t|\leq R}\left(|t| + c_1\right) dt$$

$$\leq 2R(R+c_1)\, n^{-\frac{1}{2}(\nu-2)}$$

$$\to 0\ ,\ \text{as } n\to\infty\ ,$$

uniformly in $|x| \geq 1$.

Inequality (3.6) also yields the result:

$$n^{-\frac{1}{2}(\nu-3)}J_2(n) \leq n^{-\frac{1}{2}(\nu-2)}\int_{\delta\sqrt{n}\geq|t|\geq R}(|t| + c_1)\left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-2} dt$$

$$\leq n^{-\frac{1}{2}(\nu-2)}\int_{\delta\sqrt{n}\geq|t|>R}(|t| + c_1)e^{-\frac{t^2}{4n}(n-2)} dt\ ,$$

by Lemma 2.6. Thus

$$n^{-\frac{1}{2}(\nu-3)} J_2(n) \to 0 \quad , \; n \to \infty \; ,$$

uniformly in $|x| \geq 1$.

The fact that

$$n^{-\frac{1}{2}(\nu-3)} J_3(n) \to 0 \quad , \; n \to \infty \; ,$$

uniformly in $|x| \geq 1$ follows from Lemma 2.7 and the inequality $|\alpha_n^{(1)}(t,x)| \leq \lambda_1$:-

$$n^{-\frac{1}{2}(\nu-3)} J_3(n) \leq n^{-\frac{1}{2}(\nu-3)} \cdot \mu_1 \cdot A\lambda^{n-2}\sqrt{n} \; .$$

The three limit results on J_1, J_2, J_3 reveal that

$$\frac{L(\,x\,)}{|x|^{\kappa\nu-\nu}} D_{3n}(x) \to 0 \quad , \; n \to \infty \quad ,$$

uniformly in $|x| \geq 1$. This result, together with (3.16) and (3.18) proves the desired result that

$$|x|^{\nu} M(|x|)\,|a_n(x,x) - \phi(x)| \to 0$$

as $n \to \infty$, uniformly in $|x| \geq 1$. This proves the theorem when $\nu > 2$.

CASE 2: $\nu = 2$: In the previous case the fact that $\nu > 2$ was used critically in dealing with $D_{2n}(x)$ and $D_{3n}(x)$. For $|x| \geq 1$ we observe that

$$x^2 M(|x|)\,|a_n(x,x) - \phi(x)|$$

$$= L(|x|)\,|(ix)^3\, a_n(x,x) - (ix)^3 \phi(x)|$$

$$= \frac{L(|x|)}{2\pi} \Big| \int_{-\infty}^{+\infty} e^{-itx} \{A_{1n}^{(3)}(t,x) - N^{(3)}(t)\} \; dt$$

$$+ \int_{-\infty}^{+\infty} e^{-itx} A_{2n}^{(3)} \; (t,x) \; dt \Big| \; ,$$

where we now write

$$A_{1n}^{(3)}(t,x) = n^{-\frac{3}{2}} \{n^{[3]}\{\alpha_n^{(1)}(\frac{t}{\sqrt{n}},x)\}^3\{\alpha_n(\frac{t}{\sqrt{n}},x)\}^{n-3}$$

$$+ 3n^{[2]}\{\alpha_n^{(1)}(\frac{t}{\sqrt{n}},x)\}\{\alpha_n^{(2)}(\frac{t}{\sqrt{n}},x)\}\{\alpha_n(\frac{t}{\sqrt{n}},x)\}^{n-2}\}$$

and

$$A_{2n}^{(3)}(t,n) = n^{-\frac{1}{2}} \alpha_n^{(3)}(\frac{t}{\sqrt{n}},x)\{\alpha_n(\frac{t}{\sqrt{n}},x)\}^{n-1} .$$

By Lemmas 2.2 and 2.8 we see that

(3.19) $$A_{1n}^{(3)}(t,x) \to N^{(3)}(t) \quad , \quad n \to \infty .$$

Moreover this convergence is uniform in $|x| \geq 1$. The arguments used in Case 1 to deal with $D_{1n}(x)$ will now show that

$$\frac{M(|x|)}{|x|} \int_{-\infty}^{+\infty} \left| A_{1n}^{(3)}(t,x) - N^{(3)}(t) \right| dt \to 0 ,$$

as $n \to \infty$,uniformly in $|x| \geq 1$.

Next, still supposing $|x| \geq 1$, we see that

$$\frac{L(|x|)}{\sqrt{n}} \left| \alpha_n^{(3)}(\frac{t}{\sqrt{n}},x) \right|$$

$$\leq \frac{L(|x|)}{\sqrt{n}} \int_{|u| \leq |x|\sqrt{n}} u^3 f(u)\, du$$

$$\leq \frac{L(1)R^3}{\sqrt{n}} + \frac{L(|x|)}{\sqrt{n}L(|x|\sqrt{n})} \int_{R<|u|} u^2 M(|u|) f(u)\, du$$

for any $R > 0$, all sufficiently large n .

But, as before,

$$\frac{L(|x|)}{\sqrt{n}\, L(|x|\sqrt{n})} \leq 1 ,$$

and

$$\int_{R<|u|} u^2 M(|u|)\ f(u)\ du \to 0\ ,\ \text{as } R \to \infty\ .$$

Thus we can make

$$\frac{L(|x|)}{\sqrt{n}} \left|\alpha_n^{(3)}\left(\frac{t}{\sqrt{n}},x\right)\right| = \theta_n(t,x)\ ,\ \text{say},$$

uniformly small for all real t and all $|x| \geq 1$, by choosing R large and then n sufficiently large.

Let us set

$$\varepsilon(n) = \sup_{\substack{|x|\geq 1 \\ -\infty<t<+\infty}} \theta_n(t,x)$$

Then we have shown that $\varepsilon(n) \to 0$ as $n \to \infty$.

We now note that

$$L(|x|) \int_{-\infty}^{+\infty} \left|A_{2n}^{(3)}(t,x)\right| dt$$

$$\leq \varepsilon(n) \int_{-\infty}^{+\infty} \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-1} dt\ .$$

Reference to Lemma 2.5, (with the familiar change from x to x^*, say, dictated by $x^*\sqrt{n-1} = x\sqrt{n}$) will show that

$$\int_{-\infty}^{+\infty} \left|\alpha_n\left(\frac{t}{\sqrt{n}},x\right)\right|^{n-1} dt$$

is uniformly bounded in $|x| \geq 1$. Thus

$$L(|x|) \int_{-\infty}^{+\infty} \left|A_{2n}^{(3)}(t,x)\right| dt \to 0\ ,\ \text{as } n \to \infty\ ,$$

uniformly in $|x| \geq 1$. This is sufficient to complete the proof of Case 2 of the theorem.

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