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BOOLEAN SUMS OF SETS OF CERTAIN DESIGNS*

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ABSTRACT

In this paper the author applies Boolean sum operation on sets of certain known designs to get new series of designs. In particular certain unsymmetrical balanced incomplete block designs can be constructed from symmetrical balanced incomplete block designs and triangular designs can be constructed from irreducible balanced incomplete block designs and tactical configurations $C[k, 4, \delta, v]$. The solutions obtained by our method are non-isomorphic to some of the known designs. The possible generalizations of these concepts are also considered.

1. INTRODUCTION

A tactical configuration (or simply configuration) $C[k,\ell,\delta,v]$ is defined to be a system of subsets, of a set E of v symbols, having k symbols each, such that every subset of E having ℓ symbols is contained in exactly δ sets of the system (cf. Raghavarao (1971)).

A tactical configuration $C[k,\ell,\delta, v]$ when $\ell = 2$ is known as a balanced incomplete block (BIB) design. For a BIB design it is customary to denote δ by λ. Let each of the v symbols occur in r subsets of the configuration and let there be b subsets in the system. Then v,

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b, r, k, and λ are known as the parameters of the BIB design. The BIB design is said to be symmetrical if $v = b$; otherwise unsymmetrical. The BIB design with parameters

$$v,\ b = \binom{v}{k}\ ,\quad r = \binom{v-1}{k-1}\ ,\quad k,\lambda = \binom{v-2}{k-2}\ , \tag{1.1}$$

where $\binom{n}{r}$ denotes the number of combinations of r symbols from n symbols, is called irreducible BIB design.

A triangular design is an arrangement of $v = n(n-1)/2$ symbols in b sets each of size $k(<v)$ such that

(i) every symbol occurs at most once in a set,

(ii) every symbol occurs in exactly r sets, and

(iii) the v symbols will be arranged symmetrically in a $n \times m$ square array with blank diagonals as follows:

$$\begin{matrix} x & 1 & 2 & \cdots & n-1 \\ 1 & x & n & \cdots & 2n-3 \\ 2 & n & x & \cdots & 3n-6 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ n-1 & 2n-3 & 3n-6 & \cdots & x \end{matrix}$$

and two symbols occuring in the same row (or column) of the above array occur together in λ_1 sets and two symbols not occurring in the same row (or column) occur together in λ_2 sets of the design (cf. Bose and Shimamoto (1952), Raghavarao (1971)).

Given two sets S and T, the Boolean sum of S and T, denoted by $S + T$, is the set of elements which belongs to either S or T, but not to both. In symbols,

$$S + T = \{x | x \in S \cup T, \quad x \notin S \cap T\} . \tag{1.2}$$

In this paper we discuss the application of Boolean sums on sets of BIB designs and $C[k,4,\delta,v]$ configuration to get BIB and triangular designs.

2. BOOLEAN SUMS OF SETS OF SYMMETRICAL BIB DESIGNS

Let $v = b$, $r = k$, λ be the parameters of a symmetrical BIB design on the v symbols of a set $E = \{1,2,\ldots,v\}$ and let $S_1, S_2,\ldots,S_v$ be the v sets of the designs. Since the design is symmetrical, it is well known that the cardinality of sets $S_i \cap S_j$ for $i \neq j$ is λ and hence the cardinality of sets $S_i + S_j$ for $i \neq j$ is $2(r-\lambda)$. Now if we consider the $v(v-1)/2$ sets $T_{ij} = S_i + S_j$ for $i < j$, we can easily verify that they form an unsymmetrical BIB design with parameters

$$v^* = v,\ b^* = v(v-1)/2,\ r^* = r(v-r),\ k^* = 2(r-\lambda) ,$$

$$\lambda^* = (r-\lambda)^2 + \lambda(v-2r+\lambda) \tag{2.1}$$

As an illustration, let us consider the BIB design with parameters

$$v = 7 = b , r = 3 = k, \quad \lambda = 1 , \tag{2.2}$$

whose sets are

$$\begin{array}{lcl} S_1: (1, 2, 4) & ; & S_2: (2, 3, 5) ; \\ S_3: (3, 4, 6) & ; & S_4: (4, 5, 7) ; \\ S_5: (5, 6, 1) & ; & S_6: (6, 7, 2) ; \\ & S_7: (7, 1, 3) . & \end{array} \tag{2.3}$$

The 21 sets T_{ij} $(i < j)$ obtained by taking Boolean sums of every pair of S_i $(i = 1,2,\ldots,7)$ are the following

T_{12}: (1, 3, 4, 5) ; T_{13}: (1, 2, 3, 6) ;
T_{14}: (1, 2, 5, 7) ; T_{15}: (2, 4, 5, 6) ;
T_{16}: (1, 4, 6, 7) ; T_{17}: (2, 3, 4, 7) ;
T_{23}: (2, 4, 5, 6) ; T_{24}: (2, 3, 4, 7) ;
T_{25}: (1, 2, 3, 6) ; T_{26}: (3, 5, 6, 7) ;
T_{27}: (1, 2, 5, 7) ; T_{34}: (3, 5, 6, 7) ;
T_{35}: (1, 3, 4, 5) ; T_{36}: (2, 3, 4, 7) ;
T_{37}: (1, 4, 6, 7) ; T_{45}: (1, 4, 6, 7) ;
T_{46}: (2, 4, 5, 6) ; T_{47}: (1, 3, 4, 5) ;
T_{56}: (1, 2, 5, 7) ; T_{57}: (3, 5, 6, 7) ;

T_{67}: (1, 2, 3, 6) ; (2.4)

which form a BIB design with parameters

$$v^* = 7,\ b^* = 21,\ r^* = 12,\ k^* = 4,\ \lambda^* = 6\ . \tag{2.5}$$

However, one can notice that (2.4) is obtained by repeating the sets T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17} and T_{26} and without repetitions, the sets T_{12}, T_{13}, T_{14}, T_{15}, T_{16}, T_{17} and T_{26} form a BIB design with parameters

$$v^* = 7 = b^*,\ r^* = 4 = k^*,\ \lambda^* = 2\ . \tag{2.6}$$

The BIB design with parameters

$$v^* = 13,\ b^* = 78,\ k^* = 6,\ r^* = 36,\ \lambda^* = 15\ , \tag{2.7}$$

can be obtained as Boolean sum of sets of BIB design with parameters

$$v = 13 = b,\ \ r = 4 = k,\ \ \lambda = 1\ , \tag{2.8}$$

and can also be obtained by duplicating the sets of the BIB design with parameters

$$v' = 13,\ b' = 26,\ r' = 12,\ k' = 6,\ \lambda' = 5\ , \tag{2.9}$$

two more times. However, both the solutions will be non-isomorphic, in the sense that we cannot obtain one solution from another by re-numbering the symbols and sets, as there are no repeated sets in the solution obtained as Boolean sum of sets of design with parameters (2.8).

3. TRIANGULAR DESIGNS FROM IRREDUCIBLE BIB DESIGNS

Let

$$v,\ b = \binom{v}{k}\ ,\ \ r = \binom{v-1}{k-1}\ ,\ \ k, \lambda = \binom{v-2}{k-2}\ , \tag{3.1}$$

be parameters of an irreducible BIB design on v symbols of a set $E: \{1,2,\ldots,v\}$ and let $S_1, S_2,\ldots,S_b$ be its b sets. Let T_i be the set of indices of S's in which the i-th symbol occurs.

We now form $v^* = v(v-1)/2$ symbols by taking pairs of the form $(i\ j)$ $(i < j,\ i,\ j = 1,2,\ldots,v)$. We form b new sets $S_1', S_2', \ldots, S_b'$ where the symbol (i,j) occurs in the sets S''s whose indices are $T_i + T_j$. The sets $S_1', S_2',\ldots,S_b'$ thus formed can be verified to be a triangular design with parameters

$$v^* = v(v-1)/2,\ \ b^* = b,\ \ r^* = 2(v-2),\ \ k^* = k(v-k)\ ,$$
$$\lambda_1^* = \binom{v-2}{k-1}\ ,\ \ \lambda_2^* = 4\binom{v-4}{k-2}\ , \tag{3.2}$$

where the $v(v-1)/2$ symbols are arranged according to the array

x	12	13	. . .	1v
12	x	23	. . .	2v
13	23	x	. . .	3v
.	.	.	. . . .	.
1v	2v	3v	. . .	x

(3.3)

As an illustration, let us consider the irreducible BIB design with $v = 5$, $k = 3$ whose sets are:

$$S_1: (1, 2, 3) \; ; \quad S_2: (1, 2, 4) \; ; \quad S_3: (1, 2, 5) \; ;$$
$$S_4: (1, 3, 4) \; ; \quad S_5: (1, 3, 5) \; ; \quad S_6: (1, 4, 5) \; ;$$
$$S_7: (2, 3, 4) \; ; \quad S_8: (2, 3, 5) \; ; \quad S_9: (2, 4, 5) \; ;$$
$$S_{10}: (3, 4, 5) \; . \tag{3.4}$$

From the method discussed in this section, we produce the following 10 sets:

$$S_1': (14, 15, 24, 25, 34, 35) \; ; \quad S_2': (13, 15, 23, 25, 34, 45) \; ;$$
$$S_3': (13, 14, 23, 24, 35, 45) \; ; \quad S_4': (12, 15, 23, 35, 24, 45) \; ;$$
$$S_5': (12, 14, 23, 34, 25, 45) \; ; \quad S_6': (12, 13, 24, 34, 25, 35) \; ;$$
$$S_7': (12, 25, 13, 35, 14, 45) \; ; \quad S_8': (12, 24, 13, 34, 15, 45) \; ;$$
$$S_9': (12, 23, 14, 34, 15, 35) \; ; \quad S_{10}': (13, 23, 14, 24, 15, 25) \; ; \tag{3.5}$$

which can be seen to be a triangular design with parameters

$$v^* = 10 = b^* \; , \quad r^* = 6 = k^* \; , \quad \lambda_1^* = 3 \; , \quad \lambda_2^* = 4 \; ; \tag{3.6}$$

when the symbols are arranged in a 5×5 square

x	12	13	14	15
12	x	23	24	25
13	23	x	34	35
14	24	34	x	45
15	25	35	45	x

(3.7)

The design with parameters (3.6) can also be constructed by forming 10 blocks S_{ij}^* for $i < j$ and $i, j = 1,2,\ldots,5$ where S_{ij}^* contains the other symbols occurring in the rows of (3.7) where ij symbol occurs as follows:

$$S^*_{12}: (13, 14, 15, 23, 24, 25) \; ; \quad S^*_{13}: (12, 14, 15, 23, 34, 35) \; ;$$
$$S^*_{14}: (12, 13, 15, 24, 34, 45) \; ; \quad S^*_{15}: (12, 13, 14, 25, 35, 45) \; ;$$
$$S^*_{23}: (12, 24, 25, 13, 34, 35) \; ; \quad S^*_{24}: (12, 23, 25, 13, 34, 45) \; ;$$
$$S^*_{25}: (12, 23, 24, 15, 35, 45) \; ; \quad S^*_{34}: (13, 23, 35, 14, 24, 45) \; ;$$
$$S^*_{35}: (13, 23, 34, 15, 25, 45) \; ; \quad S^*_{45}: (14, 24, 34, 15, 25, 35) \; . \tag{3.8}$$

The solution (3.8) is the one given in the Tables (1954) and the solution obtained by our present method is non-isomorphic to the known solution.

4. TRIANGULAR DESIGNS FROM C[k, 4, δ, v] CONFIGURATIONS

Let $S_1, S_2, \ldots, S_b$ be the b sets of a $C[k, 4, \delta, v]$ configuration. Let x, y, z and u be any 4 symbols and let μ_{1000}, μ_{0100}, μ_{0010} and μ_{0001} be the number of sets in which x occurs in the absence of y, z, u ; y occurs in the absence of x, z, u ; z occurs in the absence of x, y, u ; and u occurs in the absence of x, y, z. Analogously, we introduce the notation μ_{1100}, μ_{1010}, μ_{1001}, μ_{0110}, μ_{0101}, μ_{0011}, μ_{1110}, μ_{1101}, μ_{1011}, μ_{0111}, and μ_{1111}. Let r be the number of sets in which each symbol occurs and let λ_i be the number of sets in which any i symbols occur together $(i = 2,3)$. Then we have the following relations:

$$\lambda_2 = \delta\binom{v-2}{2} \Big/ \binom{k-2}{2} \; ,$$

$$\lambda_3 = \delta(v-3)/(k-3) \tag{4.1}$$

$$\mu_{1110} + \mu_{1111} = \mu_{0111} + \mu_{1111} = \mu_{1011} + \mu_{1111} = \mu_{1101} + \mu_{1111} = \lambda_3$$

$$\mu_{1100} + \mu_{1101} + \mu_{1110} + \mu_{1111} = \lambda_2 \text{ , etc.}$$

$$\mu_{1000} + \mu_{1001} + \mu_{1010} + \mu_{1011} + \mu_{1100} + \mu_{1101} + \mu_{1110} + \mu_{1111} = r \text{ , etc.}$$

By following the method of Section 3, we can generate $b(b-1)$ sets S_i' which can be verified to form a triangular design with parameters

$$v^* = v(v-1)/2,\ b^* = b,\ \ r^* = 2(r-\lambda_2),\ k^* = (v-k)k,$$

$$\lambda_1^* = \mu_{1000} + \mu_{1001} + \mu_{0110} + \mu_{0111}$$

$$\lambda_2^* = \mu_{1010} + \mu_{1001} + \mu_{0110} + \mu_{0101}$$

when the symbols are arranged in an array given by (3.3).

5. GENERALIZATIONS

The concept of Boolean sum of two sets S and T, can be generalized to define Boolean sum of three sets R, S and T, denoted by $R + S + T$, to consist of all elements belonging to exactly one of the three sets or to all three sets. In symbols,

$$R + S + T = \{x|x \in R,\ x \notin S,T\} \cup \{x|x \in S,\ x \notin R,T\} \cup \{x|x \in T,\ x \notin R,S\} \cup \{x|x \in R \cap S \cap T\} \tag{5.1}$$

Then the method of Section 4 can be generalized and from $C[k,5,\delta,v]$ configuration we can generate $v(v-1)(v-2)/6$ symbols $\alpha\beta\gamma$ ($\alpha < \beta < \gamma$ $\alpha,\beta,\gamma = 1,2,\ldots,v$) and form b sets $S_1^*, S_2^*,\ldots,S_b^*$ in such a way that the symbol $\alpha\beta\gamma$ occur in the sets S^*'s whose subscripts belong to $T_\alpha + T_\beta + T_\gamma$, where T_α is the set of indices of sets S's in which the symbol α occurs. Such designs can be seen to be extended triangular designs as defined by John (1966).

The Boolean sums of sets of partially balanced incomplete block (PBIB) designs of Bose and Nair (1939), which are linked block designs as defined by Roy and Laha (1956) can again be verified to be PBIB designs.

One can take the Boolean sums of sets of PBIB designs with certain subsets of the v symbols to get new PBIB designs. For example, the Boolean sum of each of the groups of association scheme with each of the sets of a semi-regular group divisible design (cf. Bose and Connor (1952), Raghavarao (1971)) will result in a group divisible design.

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