

A CHERNOFF-SAVAGE REPRESENTATION OF RANK ORDER STATISTICS
FOR STATIONARY ϕ -MIXING PROCESSES

By

PRANAB KUMAR SEN AND MALAY GHOSH

Department of Biostatistics, U. N. C., Chapel Hill
and
Indian Statistical Institute, Calcutta

Institute of Statistics Mimeo Series No. 862

FEBRUARY 1973

A CHERNOFF-SAVAGE REPRESENTATION OF RANK ORDER STATISTICS
FOR STATIONARY ϕ -MIXING PROCESSES

By

PRANAB KUMAR SEN¹ AND MALAY GHOSH

University of North Carolina, Chapel Hill,
and Indian Statistical Institute, Calcutta

ABSTRACT

For stationary ϕ -mixing processes, a Chernoff-Savage representation of a general class of rank order statistics is considered and suitable orders (in probability or almost surely) of the remainder terms are specified. These are then utilized in proving the asymptotic normality, weak convergence to Brownian motions and Strassen-type almost sure invariance principles for these rank statistics. The law of iterated logarithm for these statistics is also established.

1) Part of the work was completed while the author was visiting the Indian Statistical Institute, Calcutta, and partly supported by the Aerospace Research Laboratories, Air Force Systems Command, U. S. Air Force, Contract No. F33615-71-C-1927. Reproduction in whole or in part permitted for any purpose of the U. S. Government.

1. INTRODUCTION

For the problem of two independent samples, Chernoff and Savage (1958) considered an elegant decomposition of a rank order statistic into a principal term involving averages of independent random variables (where the central limit theorem applies) and a remainder term which converges (at a faster rate) to 0, in probability, as the sample sizes increase. Similar decomposition for the one-sample rank order statistics were studied by Govindarajulu (1960), Puri and Sen (1969) and Sen (1970), among others. Hájek (1968) and Hušková (1970) relaxed the regularity conditions to a certain extent by using a powerful variance inequality along with the polynomial approximation of absolutely continuous score functions [of bounded variation], and Pyke and Shorack (1968a, 1968b) attacked the problem of asymptotic normality through weak convergence of certain related empirical processes. Though these later developments weaken the regularity conditions on the score functions a little, they may not provide an order of the remainder term (holding either in probability or almost surely (a.s.)), which has certain interests of its own. For example, a refined order of stochastic convergence of the remainder term enables one to study the limiting behaviour of rank order statistics (deeper in nature than their asymptotic normality) by working with their principal terms which involve averages over independent random variables and are thereby readily adaptable to refined probabilistic analysis. Indeed, in certain weak and a.s. invariance principles for rank order statistics, to be studied in detail in sections 5 and 6, the CS (Chernoff-Savage) decomposition along with specified stochastic orders of the remainder terms are very useful. The law of iterated logarithm for rank order statistics also follows quite easily from this decomposition. Finally, when the observations are not independent, it may be quite difficult to extend the powerful variance inequality of Hájek (1968), and a comparatively easy approach

to the study of the asymptotic normality of rank order statistics can be formulated with the aid of the CS-decomposition and the stochastic order of the remainder term.

Our study in the present paper centers around the discussion of how fast the remainder term converges stochastically to 0 for the one-sample problem (see Section 4). In doing so, we consider stationary ϕ -mixing processes which include independent, m -dependent ($m \geq 1$ and fixed), autoregressive and moving average processes as special cases. Under essentially two different ϕ -mixing conditions [viz., (2.3) and (2.4)], two different orders of the stochastic convergence of the remainder term are obtained. It is seen that less restrictive ϕ -mixing conditions can be adopted by paying a premium on the growth condition of the score function [viz., (2.8)]. Under either of the two ϕ -mixing conditions, asymptotic normality of rank order statistics is studied in Section 5, while certain weak and a.s. invariance principles are established in Section 6. In passing, we may remark that for stationary ϕ -mixing processes, besides the work of Serfling (1968) on the Wilcoxon two-sample statistic, the authors are not aware of any development for unbounded score functions.

The basic study is related to certain stochastic order of fluctuations of the empirical process, studied in detail in Section 3. The preliminary notions are given in Section 2 and the order of the remainder terms are studied in Section 4. Asymptotic properties of rank statistics are then presented in Sections 5 and 6.

2. PRELIMINARY NOTIONS

Let $\{X_i, -\infty < i < \infty\}$ be a stationary sequence of ϕ -mixing random variables defined on a probability space (Ω, \mathcal{A}, P) . Thus, if $M_{-\infty}^k$ and M_{k+n}^{∞} be respectively the σ -fields generated by $\{X_i, i \leq k\}$ and $\{X_i, i \geq k+n\}$, and if $E_1 \in M_{-\infty}^k$ and $E_2 \in M_{k+n}^{\infty}$, then for all $k (-\infty < k < \infty)$ and $n (\geq 1)$,

$$|P(E_2|E_1) - P(E_2)| \leq \phi(n), \quad \phi(n) \geq 0, \quad (2.1)$$

where $1 \geq \phi(1) \geq \phi(2) \geq \dots$ and $\lim_{n \rightarrow \infty} \phi(n) = 0$. The usual ϕ -mixing condition pertaining to the applicability of the central limit theorem for sums of the X_i is

$$A_0(\phi) = \sum_{n=1}^{\infty} \phi^{\frac{1}{2}}(n) < \infty. \quad (2.2)$$

On the other hand, for weak convergence of empirical processes (to be defined later on), we may need either of the following two stronger conditions:

$$(a) \text{ for some } k \geq 1, \quad A_k(\phi) = \sum_{n=1}^{\infty} n^k \phi^{\frac{1}{2}}(n) < \infty, \quad (2.3)$$

$$(b) \text{ for some } t > 0, \quad \sum_{n=1}^{\infty} e^{tn} \phi(n) < \infty. \quad (2.4)$$

We denote the marginal distribution function (df) of X_1 by $F(x)$, $x \in \mathbb{R}$, the real line $(-\infty, \infty)$, and assume that $F(x)$ is continuous everywhere. For a sample (X_1, \dots, X_n) of size n , define the empirical df F_n by

$$F_n(x) = n^{-1} \sum_{i=1}^n u(x - X_i), \quad -\infty < x < \infty, \quad (2.5)$$

where $u(t)$ is 1 or 0 according as t is \geq or $<$ 0. The weak convergence of the empirical process

$$\{n^{\frac{1}{2}}[F_n(x) - F(x)], \quad -\infty < x < \infty\} \quad (2.6)$$

to an appropriate Gaussian process has been studied by Billingsley (1968, p. 197) under $A_2(\phi) < \infty$ and by Sen (1971) under $A_1(\phi) < \infty$. In passing, we may remark that (2.4) holds for m -dependent and a general class of autoregressive processes, while (2.3) is much more general.

Let $R_{ni} = \sum_{j=1}^n u(|X_i| - |X_j|)$ be the rank of $|X_i|$ among $|X_1|, \dots, |X_n|$, $1 \leq i \leq n$, and consider the usual one-sample rank order statistic

$$T_n = n^{-1} \sum_{i=1}^n u(X_i) J_n((n+1)^{-1} R_{ni}), \quad n \geq 1, \quad (2.7)$$

where $J_n(i/(n+1)) = EJ(U_{ni})$ or $J(i/(n+1))$, $1 \leq i \leq n$, $U_{n1} \leq \dots \leq U_{nn}$ are the ordered random variables of a sample of size n from the rectangular $(0,1)$ df, and $J(u) = J^*((1+u)/2)$, $0 \leq u < 1$, is an absolutely continuous and twice differentiable score function. As in Chernoff and Savage (1958) and Puri and Sen (1969), we assume that there are positive (finite) constants K , α ($0 < \alpha \leq \frac{1}{2}$) and δ ($0 < \delta \leq \alpha$), such that for $0 < u < 1$,

$$|J^{*(r)}(u)| = |d^r J^*(u)/du^r| \leq K[u(1-u)]^{-\alpha-r+\delta}, \quad (2.8)$$

for $r=0,1,2$. Note that (2.8) implies that

$$|J^{(r)}(u)| \leq K^*[1-u]^{-\alpha-r+\delta}, \quad 0 \leq u < 1, \quad r=0,1,2, \quad (2.9)$$

where $K^* < \infty$. The choice of α depends on the ϕ -mixing conditions (2.2), (2.3) or (2.4). In fact, under (2.4), we will let $\alpha = \frac{1}{2}$, while under (2.3), $\alpha = (2k-1)/2(2k+1)$, $k \geq 1$, or $(k-2)/2k$, $k \geq 3$, depending on whether we desire to have a bound for the remainder term holding in probability or a.s.

We write $H(x) = P\{|X_i| \leq x\} = F(x) - F(-x)$, $x \geq 0$ and $H_n(x) = F_n(x) - F_n(-x)$, $x \geq 0$. Then, we have by (2.7),

$$\begin{aligned} T_n &= \int_0^\infty J_n(nH_n(x)/(n+1)) dF_n(x) \\ &= \mu + n^{-1} \sum_{i=1}^n B(X_i) + R_n, \end{aligned} \quad (2.10)$$

where

$$\mu = \int_0^\infty J(H(x)) dF(x), \quad (2.11)$$

$$B(X_i) = u(X_i)J(H(|X_i|)) + \int_0^\infty [u(x-|X_i|) - H(x)] J'(H(x)) dF(x), \quad (2.12)$$

$$R_n = \int_0^\infty [J_n(nH_n(x)/(n+1)) - J(nH_n(x)/(n+1))] dF_n(x) + \quad (2.13)$$

$$\int_0^\infty [J(nH_n(x)/(n+1)) - J(H(x))] dF_n(x) - \int_0^\infty [H_n(x) - H(x)] J'(H(x)) dF(x).$$

We term $n^{-1}S_n = \mu + n^{-1} \sum_{i=1}^n B(X_i)$ and R_n as respectively the principal and remainder terms of T_n . Limiting behaviour of these terms and other properties are studied with the aid of certain convergence properties of the empirical process in (2.6), which we consider first in Section 3.

3. ASYMPTOTIC BEHAVIOUR OF THE EMPIRICAL PROCESS

It is known [cf. Sen (1971)] that if $A_1(\phi) < \infty$ then the empirical process in (2.6) weakly converges to an appropriate, Gaussian process and

$$\sup_{-\infty < x < \infty} n^{1/2} |F_n(x) - F(x)| = o_p(1). \quad (3.1)$$

We are interested in the following related results where we define $Y_i = F(X_i)$, $i \geq 1$, so that $P\{Y_i \leq t\} = t$: $0 \leq t \leq 1$, and we let $G_n(t) = n^{-1} \sum_{i=1}^n u(t - Y_i)$, $0 \leq t \leq 1$, $n \geq 1$.

LEMMA 3.1. If (2.3) holds for some $k > 1$, then for every $\epsilon > 0$,

$$\sup_{0 \leq t \leq 1} n^{1/2} |G_n(t) - t| \{t(1-t)\}^{-1/2 + \epsilon} = o_p(n^{1/2(2k+1)}), \text{ as } n \rightarrow \infty; \quad (3.2)$$

if (2.3) holds for some $k > 2$, then as $n \rightarrow \infty$

$$\sup_{0 \leq t \leq 1} n^{1/2} |G_n(t) - t| \{t(1-t)\}^{-1/2 + \epsilon} = o(n^{1/2k}) \text{ a.s.} \quad (3.3)$$

Finally, under (2.4), for every $\epsilon > 0$,

$$\sup_{0 \leq t \leq 1} n^{1/2} |G_n(t) - t| \{t(1-t)\}^{-1/2 + \epsilon} = O(\log n) \text{ a.s.} \quad (3.4)$$

[Note that for $k=1$, (3.3) always holds, but is of little practical interest.]

Proof. First note that $\min_{1 \leq i \leq n} Y_i = Y_{n,1}$ (say) is \surd in n , so that for every $r > 1$,

$$\begin{aligned}
 & P\{Y_{m,1} \leq m^{-r} \text{ for some } m \geq n\} \\
 &= P\left\{ \bigcup_{j=1}^{\infty} [Y_{m,1} \leq m^{-r} \text{ for some } 2^{j-1}n \leq m < 2^j n] \right\} \\
 &\leq \sum_{j=1}^{\infty} P\{Y_{m,1} \leq (2^{j-1}n)^{-r}, \text{ for some } 2^{j-1}n \leq m < 2^j n\} \\
 &\leq \sum_{j=1}^{\infty} P\{Y_{n2^j,1} \leq (n2^{j-1})^{-r}\} \\
 &= \sum_{j=1}^{\infty} P\left\{ \sum_{i=1}^{n2^j} u((n2^{j-1})^{-r} - Y_i) \geq 1 \right\} \tag{3.5} \\
 &\leq \sum_{j=1}^{\infty} E\left\{ \sum_{i=1}^{n2^j} u((n2^{j-1})^{-r} - Y_i) \right\} \\
 &= \sum_{j=1}^{\infty} n2^j \cdot n^{-r} 2^{-r(j-1)} = 2^r n^{-r+1} \sum_{j=0}^{\infty} 2^{-j(r-1)} \\
 &= 2^r n^{-r+1} (1-2^{-(r-1)})^{-1} \rightarrow 0 \text{ as } n \rightarrow \infty.
 \end{aligned}$$

Let us define

$$h_n(t) = n^{\frac{1}{2}} \{t(1-t)\}^{-\frac{1}{2} + \epsilon} |G_n(t) - t|, \quad 0 \leq t \leq 1. \tag{3.6}$$

Then, note that $G_n(t) = 0$, $0 \leq t \leq n^{-r}$, implies that $h_n(t) = O(n^{-(r-1)/2}) \rightarrow 0$ as $n \rightarrow \infty$. Thus, by (3.5), for every $r > 1$,

$$\sup_{0 \leq t \leq n^{-r}} h_n(t) \rightarrow 0 \text{ a.s., as } n \rightarrow \infty. \tag{3.7}$$

Similarly,

$$\sup_{1-n^{-r} \leq t \leq 1} h_n(t) \rightarrow 0 \text{ a.s., as } n \rightarrow \infty. \tag{3.8}$$

Next, we note that if for some positive integer k , $A_k(\phi) < \infty$, then [cf. Sen (1972b)]

$$n^{2(k+1)} \phi(n) \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (3.9)$$

$$E\left[\sum_{i=1}^n \{u(t-Y_i) - t\}\right]^{2(k+1)} \leq K_\phi \{n\tau + \dots + (n\tau)^{k+1}\}, \quad (3.10)$$

for every $0 \leq t \leq 1$, $n \geq 1$, where $\tau = t(1-t)$ and $K_\phi (< \infty)$ depends only on $\{\phi(n)\}$. Also, note that $G_n(t)$ and t are non-decreasing in t ($0 \leq t \leq 1$), and hence, if we define

$$V_{nj}^{(1)} = h_n(jn^{-r}), \quad j=1, \dots, n_0 = [n^{r-1}] + 1, \quad (3.11)$$

it follows by some routine steps that

$$\sup_{n^{-r} \leq t \leq n^{-1}} [h_n(t)] \leq \sqrt{2} \left\{ \max_{1 \leq j \leq n_0} V_{nj}^{(1)} + O(n^{-(r-1)/2}) \right\}, \quad (3.12)$$

where by (3.10), $A_k(\phi) < \infty \Rightarrow$ for every $1 \leq j \leq n_0$,

$$E[(V_{nj}^{(1)})^{2(k+1)}] \leq K_\phi (j/n^r)^{2\epsilon(k+1)} \{1 + \tau_{nj}^{-1} + \dots + \tau_{nj}^{-k}\}, \quad (3.13)$$

and $\tau_{nj} = n[j(n^r - j)/n^{2r}] \leq 1 + O(n^{-r+1})$. Thus, for every $\eta > 0$ and $a > 0$,

$$\begin{aligned} P\left\{ \max_{1 \leq j \leq n_0} V_{nj}^{(1)} > \eta n^a \right\} &\leq \sum_{j=1}^{n_0} P\{V_{nj}^{(1)} > \eta n^a\} \\ &\leq \sum_{j=1}^{n_0} \left\{ K_\phi (j/n^r)^{2\epsilon(k+1)} [1 + n^{r-1} \cdot O(j^{-1}) + \dots + n^{k(r-1)} \cdot O(j^{-k})] / (\eta n^a)^{2(k+1)} \right\} \\ &= K_\phi [n^{-2a(k+1)} / \eta^{2a(k+1)}] [O(n_0) + O(n_0 \log n_0) + \dots + O(n_0^k)] [O(n^{-2\epsilon(k+1)})]. \end{aligned} \quad (3.14)$$

Thus, on letting $r = (2k+3)/(2k+1) (> 1)$ and $a = 1/2(2k+1)$, we obtain from (3.12) and (3.14) that if $A_k(\phi) < \infty$ for some $k \geq 1$, then

$$\sup_{n^{-r} \leq t \leq n^{-1}} [h_n(t) \cdot n^{-1/2(2k+1)}] = o_p(1). \quad (3.15)$$

Also, if $A_k(\phi) < \infty$ for some $k \geq 2$, on letting $a = 1/2k$ and $r = 1 + (2k^2)^{-1}$, we obtain from (3.12), (3.14) and the Borel-Cantelli lemma that as $n \rightarrow \infty$,

$$\sup_{n^{-r} \leq t \leq n^{-1}} [n^{-1/2k} h_n(t)] = o(1) \text{ a.s.} \quad (3.16)$$

The case of $1 - n^{-1} \leq t \leq 1 - n^{-r}$, $r > 1$, follows similarly. Let us now define

$$v_{nj}^{(2)} = h_n(j/n), \quad j=1, \dots, n-1, \quad (3.17)$$

and note that as in (3.12),

$$\sup_{n^{-1} \leq t \leq 1 - n^{-1}} [h_n(t)] \leq \sqrt{2} \{ \max_{1 \leq j \leq n-1} v_{nj}^{(2)} + n^{-\varepsilon} \}. \quad (3.18)$$

Also, by (3.10), for every $1 \leq j \leq n^* = [n^{(k+1)/(2k+1)}]$ and $\eta > 0$,

$$\begin{aligned} P\{v_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} &\leq n^{-(k+1)/(2k+1)} \eta^{-2(k+1)} E[(v_{nj}^{(2)})^{2(k+1)}] \\ &\leq K_\phi \eta^{-2(k+1)} (n^*)^{-1-2\varepsilon k} [1 + O(j/n)], \end{aligned} \quad (3.19)$$

so that

$$\begin{aligned} P\{\max_{1 \leq j \leq n^*} v_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} &\leq \sum_{j=1}^{n^*} P\{v_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} \\ &\leq K_\phi \eta^{-2(k+1)} (n^*)^{-2\varepsilon k} [1 + O(n^{-k/(2k+1)})] \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \text{ for every } \varepsilon > 0, \eta > 0. \end{aligned} \quad (3.20)$$

Similarly, for every $\eta > 0$,

$$P\{\max_{n-n^* \leq j \leq n-1} v_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.21)$$

Thus, to prove (3.2), it remains only to show that for every $\eta > 0$,

$$P\{\max_{n^* < j < n - n^*} v_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.22)$$

For proving (3.22), we note that the left hand side is bounded by

$$\begin{aligned} & \sum_{j=n^*-1}^{n-n^*-1} P\{V_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} \\ & \leq (n-2n^*-1) \left[\max_{n^* < j < n-n^*} P\{V_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} \right], \end{aligned} \quad (3.23)$$

so that it is enough to show that for each $n^* < j < n-n^*$,

$$nP\{V_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (3.25)$$

Now, by (3.6) and (3.17),

$$P\{V_{nj}^{(2)} > \eta n^{1/2(2k+1)}\} = P\left\{ \left| \sum_{i=1}^n [u(\frac{j}{n} - Y_i) - \frac{j}{n}] \right| > h_{nj}^* \right\}, \quad (3.26)$$

where

$$h_{nj}^* = \eta n^{(2k+2)/2(2k+1)} [j(n-j)/n^2]^{1/2-\epsilon}, \quad n^* < j < n-n^*. \quad (3.27)$$

Since $\{u(\frac{j}{n} - Y_i), i \geq 1\}$ is ϕ -mixing where $P\{u(j/n - Y_i) = 1\} = 1 - P\{u(j/n - Y_i) = 0\} = j/n$, following the proof of Lemma 4.1 of Sen (1972a) and choosing (in his notations) $k_n = n^{1/2(2k+1)}$, we obtain that for every $t > 0$, $A_k(\phi) < \infty$ for some $k \geq 1$,

$$\begin{aligned} E[\exp\{t \sum_{i=1}^n [u(\frac{j}{n} - Y_i) - \frac{j}{n}]\}] & \leq E[\exp\{tk_n \sum_{\ell=1}^{m_n} [u(\frac{j}{n} - Y_{i+\ell k_n}) - \frac{j}{n}]\}] \\ & \leq \{1 + [\frac{j}{n} + o(n^{-(k+1)/(2k+1)})]\} [\exp(tk_n) - 1]^{m_n+1}, \end{aligned} \quad (3.28)$$

where m_n is the largest integer for which $1 + k_n m_n \leq n$ (i.e., $m_n \sim n^{1-1/2(2k+1)}$).

By (3.26), (3.27), (3.28) and the Markov inequality, we have

$$\begin{aligned} & P\left\{ \sum_{i=1}^n [u(\frac{j}{n} - Y_i) - \frac{j}{n}] > h_{nj}^* \right\} \\ & \leq \inf_{t>0} \{ (\exp[-tj - th_{nj}^*]) E[\exp(t \sum_{i=1}^n \{u(j/n - Y_i) - j/n\})] \} \\ & \leq \exp\{-\eta n^{1/4(2k+1)} + \frac{1}{2} [j(n-j)/n^2]^{2\epsilon} + o(1)\}, \end{aligned} \quad (3.29)$$

where in the last line we use $t = t_{nj} = n^{-1/2-1/4(2k+1)} [j(n-j)/n^2]^{-1/2+\epsilon}$. A similar bound holds for $P\{\sum_{i=1}^n [u(j/n - Y_i) - j/n] < -h_{nj}^*\}$. Now, for every $\eta > 0$ and $k \geq 1$,

$\eta n^{1/4(2k+1)} \rightarrow \infty$ as $n \rightarrow \infty$, and for every $s(>1)$ there exists an $n_0(s, \eta)$ such that for $n \geq n_0(s, \eta)$,

$$\eta n^{1/4(2k+1)} > s \log n + \frac{1}{2} + \log 2. \quad (3.30)$$

Consequently, by (3.26), (3.29) and (3.30), the left hand side of (3.25) is $O(n^{-s+1}) \rightarrow 0$ as $n \rightarrow \infty$. Hence, the proof of (3.2) is complete.

To prove (3.3), by virtue of (3.7), (3.8), (3.16) and (3.18), it suffices to show that as $n \rightarrow \infty$

$$\max_{1 \leq j \leq n-1} [h_n(j/n)/n^{1/2k}] = o(1) \text{ a.s.} \quad (3.31)$$

For this purpose, we define $n^* = [n^{3/4k}]$, $k \geq 2$, proceed as in (3.19) through (3.21), and obtain that for every $\eta > 0$,

$$\begin{aligned} & P\{ \max_{1 \leq j \leq n^*} V_{nj}^{(2)} > \eta n^{1/2k} \} \\ & \leq K_\phi n^{-(k+1)/k} \eta^{-2(k+1)} \sum_{j=1}^{n^*} \{1+O(j/n)\} \\ & = K_\phi n^{-(k+1)/k} \eta^{-2(k+1)} n^{3/4k} [1+O(n^{-1+3/4k})] \\ & = K_\phi n^{-1-1/4k} \eta^{-2(k+1)} [1+o(1)], \end{aligned} \quad (3.32)$$

so that by the Borel-Cantelli Lemma,

$$\begin{aligned} & P\{ \max_{1 \leq j \leq m^*} V_{mj}^{(2)} > \eta m^{1/2k} \text{ for some } m \geq n \} \\ & \leq K_\phi^* \eta^{-2(k+1)} n^{-1/4k} \rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned} \quad (3.33)$$

where $K_\phi^*(<\infty)$ depends on K_ϕ and k . Similarly,

$$P\{ \max_{m-m^* \leq j \leq m-1} V_{mj}^{(2)} > \eta m^{1/2k} \text{ for some } m \geq n \} \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (3.34)$$

For $n^* < j < n - n^*$, we repeat the steps in (3.22) through (3.30), where we take

$k_n = n^{1/2k}$ and $s > 2$, so that

$$P\left\{\max_{n^* < j < n - n^*} V_{nj}^{(2)} > \eta n^{1/2k}\right\} = o(n^{-s+1}) \text{ as } n \rightarrow \infty. \quad (3.35)$$

Thus, again by the Borel-Cantelli Lemma and $s > 2$,

$$\max_{n^* < j < n - n^*} [n^{-1/2k} h_n(j/n)] \rightarrow 0 \text{ a.s., as } n \rightarrow \infty, \quad (3.36)$$

which completes the proof of (3.3).

Finally, to prove (3.4), by (3.7) and (3.8), it suffices to show that for every $r > 1$, and some $K(1 < K < \infty)$,

$$\sup_{n^{-r} \leq t \leq 1 - n^{-r}} [(\log n)^{-1} h_n(t)] \leq K, \text{ a.s. as } n \rightarrow \infty. \quad (3.37)$$

Here, we let $r = 1 + \varepsilon$, $\varepsilon > 0$, where ε is defined in (3.4). Then, we divide the range $(n^{-1-\varepsilon}, 1 - n^{-1-\varepsilon})$ into $(n^{-1-\varepsilon}, n^{-1})$, $(n^{-1}, 1 - n^{-1})$ and $(1 - n^{-1}, 1 - n^{-1-\varepsilon})$. For the range $(n^{-1-\varepsilon}, n^{-1})$, we again use (3.12) and show that on defining n_0 as in (3.11),

$$\max_{1 \leq j \leq n_0} [h_n(j/n^{1+\varepsilon})/\log n] \rightarrow 0 \text{ a.s. as } n \rightarrow \infty. \quad (3.38)$$

Since (2.4) holds for some $t_0 > 0$, we can select a $C(< \infty)$ such that $Ct_0 > 1 + \varepsilon$.

Then on choosing $k_n = c \log n$

$$\begin{aligned} P\{u(j/n^{1+\varepsilon} - Y_{i+k_n}) = 1 \mid M_{-\infty}^i\} &\leq [j/n^{1+\varepsilon}] + o(e^{-ct_0 \log n}) \\ &= [j/n^{1+\varepsilon}] + o(n^{-1-\varepsilon}), \forall 1 \leq j \leq n_0. \end{aligned} \quad (3.39)$$

Thus, proceeding as in (3.26) through (3.30), we have for every $\eta > 0$,

$$\begin{aligned} &P\{h_n(j/n^{1+\varepsilon}) > \eta \log n\} \\ &\leq 2[\exp\{-\eta(\log)^{\frac{1}{2}}(n^{1+\varepsilon} j^{-1})^\varepsilon + \frac{1}{2}c^2 [n^{-2-2\varepsilon} j(n^{1+\varepsilon} - j)]^{2\varepsilon} + o(1)\}] \\ &= 2[\exp\{-\eta(\log n)^{\frac{1}{2}} n^\varepsilon + o(1)\}], \forall 1 \leq j \leq n_0 \wedge n^\varepsilon. \end{aligned} \quad (3.40)$$

Again, for every $\varepsilon > 0$ and $\eta > 0$, there exists an $n_0(\varepsilon, \eta)$, such that

$$\eta(\log n)^{\frac{1}{2}} n^\epsilon \geq s \log n, \text{ where } s > 2, \forall n \geq n_0(\epsilon, \eta). \quad (3.41)$$

Consequently, by the Borel-Cantelli Lemma,

$$\max_{1 \leq j \leq n_0} [h_n(j/n^{1+\epsilon}) / \log n] \rightarrow 0 \text{ a.s. as } n \rightarrow \infty. \quad (3.42)$$

A similar proof holds for the range $1-n^{-1} \leq t \leq 1-n^{-1-\epsilon}$. Finally, for the range $n^{-1} \leq t \leq 1-n^{-1}$, we again use (3.18) for each j ($1 \leq j \leq n-1$), and note that (2.4) implies that as $n \rightarrow \infty$, $ne^{t_0 n} \phi(n) \rightarrow 0$ for some $t_0 > 0$ i.e., $\phi(n) = o(n^{-1} e^{-t_0 n})$ for some $t_0 > 0$. Thus, $\phi(C \log n) = o(e^{-C t_0 \log n} (C \log n)^{-1}) = o((\log n)^{-1} \cdot n^{-1})$, where $C t_0 > 1$. Thus, if we repeat the steps (3.26)-(3.29) where we choose $k_n = C \log n$, we obtain on choosing $t = t_{nj} = n^{-\frac{1}{2}} [j(n-j)/n^2]^{-\frac{1}{2} + \epsilon}$, that as $n \rightarrow \infty$,

$$\begin{aligned} P\{ \max_{1 \leq j \leq n-1} h_n(j/n) \geq K \log n \} \\ \leq 2(n-1) [\exp\{-K \log n + \frac{1}{2}(C \log n) [j(n-j)/n^2]^{2\epsilon} + o(1)\}] \quad (1 \leq j \leq n-1) \\ \leq 2 n^{-s+1}, \quad s > 2, \end{aligned} \quad (3.43)$$

by choosing $K (> \frac{1}{2}C + s)$ adequately large. Thus, again by the Borel-Cantelli Lemma,

$$\max_{1 \leq j \leq n-1} h_n(j/n) / \log n \leq K \text{ a.s. as } n \rightarrow \infty, \quad (3.44)$$

and the proof of (3.4) is complete. Note that (3.4) extends a lemma of Ghosh (1972) to a class of ϕ -mixing processes.

4. STOCHASTIC ORDER OF THE REMAINDER TERM

We rewrite R_n in (2.13) as

$$R_n = C_{1n} + \dots + C_{6n}; \quad (4.1)$$

$$C_{1n} = \int_0^\infty [J_n(nH_n(x)/(n+1)) - J(nH_n(x)/(n+1))] dF_n(x), \quad (4.2)$$

$$C_{2n} = \int_{a_n}^\infty [J(nH_n(x)/(n+1)) - J(H(x))] dF_n(x), \quad (4.3)$$

$$C_{3n} = \int_0^{a_n} [J(nH_n(x)/(n+1)) - J(H(x)) - (nH_n(x)/(n+1) - H(x))J'(H(x))] dF_n(x), \quad (4.4)$$

$$C_{4n} = -(n+1)^{-1} \int_0^{a_n} H_n(x)J'(H(x))dF_n(x), \quad (4.5)$$

$$C_{5n} = \int_0^{a_n} [H_n(x) - H(x)]J'(H(x))d[F_n(x) - F(x)], \quad (4.6)$$

$$C_{6n} = - \int_{a_n}^{\infty} [H_n(x) - H(x)]J'(H(x))dF(x), \quad (4.7)$$

where we define a_n by $H(a_n) = 1 - n^{-1+\delta}$, $\delta(>0)$ is defined in (2.8). Note that $dF_n \leq dH_n$, so that

$$|C_{1n}| \leq n^{-1} \sum_{i=1}^n |J_n(i/(n+1)) - J(i/(n+1))| = O(n^{-\frac{1}{2}-\eta}), \quad \eta > 0, \quad (4.8)$$

by Theorem 2 of Charnoff and Savage (1958); for a proof of (4.8) under less restrictive regularity conditions, we may refer to Theorem 3.6.6. of Puri and Sen (1971).

THEOREM 4.1. If for some $k \geq 1$, $A_k(\phi) < \infty$ and (2.8) holds for $\alpha = (2k-1)/2(2k+1)$, then

$$n^{\frac{1}{2}} R_n = o_p(n^{-\eta}) \text{ for some } \eta > 0. \quad (4.9)$$

If for some $k \geq 3$, $A_k(\phi) < \infty$ and (2.8) holds for $\alpha = (k-2)/2k$, then

$$n^{\frac{1}{2}} R_n = o(n^{-\eta}) \text{ a.s. as } n \rightarrow \infty, \text{ for some } \eta > 0. \quad (4.10)$$

If (2.4) holds for some $t > 0$ and (2.8) holds for $\alpha = \frac{1}{2}$, then

$$n^{\frac{1}{2}} R_n = o(n^{-\eta}) \text{ a.s. as } n \rightarrow \infty, \text{ for some } \eta > 0. \quad (4.11)$$

Proof. We only prove (4.11) as (4.9) and (4.8) follow on similar lines. By virtue of (4.8), we only show that $|C_{kn}| = o(n^{-\eta})$ a.s. (as $n \rightarrow \infty$) for $2 \leq k \leq 6$. First, consider C_{2n} . Using the fact $dF_n \leq dH_n$, we can write

$$|C_{2n}| \leq \left| \int_{a_n}^{\infty} \left[J\left(\frac{n}{n+1}H_n(x)\right) \right] dH_n(x) \right| + \left| \int_{a_n}^{\infty} J(H(x)) dH_n(x) \right| \quad (4.12)$$

Integrating by parts,

$$\begin{aligned} \int_{a_n}^{\infty} J(H(x)) dH_n(x) &= - \int_{a_n}^{\infty} J(H(x)) d[1-H_n(x)] = \\ &[1-H_n(a_n)]J(H(a_n)) + \int_{a_n}^{\infty} [1-H_n(x)]J'(H(x))dH(x). \end{aligned}$$

Hence, by (4.12),

$$\begin{aligned} |C_{2n}| \leq \int_{a_n}^{\infty} \left| J\left(\frac{n}{n+1}H_n(x)\right) \right| dH_n(x) + \int_{a_n}^{\infty} (1-H_n(x)) |J'(H(x))| dH(x) \\ + [1-H_n(a_n)] |J(H(a_n))|. \end{aligned} \quad (4.13)$$

By the same arguments as in Lemma 4.1 of Sen (1972a), it can be shown that under (2.4),

$$H_n(a_n) - H(a_n) = O(n^{-1+\frac{1}{2}\delta} \log n) \text{ a.s. as } n \rightarrow \infty. \quad (4.14)$$

In the sequel, we assume that n is so large that $H(a_n) > \frac{1}{2}$. Now, by (2.9) with $\alpha = \frac{1}{2}$ and (4.14), the first term on the right hand side of (4.13) is bounded by

$$\begin{aligned} &\int_{a_n}^{\infty} [1-nH_n(x)/(n+1)]^{-\frac{1}{2}+\delta} dH_n(x) \\ &\leq [1-nH_n(a_n)/(n+1)]^{\frac{1}{2}+\delta} \int_{a_n}^{\infty} [1-nH_n(x)/(n+1)]^{-1} dH_n(x) \\ &\leq [1-nH_n(a_n)/(n+1)]^{\frac{1}{2}+\delta} \int_0^{\infty} [1-nH_n(x)/(n+1)]^{-1} dH_n(x) \\ &\leq [0(n^{-1+\delta}) + 0(n^{-1+\frac{1}{2}\delta} \log n)]^{\frac{1}{2}+\delta} \cdot \sum_{i=1}^n (1-i/(n+1))^{-1} \text{ a.s.} \\ &= [0(n^{-1+\delta})^{\frac{1}{2}+\delta}] [0(\log n)] \text{ a.s.} \end{aligned}$$

Thus, the right hand side of (4.15) is

$$O(n^{-\frac{1}{2}-\frac{1}{2}\delta(1-\delta)} \log n) \text{ a.s.} = O(n^{-\frac{1}{2}-\eta}) \text{ a.s.} \quad (4.16)$$

where $0 < \eta < \frac{1}{2}\delta(1-\delta)$. Now, by (2.9) with $\alpha = \frac{1}{2}$ and (3.4), the second term on the right hand side of (4.13) is bounded by

$$\begin{aligned} & \int_{a_n}^{\infty} (1-H(x)) |J'(H(x))| dH(x) + [O(n^{-\frac{1}{2}}(\log n))] \int_{a_n}^{\infty} [1-H(x)]^{\frac{1}{2}-\epsilon} |J'(H(x))| dH(x) \text{ a.s.} \\ & \leq K \int_{a_n}^{\infty} (1-H(x))^{-\frac{1}{2}+\delta} dH(x) + [O(n^{-\frac{1}{2}} \log n)] \int_{a_n}^{\infty} [1-H(x)]^{-1-\epsilon+\delta} dH(x) \text{ (a.s.)} \\ & = O([1-H(a_n)]^{\frac{1}{2}+\delta}) + [O(n^{-\frac{1}{2}} \log n)] [O([1-H(a_n)])^{\delta-\epsilon}] \text{ (a.s.)} \\ & = O(n^{(-1+\delta)(\frac{1}{2}+\delta)}) + [O(n^{-\frac{1}{2}} \log n)] [O(n^{(-1+\delta)(\delta-\epsilon)})] \text{ (a.s.)} \\ & = O(n^{-\frac{1}{2}-\frac{1}{2}\delta(1-2\delta)}) + [O(n^{-\frac{1}{2}} \log n)] [O(n^{-(1-\delta)(\delta-\epsilon)})] \text{ (a.s.)} \\ & = O(n^{-\frac{1}{2}-\eta}) \text{ a.s.} \end{aligned} \quad (4.17)$$

where we choose $0 < \epsilon < \delta/2$ and $0 < \eta < \frac{1}{2}\delta(1-2\delta)$. Finally, by (4.14) and (2.9) with $\alpha = \frac{1}{2}$, the last term on the right hand side of (4.13) is bounded by

$$\begin{aligned} & |[1-H(a_n) + \{H_n(a_n) - H(a_n)\}]| |J(H(a_n))| \\ & \leq [O(n^{-1+\delta}) + O(n^{-1+\frac{1}{2}\delta} \log n)] [O(n^{-(1-\delta)(\frac{1}{2}-\delta)})] \text{ (a.s.)} \\ & = [O(n^{-1+\delta})] [O(n^{-(1-\delta)(\frac{1}{2}-\delta)})] \text{ (a.s.)} \\ & = [O(n^{-3/2-3\delta/2-\delta^2})] = O(n^{-\frac{1}{2}-\eta}) \text{ a.s., } \eta > 0, \end{aligned} \quad (4.18)$$

as $0 < \delta < \frac{1}{2}$. Hence, $|C_{2n}| = O(n^{-\frac{1}{2}-\eta})$ a.s. Next, we consider C_{3n} . Using the fact that $dF_n \leq dH_n$ and the mean value theorem, we obtain from (4.4) that

$$|C_{3n}| \leq \int_0^{a_n} \left[\frac{n}{n+1} H_n(x) - H(x) \right]^2 |J''(H_{n,\theta}(x))| dH_n(x), \quad (4.19)$$

where $H_{n,\theta} = \theta n H_n(x)/(n+1) + (1-\theta)H(x)$, $0 < \theta < 1$. Since $H(a_n) = 1 - n^{-1+\delta}$, $\delta > 0$, and by our choice $0 < \epsilon < \frac{1}{2}$, by (3.4), for every $0 < \theta < 1$, $0 < x < a_n$,

$$\begin{aligned}
1 - H_{n,\theta}(x) &\geq 1 - H(x) - [O(n^{-\frac{1}{2}} \log n)] \{H(x) [1 - H(x)]\}^{\frac{1}{2} - \epsilon} \quad (\text{a.s.}) \\
&\geq \{1 - H(x)\} [1 - [O(n^{-\frac{1}{2}} \log n)] \{1 - H(x)\}^{\frac{1}{2} - \epsilon}] \\
&\geq \{1 - H(x)\} [1 - [O(n^{-\frac{1}{2}} \log n)] \{1 - H(a_n)\}^{\frac{1}{2} - \epsilon}] \\
&= \{1 - H(x)\} \{1 - [O(n^{-\frac{1}{2}} \log n)] [O(n^{-(\frac{1}{2} + \epsilon)(1 - \delta)})]\} \\
&= \{1 - H(x)\} \{1 - O(n^{-(\delta/2 - \epsilon) - \delta \epsilon} \log n)\} \\
&= \{1 - H(x)\} \{1 - o(1)\}, \text{ a.s. as } n \rightarrow \infty. \\
&= \{1 - H_n(x)\} \{1 - o(1)\}, \text{ a.s. as } n \rightarrow \infty.
\end{aligned} \tag{4.20}$$

[Putting $\theta = 1$, the same inequality applies to $1 - H_n(x)$.] Using then (2.9) with $\alpha = \frac{1}{2}$, (3.4), (4.19) and (4.20), we obtain that for large n ,

$$\begin{aligned}
|c_{3n}| &\leq [O(n^{-1}(\log n)^2)] \int_0^{a_n} \{H(x) [1 - H(x)]\}^{1 - 2\epsilon} [1 - H_{n,\theta}(x)]^{-\frac{5}{2} + \delta} dH_n(x) \quad (\text{a.s.}) \\
&\leq [O(n^{-1}(\log n)^2)] \int_0^{a_n} [1 - H_n(x)]^{-\frac{3}{2} + \delta - 2\epsilon} dH_n(x) \\
&= [O(n^{-1}(\log n)^2)] [n^{-1} \sum_{j=1}^{n^*} [(n-j)/n]^{-\frac{3}{2} + \delta - 2\epsilon}] \quad (\text{where } n^* = n H_n(a_n)).
\end{aligned} \tag{4.21}$$

Now, by (4.14), $n^* = n - n^\delta \pm O(n^{\delta/2} \log n)$ a.s., while for every $k < n$,

$$\begin{aligned}
n^{-1} \sum_{j=1}^k [(n-j)/n]^{-\frac{3}{2} + \delta - 2\epsilon} &= n^{\frac{1}{2} - \delta + 2\epsilon} \sum_{j=1}^k (n-j)^{-3/2 + \delta - 2\epsilon} \\
&= n^{\frac{1}{2} - \delta + 2\epsilon} \cdot [O((n-k)^{-\frac{1}{2} + \delta - 2\epsilon})] \\
&= O(n^{(\frac{1}{2} - \delta + 2\epsilon)(1 - \delta)}) \text{ a.s., for } k = n^* = n - n^\delta \pm O(n^{-\delta/2} \log n) \text{ a.s.}
\end{aligned} \tag{4.22}$$

Thus, the right hand side of (4.21) is

$$\begin{aligned}
& O(n^{-1}(\log n)^2)O(n^{(\frac{1}{2}-\delta+2\varepsilon)(1-\delta)}) \quad \text{a.s.} \\
& = O(n^{-\frac{1}{2}-\eta}) \quad \text{a.s. for some } \eta > 0.
\end{aligned} \tag{4.23}$$

Thus, $|C_{3n}| = O(n^{-\frac{1}{2}-\eta})$ a.s., as $n \rightarrow \infty$. Next we consider C_{4n} . By (4.5), (2.9), with $\alpha = \frac{1}{2}$, (4.14) and (4.20), we have

$$\begin{aligned}
|C_{4n}| & \leq (n+1)^{-1} \int_0^{a_n} [1-H(x)]^{-\frac{3}{2} + \delta} dH_n(x) \\
& \leq (n+1)^{-1} \int_0^{a_n} [1-H_n(x)]^{-\frac{3}{2} + \delta} (1+o(1)) dH_n(x) \\
& = O(n^{-1}) \cdot O(n^{-\frac{1}{2} - \delta)(1-\delta}) \quad \text{a.s.} \\
& = O(n^{-\frac{1}{2} - \eta}) \quad \text{a.s., where } \eta > 0.
\end{aligned} \tag{4.24}$$

For the study of the stochastic order of C_{5n} , for the convenience of manipulations, we assume by virtue of the probability integral transformation that F is rectangular $(0,1)$ df, and we write $t_0 = F(0)$, so that $0 \leq t_0 \leq 1$. Then,

$$C_{5n} = \int_{t_0}^{a_n^*} [H_n(u) - H(u)] J'(H(u)) d[F_n(u) - u], \tag{4.25}$$

where $H(a_n^*) = 1 - n^{-1+\delta}$, so that $1 - n^{-1+\delta} \leq a_n^* < 1$ ($a_n^* = F^{-1}(a_n)$). Since, for the rectangular df, the density function is a positive constant for the entire positive support of the df, proceeding as in Theorem 3.1 of Sen (1972a) and using Theorem 4.2 of Sen and Ghosh (1971) [namely, their (4.13)], it follows that as $n \rightarrow \infty$,

$$\begin{aligned}
& \sup_{0 < u < 1} \sup_{t: |t-u| \leq n^{-\frac{1}{2}}} \{|F_n(t) - F_n(u) - t + u|\} \\
& = O(n^{-3/4} \log n) \quad \text{a.s.}
\end{aligned} \tag{4.26}$$

Now, we can rewrite C_{5n} as

$$\sum_{j=1}^{n^*} \int_{I_{nj}} [H_n(u) - H(u)] J'(H(u)) d[F_n(u) - u], \quad (4.27)$$

where

$$I_{nj} = \{u: t_0 + (j-1)/\sqrt{n} \leq u \leq t_0 + j/\sqrt{n}\}, \quad j=1, \dots, n^*-1, \quad (4.28)$$

$$I_{nn^*} = \{u: t_0 + (n^*-1)/\sqrt{n} \leq u \leq a_n^*\},$$

and n^* is the largest positive integer such that $t_0 + (n^*-1)/\sqrt{n} < a_n^*$. Note that by definition $n^* = O(n^{1/2})$. Let us then write

$$t_{n0} = t_0, \quad t_{nj} = t_0 + jn^{-1/2}, \quad 1 \leq j \leq n^*-1, \quad t_{nn^*} = a_n^*. \quad (4.29)$$

Then, by (4.26), for every $u \in I_{nj}$, as $n \rightarrow \infty$,

$$H_n(u) - H(u) = H_n(t_{nj}) - H(t_{nj}) + O(n^{-3/4} \log n) \text{ a.s.} \quad (4.30)$$

$$F_n(u) - u = F_n(t_{nj}) - t_{nj} + O(n^{-3/4} \log n) \text{ a.s.} \quad (4.31)$$

and by (2.9) (with $\alpha = 1/2$), for $u \in I_{nj}$,

$$J'(H(u)) = J'(H(t_{nj})) + [O(n^{-1/2})][O(\{1-H(t_{nj})\}^{-5/2+\delta})] \quad (4.32)$$

Consequently,

$$\begin{aligned} & \left| \int_{I_{nj}} [H_n(u) - H(u)] J'(H(u)) d[F_n(u) - u] \right| \\ & \leq \left| [H_n(t_{nj}) - H(t_{nj})] J'(H(t_{nj})) [F_n(t_{nj}) - F_n(t_{nj-1}) - \frac{1}{\sqrt{n}}] \right| \\ & \quad + O(n^{-3/4} \log n) |J'(H(t_{nj}))| \int_{I_{nj}} |d[F_n(u) - u]| + \\ & \quad + O(n^{-1/2}) \int_{I_{nj}} |H_n(t_{nj}) - H(t_{nj})| [1-H(t_{nj})]^{-\frac{5}{2} + \delta} |d[F_n(u) - u]| \\ & \quad + O(n^{-5/4} \log n) \int_{I_{nj}} [1-H(t_{nj})]^{-\frac{5}{2} + \delta} |d[F_n(u) - u]|, \quad \text{a.s.} \end{aligned} \quad (4.33)$$

By (2.9) with $\alpha = \frac{1}{2}$, Lemma 3.1 and (4.26), the first term on the right hand side of (4.33) is bounded by

$$[O(n^{-5/4}(\log n)^2)][1-H(t_{nj})]^{-3/2+\delta}, \text{ a.s.} \quad (4.34)$$

Also, by (4.26),

$$\begin{aligned} \int_{I_{nj}} |d[F_n(u)-u]| &\leq \int_{I_{nj}} dF_n(u) + \int_{I_{nj}} du \\ &= F_n(t_{nj}) - F_n(t_{nj-1}) + \frac{1}{\sqrt{n}} \leq 2n^{-1/2} + O(n^{-3/4} \log n) \text{ a.s.}, \end{aligned} \quad (4.35)$$

so that the second term on the right hand side of (4.33) is of the same order as of (4.34). Similarly, the third and the fourth terms are bounded by, respectively,

$$[O(n^{-3/2} \log n)][1-H(t_{nj})]^{-2+\delta} \text{ a.s.} \quad (4.36)$$

$$[O(n^{-7/4} \log n)][1-H(t_{nj})]^{-5/2+\delta} \text{ a.s.} \quad (4.37)$$

Now, for $\beta > 1$,

$$\sum_{j=1}^{n^*} [1-H(t_{nj})]^{-\beta} \leq \sum_{j=1}^{n^*} [F(t_{nj})]^{-\beta} = n^{\beta/2} \sum_{j=1}^{n^*} (t_{oj} \sqrt{n+j})^{-\beta} = O(n^{\beta/2}). \quad (4.38)$$

On summing over $j (=1, \dots, n^*)$, we have from (4.33) through (4.38), as $n \rightarrow \infty$,

$$\begin{aligned} |C_{5n}| &\leq [O(n^{-5/4}(\log n)^2)][O(n^{3/4-\delta/2})] + \\ &[O(n^{-3/2} \log n)][O(n^{1-\delta/2})] + [O(n^{-7/4} \log n)][O(n^{5/4-\delta/2})] \text{ a.s.} \\ &= O(n^{-1/2-\delta/2}(\log n)^2) = O(n^{-1/2-\eta}) \text{ a.s.}, \eta > 0, \end{aligned} \quad (4.39)$$

where $0 < \eta < \frac{1}{2}\delta$. Thus, $C_{5n} = O(n^{-1/2-\eta})$ a.s. as $n \rightarrow \infty$. Finally,

$$\begin{aligned}
|C_{6n}| &\leq \int_{a_n}^{\infty} |H_n(x) - H(x)| |J'(H(x))| dH(x) \\
&\leq [O(n^{-\frac{1}{2}} \log n)] \int_{a_n}^{\infty} \{H(x)[1-H(x)]\}^{\frac{1}{2}-\epsilon} [1-H(x)]^{-\frac{3}{2}+\delta} dH(x) \quad (\text{a.s.}) \\
&\leq [O(n^{-\frac{1}{2}} \log n)] \int_{a_n}^{\infty} [1-H(x)]^{-1+\delta-\epsilon} dH(x) \\
&= [O(n^{-\frac{1}{2}} \log n)] [O([1-H(a_n)]^{\delta-\epsilon})] \quad (\text{as } 0 < \epsilon < \frac{1}{2}\delta) \\
&= [O(n^{-\frac{1}{2}} \log n)] [O(n^{-(1-\delta)(\delta-\epsilon)})] \\
&= O(n^{-\frac{1}{2}-\eta}) \text{ a.s.}, \quad \eta > 0. \tag{4.40}
\end{aligned}$$

Hence, the proof of (4.11) is complete. The proof of (4.9) and (4.10) follow on parallel lines; for intended brevity, the details are therefore omitted.

Remarks. For independent random variables, the proof of $C_{n5} = o_p(n^{-\frac{1}{2}})$ follows more easily by showing that $E[C_{n5}^2] = o(n^{-1})$. However, for ϕ -mixing processes and a.s. convergence, evaluation of higher order moments of C_{n5} becomes cumbersome, and the present proof appears to be simpler.

For independent random variables, the assumption on $J''(u)$ in (2.8)-(2.9) has been dispensed with [See Govindarajulu, LeCam, Raghavachari (1967) and Puri and Sen (1969)]. These results depend on the property that for independent random variables,

$$\sup_{0 < t < 1} \sqrt{n} |G_n(t) - t| \{t(1-t)\}^{-\frac{1}{2}+\epsilon} = o_p(1), \quad \forall \epsilon > 0. \tag{4.41}$$

On the other hand, for ϕ -mixing processes, our results (3.2) is not as strong as (4.41), and this necessitates the assumption on $J''(u)$. In any case, for a.s. convergence in (4.10) or (4.11), (4.41) does not suffice and we may need the condition on $J''(u)$.

We also note that by the same technique as in Lemma 3.1, it can be shown that if $A_1(\phi) < \infty$, then for every $\eta > 0$,

$$\sup_{0 \leq t \leq 1} \{n^{\frac{1}{2}-\eta} |G_n(t) - t|\} = o(1) \text{ a.s., as } n \rightarrow \infty. \quad (4.42)$$

Hence, if $J(u)$ has a bounded first derivative for all $u \in (0,1)$, we may repeat the steps for the proof of (4.11), use the boundedness of $J'(u)$ and obtain more easily that $|R_n| = o(n^{-\frac{1}{2}-\eta})$ a.s. as $n \rightarrow \infty$. Hence, we have the following.

THEOREM 4.2. If $J(u)$ has a bounded first derivative inside $(0,1)$, and $A_1(\phi) < \infty$, then $|R_n| = o(n^{-\frac{1}{2}})$ a.s., as $n \rightarrow \infty$.

5. ASYMPTOTIC NORMALITY OF RANK ORDER TESTS AND ESTIMATES

Let us define $B(X_1)$ as in (2.12) and let

$$\sigma^2 = V[B(X_1)] + 2 \sum_{h=2}^{\infty} \text{Cov}[B(X_1), B(X_h)], \quad (5.1)$$

where we know [viz., Puri and Sen (1971, Section 3.6)] that under (2.8),

$$0 \leq V[B(X_1)] < \infty. \quad (5.2)$$

Further, $A_0(\phi) = \sum_{n=1}^{\infty} \phi^{\frac{1}{2}}(n) < \infty \Rightarrow \sigma^2 < \infty$ [viz., Billingsley (1968, p. 172)]. In the sequel, we assume that

$$0 < \sigma^2 < \infty. \quad (5.3)$$

Now, by Theorem 21.1 of Billingsley (1968) under $A_0(\phi) < \infty$, as $n \rightarrow \infty$

$$\mathcal{L}(n^{-\frac{1}{2}} [\sum_{i=1}^n B(X_i)] / \sigma) \rightarrow \mathcal{N}(0,1). \quad (5.4)$$

On the other hand, by Theorem 4.1, $A_1(\phi) < \infty \Rightarrow R_n = o_p(n^{-\frac{1}{2}})$. Consequently, we have the following.

THEOREM 5.2. If $A_k(\phi) < \infty$ and (2.8) holds for $\alpha = (2k-1)/2(2k+1)$ for some $k > 1$, then
under (5.3), as $n \rightarrow \infty$

$$\mathfrak{L}(n^{\frac{1}{2}}(T_n - \mu)/\sigma) \rightarrow \mathcal{N}(0,1). \quad (5.5)$$

Under more restrictive ϕ -mixing conditions, Philipp (1969) has studied the order of the remainder term in the central limit theorem for sample averages. By virtue of our Theorem 4.1 and (5.4), under similar ϕ -mixing conditions, we have analogous expressions for $\sup_x |P\{n^{\frac{1}{2}}(T_n - \mu)/\sigma \leq x\} - \Phi(x)|$, where $\Phi(x)$ is the standard normal df.

Our next result concerns the asymptotic distribution of point estimators based on rank order statistics. Here, we assume that for the stationary ϕ -mixing process $\{X_i, i \geq 1\}$, the marginal df $F(x)$ is

$$F(x) = F_\theta(x) = F_0(x - \theta), \quad (5.6)$$

where $\theta (-\infty < \theta < \infty)$ is a location parameter and F_0 is symmetric about 0. Our interest is to estimate θ . With this end, define $T_n(b)$ similar as in T_n in (2.7) but the X_i being replaced by $X_i - b$, $1 \leq i \leq n$, where b is a real variable. Then $T_n(b)$ is defined for $-\infty < b < \infty$ and when $J(u)$ is non-decreasing, $T_n(b)$ is \downarrow in b . Define

$$\hat{\theta}_{n1} = \sup\{b: T_n(b) > \mu_0\}, \quad \hat{\theta}_{n2} = \inf\{b: T_n(b) < \mu_0\} \quad (5.7)$$

$$\hat{\theta}_n = \frac{1}{2}(\hat{\theta}_{n1} + \hat{\theta}_{n2}), \quad (5.8)$$

where $\mu_0 = \frac{1}{2} \int_0^1 J(u) du = (2n)^{-1} \sum_{i=1}^n J_n(i/(n+1))$ when $J_n(i/(n+1)) = EJ(U_{ni})$, $1 \leq i \leq n$.

As in Hodges and Lehmann (1963) and Sen (1963), we consider $\hat{\theta}_n$ as an estimator of θ . In view of our Theorem 5.1, an argument analogous to Theorems 1 and 5 of Hodges and Lehmann (1963) leads to the following.

THEOREM 5.2. If for a non-decreasing score function (2.8) holds and $A_1(\phi) < \infty$, then under (5.3),

$$P\{n^{1/2}(\hat{\theta}_n - \theta)B(F_0)/\sigma \leq x\} \rightarrow (2\pi)^{-1/2} \int_{-\infty}^x e^{-1/2 t^2} dt, \quad (5.9)$$

for every $-\infty < x < \infty$, where

$$B(F_0) = \int_{-\infty}^{\infty} [dJ^*(F_0(x))/dx] f_0(x) dx, \quad f_0 = F_0', \quad (5.10)$$

and it is assumed that as $x \rightarrow \infty$, $\frac{d}{dx} J(F_0(x))$ is bounded.

A particular estimator not covered in (5.7) is the sample median for which $J(u) = 1 \forall 0 \leq u \leq 1$. However, in this case, or in the general case of sample quantiles, results similar to (5.9) are already derived in Sen (1972a). So far, we have assumed that $\{X_i\}$ is stationary ϕ -mixing; non-stationarity can be handled in the same manner as in pages 179-180 of Billingsley (1968), provided the probability measure governing the X_i is dominated by a stationary measure.

6. INVARIANCE PRINCIPLES FOR RANK STATISTICS

Here we consider a Donsker-type invariance principle for rank statistics as well as a law of iterated logarithm for these statistics.

Consider the space $C[0,1]$ of all real, continuous functions on the unit interval $[0,1]$, and associate with it the uniform topology

$$\rho(x,y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)|, \quad (6.1)$$

where both x and y belong $C[0,1]$. For every $n \geq 2$, define a process $Z_n = \{Z_n(t), 0 \leq t \leq 1\}$ by

$$Z_n(k/n) = k(T_k - \mu) / \sigma \sqrt{n}, \quad k=0, \dots, n, \quad (6.2)$$

where $T_0=0$, T_k , $k \geq 1$, are defined in (2.7), μ in (2.11) and σ^2 in (5.1). By linear interpolation, we complete the definition of $Z_n(t)$ for $t \in [k/n, (k+1)/n]$, $k=0, \dots, n-1$.

THEOREM 6.1. If $A_k(\phi) < \infty$ for some $k > 3$, and in (2.8), $\alpha = (k-2)/2k$, [or if (2.4) holds, and in (2.8), $\alpha = \frac{1}{2}$], then under (5.3), Z_n converges in law to a standard Brownian motion $W = \{W(t), 0 \leq t \leq 1\}$ in the uniform topology on $C[0,1]$.

Proof. Let us define $B(X_i)$, $i \geq 1$, as in (2.12), and let

$$Z_n^*(k/n) = [\sum_{j=1}^k B(X_j)]/\sigma\sqrt{n}, \quad k=1, \dots, n, \quad Z_n^*(0)=0, \quad (6.3)$$

and by linear interpolation, we complete the definition of $Z_n^*(t)$ for $t \in [k/n, (k+1)/n]$, $k=0, \dots, n-1$. Let $Z_n^* = \{Z_n^*(t), 0 \leq t \leq 1\}$. Then, by (2.10), (2.11), (6.1), (6.2) and (6.3),

$$\rho(Z_n, Z_n^*) \leq \{ \max_{1 \leq k \leq n} |R_k| \} / \sigma\sqrt{n}. \quad (6.4)$$

Therefore, by (4.10), under the hypothesis of the theorem,

$$\rho(Z_n, Z_n^*) \xrightarrow{P} 0 \text{ as } n \rightarrow \infty. \quad (6.5)$$

On the other hand, by Theorem 21.1 of Billingsley (1968),

$$Z_n^* \xrightarrow{D} W, \text{ in the uniform topology on } C[0,1]. \quad (6.6)$$

The theorem follows from (6.5) and (6.6). Q.E.D.

Note that by virtue of Theorem 6.1, for every $\varepsilon > 0$ and $\eta > 0$, there exist a $\delta > 0$ and an $n_0(\varepsilon, \eta)$, such that for $n \geq n_0(\varepsilon, \eta)$,

$$P\{ \sup_{|t-s| < \delta} |Z_n(t) - Z_n(s)| > \varepsilon \} < \eta. \quad (6.7)$$

Consequently, by Theorem 5.1, (6.7) and by Theorem 2 of Mogyorodi (1967), we obtain the following.

THEOREM 6.2. If $\{N_r, r \geq 1\}$ be a sequence of positive integer valued random variables, such that as $r \rightarrow \infty$

$$r^{-1} N_r \xrightarrow{P} \lambda, \quad (6.8)$$

where λ is a positive random variable defined on the same probability space (Ω, \mathcal{A}, P) , then under the conditions of Theorem 6.1, as $r \rightarrow \infty$

$$\mathcal{L}(N_r^{1/2} [T_{N_r} - \mu] / \sigma) \rightarrow \mathcal{N}(0, 1). \quad (6.9)$$

On defining $B(X_i)$, $i \geq 1$, as in (2.12) and noting that these form a stationary ϕ -mixing sequence, so that under (5.3), by the results of Reznik (1968),

$$\limsup_{n \rightarrow \infty} n^{-1/2} [\sum_{i=1}^n B(X_i)] / \sigma (2 \log \log n)^{1/2} = 1 \text{ a.s.}, \quad (6.10)$$

$$\liminf_{n \rightarrow \infty} n^{-1/2} [\sum_{i=1}^n B(X_i)] / \sigma (2 \log \log n)^{1/2} = -1 \text{ a.s.} \quad (6.11)$$

On the other hand, under the hypothesis of (4.10) or (4.11),

$$\limsup_{n \rightarrow \infty} n^{1/2} |R_n| = 0 \text{ a.s.} \quad (6.12)$$

Consequently, from (2.10), (6.10), (6.11) and (6.12), we arrive at the following.

THEOREM 6.3. Under the hypothesis of Theorem 6.1, the law of iterated logarithm holds for $\{T_n - \mu, n \geq 1\}$, i.e.,

$$P\{\limsup_{n \rightarrow \infty} n^{1/2} (T_n - \mu) / \sigma (2 \log \log n)^{1/2} = 1\} = 1, \quad (6.13)$$

$$P\{\liminf_{n \rightarrow \infty} n^{1/2} (T_n - \mu) / \sigma (2 \log \log n)^{1/2} = -1\} = 1. \quad (6.14)$$

Remark. For independent and identically distributed random variables, Sen and Ghosh (1973) have established Theorem 6.3 (under the hypothesis (5.6) with $\theta=0$), and Sen (1972c) has considered Theorem 6.1. Both of these results are based on a fundamental martingale property of $\{n[T_n - ET_n], n \geq 1\}$. For ϕ -mixing processes,

the martingale property does not hold. Nevertheless, our theorem 4.1 and the decomposition (2.10) provide the parallel results. For independent random variables or martingales, Strassen (1967) has established an a.s. invariance principle. By virtue of (3.4) [always holding for independent processes], under (2.8) (for $\alpha = \frac{1}{2}$), $|R_n| = O(n^{-\frac{1}{2}-\eta})$ a.s. for some $\eta > 0$ (see (4.11)). On the other hand, for independent processes, Strassen's (1967) Theorem 4.4 holds for $\{B(X_i), i \geq 1\}$. Consequently, if $\xi(t)$ be a standard Wiener process on $[0, \infty)$, then by Theorem 4.4 of Strassen (1967), as $n \rightarrow \infty$,

$$\sum_{i=1}^n B(X_i) = \xi(n) + o(n^{\frac{1}{2}}) \text{ a.s.}, \quad (6.15)$$

and hence, for independent and identically distributed random variables, under (2.8) for $\alpha = \frac{1}{2}$

$$n(T_n - \mu) = \xi(n) + o(n^{\frac{1}{2}}) \text{ a.s.}, \text{ as } n \rightarrow \infty. \quad (6.16)$$

Under (5.6) with $\theta = 0$, a similar result has been deduced earlier by Sen and Ghosh (1973).

REFERENCES

- BILLINGSLEY, P. (1968). Convergence of Probability Measures. John Wiley, New York.
- CHERNOFF, H., and SAVAGE, I. R. (1958). Asymptotic normality and efficiency of certain nonparametric test statistics. Ann. Math. Stat. 29, 972-994.
- GHOSH, M. (1972). On the representation of linear functions of order statistics. Sankhya, Ser. A. 34, in press.
- GOVINDARAJULU, Z. (1960). Central limit theorems and asymptotic efficiency for one-sample nonparametric procedures. Tech. Report No. 11, Dept. of Statistics, Univ. Minnesota.
- GOVINDARAJULU, Z., LECAM, L., and RAGHAVACHARI, M. (1967). Generalization of Theorems of Chernoff and Savage on the asymptotic normality of test statistics. Proc. 5th Berkeley Symp. Math. Statist. Prob. 1, 609-638.
- HÁJEK, J. (1968). Asymptotic normality of simple linear rank statistics under alternatives. Ann. Math. Stat. 39, 325-346.
- HODGES, J. L. JR., and LEHMANN, E. L. (1963). Estimates of location based on rank tests. Ann. Math. Stat. 34, 598-611.
- HUŠKOVÁ, M. (1970). Asymptotic distribution of simple linear rank statistics for testing symmetry. Zeit für Wahrsch Verw Geb. 12, 308-322.
- MOGYORODI, J. (1967). Limit distributions for sequences of random variables with random indices. Trans. 4th Prague Conference on Information Theory, Statistical Decision functions and Random Processes. 462-470.
- PHILIPP, W. (1969). The remainder of the central limit theorem for mixing stochastic processes. Ann. Math. Stat. 40, 601-605.
- PURI, M. L., and SEN, P. K. (1969). On the asymptotic normality of one sample rank order test statistics. Teoria Veroy. i ee Primen. 14, 167-172.
- PURI, M. L., and SEN, P. K. (1971). Nonparametric Methods in Multivariate Analysis. John Wiley, New York.
- PYKE, R., and SHORACK, G. R. (1968a). Weak convergence of a two-sample empirical process and a new approach to Chernoff-Savage theorems. Ann. Math. Stat. 39, 755-771.
- PYKE, R., and SHORACK, G. R. (1968b). Weak convergence and a Chernoff-Savage theorem for random sample sizes. Ann. Math. Stat. 39, 1675-1685.
- REZNIK, M. KH. (1968). The law of iterated logarithm for some class of stationary processes. Teor. Veroy. i ee Primen. 13, 606-621.

- SEN, P. K. (1963). On the estimation of relative potency in dilution (-direct) assays by distribution-free methods. Biometrics 19, 532-552.
- SEN, P. K. (1970). On the distribution of the one sample rank order statistics. Nonparametric Tech. in Stat. Infer. (Ed: M. L. Puri). Cambridge Univ. Press 53-72.
- SEN, P. K. (1971). A note on weak convergence of empirical processes for sequences of ϕ -mixing random variables. Ann. Math. Stat. 42, 2131-2133.
- SEN, P. K. (1972a). On the Bahadur representation of sample quantiles for sequences of ϕ -mixing random variables. Jour. Multivar. Anal. 2, 77-95.
- SEN, P. K. (1972b). Weak convergence of multidimensional empirical processes for stationary ϕ -mixing processes. Inst. Stat. Univ. N. Carolina, Mimeo Rept. No. 846.
- SEN, P. K. (1972c). The invariance principle for one-sample rank order statistics. Ann. Math. Stat. (submitted)
- SEN, P. K. and GHOSH, M. (1971). On bounded length sequential confidence intervals based on one-sample rank order statistics. Ann. Math. Stat. 42, 189-203.
- SEN, P. K., and GHOSH, M. (1973). A law of iterated logarithm for one sample rank order statistics and an application. Ann. Stat. 1, in press.
- SERFLING, R. J. (1968). The Wilcoxon two-sample statistic on strongly mixing processes. Ann. Math. Stat. 39, 1202-1209.
- STRASSEN, V. (1967). Almost sure behavior of sums of independent random variables and martingales. Proc. 5th Berkeley Sump. Math. Stat. Prob. 2, 315-343.