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# A VECTOR VALUED MULTIVARIATE HAZARD RATE - 1

N.L. Johnson

*University of North Carolina at Chapel Hill*

and

S. Kotz

*Temple University, Philadelphia Pa.*

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# A VECTOR VALUED MULTIVARIATE HAZARD RATE - I

by

N.L. Johnson
University of North Carolina at Chapel Hill

and

S. Kotz
Temple University, Philadelphia Pa.

## 1. Introduction

The *hazard rate* $h_X(x)$ of the distribution of an absolutely continuous random variable $X$ with cumulative distribution

$$F_X(x) = \Pr[X \le x]$$

and density function

$$f_X(x) = F_X'(x)$$

is defined as

$$h_X(x) = -\frac{d}{dx}\log(1 - F_X(x)) = \frac{f_X(x)}{1-F_X(x)} = f_X(x)/G_X(x)$$

with $G_X(x) = 1 - F_X(x)$, in the interval $0 < G_X(x) < 1$ and is undefined otherwise.

The hazard rate has been used (see, for example, [1]) as a basis for certain kinds of classification of univariate distributions. If $h_X(x)$ is an increasing (decreasing) function of $x$ (for those values for which it is defined), the distribution is termed *increasing (decreasing) hazard rate*, denoted by IHR (DHR).

Of course, most distributions are neither IHR nor DHR - $h_X(x)$ may increase over certain ranges of $x$, and decrease over others.

## 2. Multivariate Hazard Rate

Some attempts have been made to extend the definition of IHR and DHR to multivariate distributions [2], [4], [6]. In each of these investigations, multivariate hazard rate has been defined as a single scalar quantity.

These do not always involve the explicit definition of a multivariate hazard rate. Puri and Rubin [7] define multivariate hazard rate $X_1,\ldots,X_m$ as

$$\text{(Joint density function)}/G_{\underset{\sim}{X}}(\underset{\sim}{x})$$

where $G_{\underset{\sim}{X}}(\underset{\sim}{x}) = \Pr[\bigcap_{j=1}^{m}(X_j > x_j]$ .

Goodman [4] suggests that if the joint density function of $X_1,\ldots,X_m$ is a decreasing function of a *homogeneous* function $g(x_1,\ldots,x_m)$ of the arguments $\underset{\sim}{x}$ then the (univariate) hazard rate of the distribution of $g(\underset{\sim}{x})$ be regarded as the multivariate hazard rate of $\underset{\sim}{X}$. Of course, this definition can only be applied to limited classes of distributions.

Harris [6] defines multivariate IHR to require

(i) $\Pr[\bigcap_{j=1}^{m}(X_j > x_j + \Delta) \mid \bigcap_{j=1}^{m}(X_j > x_j)]$ to be a nondecreasing function of $x_1,x_2,\ldots,x_m$ and

(ii) $\Pr[\bigcap_{j=1}^{m}(X_j > x_j') \mid \bigcap_{j=1}^{m}(X_j > x_j)]$ to be a nondecreasing function of $x_1,x_2,\ldots,x_m$ for *all* $x_1,\ldots,x_m$ (including $x_j' < x_j$).

Brindley and Thompson [2] say that $X_1,\ldots,X_m$ have a joint IFR (increasing failure rate) distribution if a condition like (i) is satisfied for every subset of the $m$ variables.

This last definition appears to imply, in effect, a set of failure (or hazard) rates relevant to the joint distribution. We will take a similar point of view in the present paper.

For a concept such as "hazard rate" it is unreasonable to expect a single value to represent this aspect of a multivariate distribution. The basic idea underlying the univariate definition is that of rate of decrease in "survivors" with increase in value (x) of X (as in a life table where the hazard rate is in fact the force of mortality). When there are two or more variates this rate depends on which variate is changed (or more generally, the proportions in which different variates are changed) and we need a different "rate" for each variate.

We define the (joint) *multivariate hazard rate of* m absolutely continuous random variables $X_1,\ldots,X_m$ as the vector

$$h_{\underset{\sim}{X}}(\underset{\sim}{x}) = \left(-\frac{\partial}{\partial x_1},\ldots,-\frac{\partial}{\partial x_m}\right) \log G_{\underset{\sim}{X}}(\underset{\sim}{x}) = -\operatorname{grad} \log G_{\underset{\sim}{X}}(\underset{\sim}{x}) \tag{1}$$

For convenience we will write

$$-\frac{\partial}{\partial x_j} \log G_{\underset{\sim}{X}}(\underset{\sim}{x}) = h_{\underset{\sim}{X}}(\underset{\sim}{x})_j .$$

If for *all* values of $\underset{\sim}{x}$, *all* components of $h_{\underset{\sim}{X}}(\underset{\sim}{x})$ are increasing (decreasing) functions of the corresponding variable - i.e., $h_{\underset{\sim}{X}}(\underset{\sim}{x})_j$ is an increasing (decreasing) function of $x_j$ for $j = 1,2,\ldots,m$ - then the distribution is called *(multivariate) IHR (DHR)*.

If we wish to emphasize this particular definition we will call it *vector multivariate IHR (DHR)*.

3. <u>Some Properties</u>

(i) If $X_1,\ldots,X_m$ are mutually independent then

$$G_{\underset{\sim}{X}}(\underset{\sim}{x}) = \prod_{j=1}^{m} G_{X_j}(x_j)$$

and

$$h_{\underset{\sim}{X}}(\underset{\sim}{x})_j = h_{X_j}(x_j)$$

where, of course, the left hand side is a component of a multivariate hazard rate and the right hand side is a univariate hazard rate.

So, if $X_1,\dots,X_m$ are independent, their joint distribution is IHR (DHR) if and only if the distribution of each $X_1,\dots,X_m$ is IHR (DHR).

(ii) Suppose that $X_1,\dots,X_m$ are *exchangeable*, i.e. $G_{\underset{\sim}{X}}(\underset{\sim}{x})$ is unchanged if $X_1,\dots,X_m$ are permuted in any order. This implies that if the joint distribution is IHR (DHR) for some particular set of values $x_1,\dots,x_m$ it will also be IHR (DHR) for any permutation of these values.

(iii) If $Y_1 = g(X_1)$ is strictly monotonic function of $X_1$ then the values of $h_{\underset{\sim}{S}}(\underset{\sim}{s})_j$ $(j = 2,\dots,m)$ are the same for

$$\underset{\sim}{S} = \underset{\sim}{X} \quad \text{and} \quad \underset{\sim}{S} = (Y_1,X_2,\dots,X_m)$$

with $s_1 = g^{-1}(x_1)$ in the second case.

It follows that if the distribution of $X_1,X_2,\dots,X_m$ is IHR (DHR) so will be that of $Y_1,X_2,\dots,X_m$ provided that the first component of $h_{Y_1,X_2,\dots,X_m}(y_1,x_2,\dots,x_m)$ has the appropriate sign.

It is trivial that if the conditional joint distribution of $X_2,\dots,X_m$, given $x_1$ is IHR (DHR) for all $x_1$, so is that of $X_2,\dots,X_m$, given $y_1$.

(iv) If the joint distribution of $X$ is IHR (DHR) so is the joint distribution of $\underset{\sim}{a} + (b_1X_1,\dots,b_mX_m)$ provided $\underset{\sim}{b} > \underset{\sim}{0}$.

(v) If the multivariate hazard rate is constant, i.e.

$$h_{\underset{\sim}{X}}(\underset{\sim}{x}) = \underset{\sim}{c}$$

this means that (whenever the hazard rate exists)

$$\frac{\partial \log G_{\underset{\sim}{X}}(\underset{\sim}{x})}{\partial x_j} = - c_j \qquad (j = 1,\ldots,m) .$$

Hence

$$G_{\underset{\sim}{X}}(\underset{\sim}{x}) = e^{-c_j x_j} g_j \, (x_1,\ldots,x_{j-1}, x_{j+1},\ldots,x_m) \qquad (j = 1,\ldots,m)$$

whence

$$G_{\underset{\sim}{X}}(\underset{\sim}{x}) \propto \exp(- \sum_{j=1}^{m} c_j x_j) \quad .$$

Thus, the X's are mutually independent exponential variables if and only if the multivariate hazard rate is constant.

By contrast, Puri and Rubin [7], using the (single-valued) definition of multivariate hazard rate as (density function)/$G_{\underset{\sim}{X}}(\underset{\sim}{x})$, show that constancy of of multivariate hazard rate implies that

$$G_{\underset{\sim}{X}}(\underset{\sim}{x}) = \sum_{i=1}^{k} p_i \exp(- \sum_{j=1}^{m} \lambda_{ij} x_j)$$

with $\prod_{j=1}^{m} \lambda_{ij}$ independent of $i$ and $0 < p_i < 1$, $\sum_{i=1}^{k} p_i = 1$ .

This includes the case when $X_1,\ldots,X_m$ have independent exponential distributions, but is not limited thereto.

(vi) Noting that

$$G_{\underset{\sim}{X}}(\underset{\sim}{x}) = G_{X_2,\ldots,X_m}(x_2,\ldots,x_m)\tilde{G}_{X_1|X_2,\ldots,X_m}(x_1|x_2,\ldots,x_m)$$

where $\tilde{G}_{X_1|X_2,\ldots,X_m}(x_1,\ldots,x_m) = \Pr[X_1 > x_1 | \bigcap_{j=2}^{m} (X_j > x_j)]$

we see that

$$h_{\underset{\sim}{X}}(\underset{\sim}{x})_1 = - \frac{\partial}{\partial x_1} \log \tilde{G}_{X_1|X_2,\ldots,X_m}(x_1|x_2,\ldots,x_m)$$

$$= \tilde{h}_{X_1|X_2,\ldots,X_m}(x_1|x_2,\ldots,x_m)$$

is an obvious notation. Generally

(2) $$h_{\underset{\sim}{X}}(\underset{\sim}{x})_j = \tilde{h}_{X_j|(X_1,\ldots,X_m)_j}(x_j|(x_1,\ldots,x_m)_j)$$

where $(X_2,\ldots,X_m)_j$ denotes $X_2,\ldots,X_{j-1}, X_{j+1},\ldots,X_m$ and similarly for $(x_2,\ldots,x_m)_j$.

Thus the components of the vector multivariate hazard rate are in fact univariate hazard rates of conditional distributions of each variate, given certain inequalities on the remainder.

A further result, which follows directly, is that the vector multivariate hazard rate of the conditional joint distribution of $X_1,X_2,\ldots,X_k$ given $\bigcap_{j=k+1}^{m}(X_j > x_j)$ has the same components as the first $k$ components of $h_{\underset{\sim}{X}}(\underset{\sim}{x})$. (This is because the conditional distribution of $X_1$ given $\bigcap_{j=2}^{k}(X_j > x_j)$ *and* $\bigcap_{j=k+1}^{m}(X_j > x_j)$ is just the same as that of $X_1$ given $\bigcap_{j=2}^{m}(X_j > x_j)$.)

We can write (again in an obvious notation)

$$h_{\underset{\sim}{X}}(\underset{\sim}{x}) = (\tilde{h}_{\underset{\sim}{X}_1|\underset{\sim}{X}_2}(\underset{\sim}{x}_1|\underset{\sim}{x}_2),\ \tilde{h}_{\underset{\sim}{X}_2|\underset{\sim}{X}_1}(\underset{\sim}{x}_2|\underset{\sim}{x}_1))$$

where $\underset{\sim}{X} = \begin{pmatrix}\underset{\sim}{X}_1\\ \underset{\sim}{X}_2\end{pmatrix}$ is partitioned between the k-th and (k+1)-th rows. Similar results apply, of course, for any dichotomy of $\underset{\sim}{X}$.

A direct consequence of these results is that if the joint distribution of $X_1,\ldots,X_m$ is IHR (DHR) so is the conditional joint distribution of any subset of the X's, given inequalities of form $(X_j > x_j)$ on the remainder.

Since $\tilde{h}_{X_1|X_2,\ldots,X_m}(x_1|x_2,\ldots,x_m)$ is the expected value, with respect to variation of $X_2,\ldots,X_m$ of the hazard rate $h_{X_1|X_2,\ldots,X_m}(x_1|X_2,\ldots,X_m)$ of the distribution of $X_1$ given $X_2,X_3,\ldots,X_m$, we might expect the joint distribution of $X_1,\ldots,X_m$ to be IHR (DHR) if each of the conditional joint distributions of $X_j$ given the other (m-1) X's ($j = 1,\ldots,m$) is IHR (DHR). This would certainly be so if the appropriate distribution of the other variables did not depend on the value of $x_j$. However, in the equation

$$\tilde{h}(x_1|x_2,\ldots,x_m) = E[h(x_1|X_2,\ldots,X_m)]$$

(subscripts omitted for convenience) the relevant joint distribution of $X_2,\ldots,X_m$ is that *conditioned on* $X_1 > x_1$ and truncated by $\bigcap_{j=2}^{m}(X_j > x_j)$.

## 4. Examples

### 4.1 Multinormal Distributions

It is known that all normal distributions are IHR. For a multinormal distribution, the conditional distribution of any variable, given the remainder, is normal, and so IHR. We would thus expect that all multivariate normal distributions are IHR, according to the vector multivariate hazard rate defined in this paper.

Tables 1 and 2 provide some relevant numerical illustration, for bivariate normal distributions with positive correlation coefficients ($\rho > 0$). For the unit normal distribution, the hazard rate at variable value $x_1$ is

$$h_{X_1}(x_1) = \phi(x_1)/\{1 - \Phi(x_1)\} \tag{3}$$

where

$$\phi(x) = (\sqrt{2\pi})^{-1}\exp(-\tfrac{1}{2}x^2); \quad \Phi(x) = \int_{-\infty}^{x}\phi(t)dt \quad .$$

For the standard bivariate normal distribution with correlation coefficient $\rho$ , the $X_1$ component of the vector multivariate hazard rate at variable values $(x_1,x_2)$ is

(4) $$h_{\underset{\sim}{X}}(\underset{\sim}{x})_1 = \frac{\left\{1 - \Phi\left(\frac{x_2 - \rho x_1}{\sqrt{(1-\rho^2)}}\right)\right\} \phi(x_1)}{L_\rho(x_1,x_2)}$$

where

$$L_\rho(x_1,x_2) = [2\pi\sqrt{(1-\rho^2)}]^{-1}\int_{x_2}^{\infty}\int_{x_1}^{\infty} \exp[-\tfrac{1}{2}(1-\rho^2)^{-1}(t_1^2-2\rho t_1 t_2+t_2^2)]dt_1\, dt_2 .$$

In Table 1 values of $h_{\underset{\sim}{X}}(\underset{\sim}{x})_1$ are given.

In Table 2 are given ratios of this component of the multivariate hazard rate to the univariate hazard rate $h_{X_1}(x_1)$, i.e. of

$$\frac{h_{\underset{\sim}{X}}(\underset{\sim}{x})_1}{h_{X_1}(x_1)} = \frac{\left\{1 - \Phi\left(\frac{x_2 - \rho x_1}{\sqrt{(1-\rho^2)}}\right)\right\}\{1 - \Phi(x_1)\}}{L_\rho(x_1,x_2)}$$

Clearly, for all proper joint distributions

(5) $$\lim_{x_2 \to -\infty} h_{\underset{\sim}{X}}(\underset{\sim}{x})_1 = h_{X_1}(x_1)$$

reflecting the fact that $\lim_{x_2 \to -\infty} G_{X_1,X_2}(x_1,x_2) = G_{X_1}(x_1)$ .

From Tables 1 and 2 it can be seen that this limit is approached more rapidly when

(i) $\rho$ is large and positive,

or (ii) $x_1$ is large and positive.

TABLE 1

Bivariate Normal: First Component of Multivariate Hazard Rate

$$h_{\underset{\sim}{X}}(\underset{\sim}{x})_1 = \frac{\{1 - \Phi(\frac{x_2 - \rho x_1}{\sqrt{(1-\rho^2)}})\}\phi(x_1)}{L\ (x_1,x_2)}$$

$\rho = 0.2$

| $x_2 \backslash x_1$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.0536 | 0.1358 | 0.2835 | 0.5043 | 0.7937 | 1.136 | 1.521 | 1.935 | 2.370 |
| -1.5 | 0.0513 | 0.1317 | 0.2772 | 0.4964 | 0.7840 | 1.127 | 1.512 | 1.927 | 2.363 |
| -1.0 | 0.0477 | 0.1246 | 0.2662 | 0.4819 | 0.7673 | 1.110 | 1.495 | 1.911 | 2.348 |
| -0.5 | 0.0429 | 0.1149 | 0.2503 | 0.4603 | 0.7415 | 1.082 | 1.466 | 1.881 | 2.321 |
| 0 | 0.0374 | 0.1031 | 0.2305 | 0.4324 | 0.7072 | 1.043 | 1.425 | 1.841 | 2.280 |
| 0.5 | 0.0317 | 0.0905 | 0.2084 | 0.4003 | 0.6666 | 0.9962 | 1.374 | 1.788 | 2.212 |
| 1.0 | 0.0263 | 0.0779 | 0.1854 | 0.3658 | 0.6217 | 0.9434 | 1.316 | 1.727 | 2.163 |
| 1.5 | 0.0214 | 0.0660 | 0.1627 | 0.3304 | 0.5748 | 0.8873 | 1.254 | 1.659 | 2.094 |
| 2.0 | 0.0170 | 0.0550 | 0.1410 | 0.2958 | 0.5273 | 0.8292 | 1.188 | 1.589 | 1.945 |

$\rho = 0.4$

| $x_2 \backslash x_1$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.0510 | 0.1323 | 0.2799 | 0.5017 | 0.7918 | 1.137 | 1.522 | 1.937 | 2.372 |
| -1.5 | 0.0458 | 0.1231 | 0.2672 | 0.4898 | 0.7783 | 1.125 | 1.514 | 1.931 | 2.368 |
| -1.0 | 0.0382 | 0.1081 | 0.2444 | 0.4596 | 0.7491 | 1.099 | 1.491 | 1.914 | 2.356 |
| -0.5 | 0.0293 | 0.0888 | 0.2123 | 0.4164 | 0.7000 | 1.049 | 1.446 | 1.875 | 2.335 |
| 0 | 0.0208 | 0.0682 | 0.1745 | 0.3613 | 0.6322 | 0.9755 | 1.373 | 1.807 | 2.265 |
| 0.5 | 0.0130 | 0.0492 | 0.1361 | 0.3002 | 0.5517 | 0.8823 | 1.275 | 1.710 | 2.174 |
| 1.0 | 0.0084 | 0.0334 | 0.1008 | 0.2393 | 0.4657 | 0.7773 | 1.158 | 1.589 | 2.054 |
| 1.5 | 0.0049 | 0.0214 | 0.0711 | 0.1831 | 0.3806 | 0.6308 | 1.031 | 1.452 | 1.914 |
| 2.0 | 0.0027 | 0.0130 | 0.0478 | 0.1346 | 0.3010 | 0.5588 | 0.8998 | 1.307 | 1.760 |

Bivariate Normal $h_{\underset{\sim}{X}}(\underset{\sim}{x})_1$ continued

$\rho = 0.6$

| $x_2 \backslash x_1$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.0473 | 0.1287 | 0.2784 | 0.5028 | 0.7945 | 1.140 | 1.525 | 1.938 | 2.373 |
| -1.5 | 0.0379 | 0.1127 | 0.2588 | 0.4851 | 0.7820 | 1.133 | 1.521 | 1.937 | 2.373 |
| -1.0 | 0.0260 | 0.0876 | 0.2215 | 0.4439 | 0.7463 | 1.108 | 1.507 | 1.930 | 2.370 |
| -0.5 | 0.0150 | 0.0590 | 0.1697 | 0.3734 | 0.6743 | 1.046 | 1.463 | 1.903 | 2.356 |
| 0 | 0.0072 | 0.0341 | 0.1147 | 0.2869 | 0.5660 | 0.9369 | 1.370 | 1.834 | 2.311 |
| 0.5 | 0.0029 | 0.0169 | 0.0679 | 0.1973 | 0.4386 | 0.7868 | 1.220 | 1.704 | 2.210 |
| 1.0 | 0.0010 | 0.0072 | 0.0351 | 0.1211 | 0.3084 | 0.6161 | 1.029 | 1.516 | 2.043 |
| 1.5 | 0.0003 | 0.0026 | 0.0158 | 0.0659 | 0.1974 | 0.4474 | 0.8195 | 1.288 | 1.818 |
| 2.0 | 0.0001 | 0.0008 | 0.0062 | 0.0316 | 0.1136 | 0.2991 | 0.6126 | 1.042 | 1.558 |

$\rho = 0.8$

| $x_1 \backslash x_2$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.0419 | 0.1269 | 0.2820 | 0.5074 | 0.7976 | 1.141 | 1.525 | 1.939 | 2.373 |
| -1.5 | 0.0253 | 0.0994 | 0.2568 | 0.4948 | 0.7938 | 1.140 | 1.525 | 1.939 | 2.373 |
| -1.0 | 0.0102 | 0.0578 | 0.1955 | 0.4435 | 0.7683 | 1.133 | 1.524 | 1.938 | 2.373 |
| -0.5 | 0.0026 | 0.0229 | 0.0760 | 0.3303 | 0.6782 | 1.087 | 1.509 | 1.936 | 2.373 |
| 0 | 0.0004 | 0.0059 | 0.0303 | 0.1895 | 0.5017 | 0.9474 | 1.436 | 1.911 | 2.368 |
| 0.5 | - | 0.0010 | 0.0081 | 0.0779 | 0.2906 | 0.6988 | 1.239 | 1.803 | 2.328 |
| 1.0 | - | 0.0001 | 0.0014 | 0.0219 | 0.1245 | 0.4135 | 0.9155 | 1.542 | 2.178 |
| 1.5 | - | - | 0.0002 | 0.0041 | 0.0374 | 0.1362 | 0.5559 | 1.146 | 1.352 |
| 2.0 | - | - | - | 0.0001 | 0.0075 | 0.0602 | 0.2638 | 0.7158 | 1.388 |

(- means < 0.00005)

TABLE 2

Bivariate Normal: Ratio of Multivariate Component to Univariate Hazard Rate

$$\frac{h_{\underset{\sim}{X}}(\underset{\sim}{x})_1}{h_{X_1}(x_1)} = \frac{\left\{1 - \Phi\left(\frac{x_2 - \rho x_1}{\sqrt{(1-\rho^2)}}\right)\right\}\left\{1 - \Phi(x_1)\right\}}{L_\rho(x_1,x_2)}$$

$\rho = 0.2$

| $x_2 \backslash x_1$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.9700 | 0.9786 | 0.9857 | 0.9904 | 0.9936 | 0.9957 | 0.9972 | 0.9981 | 0.9987 |
| -1.5 | 0.9292 | 0.9488 | 0.9638 | 0.9749 | 0.9826 | 0.9879 | 0.9916 | 0.9941 | 0.9958 |
| -1.0 | 0.8642 | 0.8981 | 0.9255 | 0.9465 | 0.9617 | 0.9725 | 0.9802 | 0.9858 | 0.9894 |
| -0.5 | 0.7771 | 0.8276 | 0.8703 | 0.9040 | 0.9293 | 0.9478 | 0.9611 | 0.9705 | 0.9778 |
| 0 | 0.6772 | 0.7431 | 0.8015 | 0.8493 | 0.8864 | 0.9140 | 0.9342 | 0.9494 | 0.9605 |
| 0.5 | 0.5743 | 0.6523 | 0.7246 | 0.7862 | 0.8354 | 0.8730 | 0.9011 | 0.9222 | 0.9381 |
| 1.0 | 0.4758 | 0.5615 | 0.6446 | 0.7184 | 0.7792 | 0.8268 | 0.8632 | 0.8907 | 0.9116 |
| 1.5 | 0.3868 | 0.4753 | 0.5656 | 0.6490 | 0.7204 | 0.7776 | 0.8220 | 0.8560 | 0.8824 |
| 2.0 | 0.3091 | 0.3965 | 0.4902 | 0.5809 | 0.6609 | 0.7267 | 0.7788 | 0.8195 | 0.8508 |

$\rho = 0.4$

| $x_2 \backslash x_1$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.9235 | 0.9532 | 0.9733 | 0.9854 | 0.9924 | 0.9961 | 0.9981 | 0.9991 | 0.9996 |
| -1.5 | 0.8285 | 0.8866 | 0.9289 | 0.9619 | 0.9755 | 0.9863 | 0.9926 | 0.9961 | 0.9980 |
| -1.0 | 0.6908 | 0.7786 | 0.8498 | 0.9026 | 0.9389 | 0.9628 | 0.9778 | 0.9871 | 0.9927 |
| -0.5 | 0.5308 | 0.6399 | 0.7381 | 0.8178 | 0.8773 | 0.9193 | 0.9479 | 0.9670 | 0.9840 |
| 0 | 0.3766 | 0.4917 | 0.6069 | 0.7095 | 0.7924 | 0.8549 | 0.9000 | 0.9320 | 0.9544 |
| 0.5 | 0.2346 | 0.3544 | 0.4731 | 0.5896 | 0.6915 | 0.7732 | 0.8357 | 0.8820 | 0.9160 |
| 1.0 | 0.1529 | 0.2406 | 0.3505 | 0.4700 | 0.5837 | 0.6812 | 0.7593 | 0.8197 | 0.8656 |
| 1.5 | 0.0885 | 0.1544 | 0.2472 | 0.3597 | 0.4770 | 0.5528 | 0.6760 | 0.7491 | 0.8064 |
| 2.0 | 0.0484 | 0.0938 | 0.1662 | 0.2644 | 0.3772 | 0.4897 | 0.5900 | 0.6739 | 0.7416 |

$\rho = 0.6$

| $x_2 \backslash x_1$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.8564 | 0.9273 | 0.9679 | 0.9875 | 0.9957 | 0.9987 | 0.9996 | 0.9999 | 1.0000 |
| -1.5 | 0.6854 | 0.8117 | 0.9000 | 0.9528 | 0.9801 | 0.9925 | 0.9975 | 0.9993 | 0.9998 |
| -1.0 | 0.4700 | 0.6310 | 0.7703 | 0.8718 | 0.9354 | 0.9706 | 0.9880 | 0.9956 | 0.9986 |
| -0.5 | 0.2708 | 0.4252 | 0.5902 | 0.7334 | 0.8451 | 0.9167 | 0.9591 | 0.9817 | 0.9926 |
| 0 | 0.1308 | 0.2458 | 0.3989 | 0.5634 | 0.7094 | 0.8211 | 0.8981 | 0.9458 | 0.9737 |
| 0.5 | 0.0532 | 0.1217 | 0.2361 | 0.3874 | 0.5477 | 0.6895 | 0.8000 | 0.8788 | 0.9313 |
| 1.0 | 0.0184 | 0.0517 | 0.1220 | 0.2378 | 0.3865 | 0.5399 | 0.6749 | 0.7818 | 0.8608 |
| 1.5 | 0.0054 | 0.0189 | 0.0548 | 0.1294 | 0.2474 | 0.3921 | 0.5373 | 0.6642 | 0.7662 |
| 2.0 | 0.0014 | 0.0058 | 0.0214 | 0.0620 | 0.1424 | 0.2621 | 0.4017 | 0.5377 | 0.6563 |

$\rho = 0.8$

| $x_2 \backslash x_1$ | -2.0 | -1.5 | -1.0 | -0.5 | 0 | 0.5 | 1.0 | 1.5 | 2.0 |
|---|---|---|---|---|---|---|---|---|---|
| -2.0 | 0.7575 | 0.9146 | 0.9806 | 0.9966 | 0.9996 | 1.0000 | 1.0000 | 1.0000 | 1.0000 |
| -1.5 | 0.4572 | 0.7160 | 0.8930 | 0.9718 | 0.9949 | 0.9994 | 1.0000 | 1.0000 | 1.0000 |
| -1.0 | 0.1847 | 0.4166 | 0.6799 | 0.8710 | 0.9629 | 0.9926 | 0.9990 | 0.9999 | 1.0000 |
| -0.5 | 0.0472 | 0.1651 | 0.3887 | 0.6487 | 0.8500 | 0.9529 | 0.9896 | 0.9984 | 0.9998 |
| 0 | 0.0075 | 0.0425 | 0.1552 | 0.3721 | 0.6288 | 0.8303 | 0.9418 | 0.9857 | 0.9976 |
| 0.5 | 0.0007 | 0.0070 | 0.0414 | 0.1529 | 0.3642 | 0.6124 | 0.8121 | 0.9300 | 0.9810 |
| 1.0 | - | 0.0007 | 0.0072 | 0.0430 | 0.1561 | 0.3624 | 0.6003 | 0.7955 | 0.9176 |
| 1.5 | - | - | 0.0008 | 0.0080 | 0.0469 | 0.1632 | 0.3645 | 0.5913 | 0.7804 |
| 2.0 | - | - | 0.0001 | 0.0010 | 0.0094 | 0.0527 | 0.1730 | 0.3692 | 0.5847 |

### 4.2 Multivariate Pareto Distributions

Consider the m-dimensional multivariate Pareto distribution, with density function

$$p_{\underset{\sim}{X}}(\underset{\sim}{x}) = a(a+1)\dots(a+m-1)\left(\prod_{j=1}^{m} \Theta_j\right)^{-1}\left(\sum_{j=1}^{m} \Theta_j^{-1}x_j - m + 1\right)^{-(a+m)}$$

$$(a>0;\ x_j>\Theta_j>0)\ .$$

For this distribution

$$G_{\underset{\sim}{X}}(\underset{\sim}{x}) = \left(\sum_{j=1}^{m} \Theta_j^{-1}x_j - m + 1\right)^{-a} .$$

Hence

$$\partial \log G_{\underset{\sim}{X}}(\underset{\sim}{x})/\partial x_r = a\Theta_r^{-1}\left(\sum_{j=1}^{m} \Theta_j^{-1}x_j - m + 1\right) \quad .$$

This is a decreasing function of $x_r$ for all r (=1,...,m). Hence (according to our definition) this is a decreasing hazard rate distribution.

### 4.3 Morgenstern-Gumbel-Farlie Distributions

Consider the family of bivariate distributions ([3], [5]) for which

(6) $$G_{X_1,X_2}(x_1,x_2) = G_{X_1}(x_1)\ G_{X_2}(x_2)[1 + \alpha(1-G_{X_1}(x_1))(1-G_{X_2}(x_2))] \quad \text{with } |\alpha|<1.$$

(These were originally defined in terms of a similar equivalent relation among the cumulative distribution functions.) For this family

(7) $$h_{X_1,X_2}(x)_j = h_{X_j}(x_j) - \frac{\alpha\{1 - G_{X_{3-j}}(x_{3-j})\}f_{X_j}(x_j)}{1+\alpha\{1-G_{X_1}(x_1)\}\{1-G_{X_2}(x_2)\}}$$

$$= [1 - \{\beta(G_{X_j}(x_j))^{-1} - 1\}^{-1}]h_{X_j}(x_j)$$

where $\beta = 1 + [\alpha\{1-G_{X_{3-j}}(x_{3-j})\}]^{-1}$ .

Note that $\beta$ has the same sign as $\alpha$ , since $|\alpha|^{-1} > 1$ and $|1-G_{X_{3-j}}(x_{3-j})|^{-1} > 1$ .

If $\alpha$ is positive (negative) then (noting that $G_{X_j}(x_j)$ is a decreasing function of $x_j$) $h_{\underset{\sim}{X}}(\underset{\sim}{x})_j$ is an increasing (decreasing) function of $x_j$.

It follows that if both $X_1$ and $X_2$ are IHR (DHR) then $\underset{\sim}{X}$ is multivariate IHR (DHR) if $\alpha$ is positive (negative).

In the special case when both $X_1$ and $X_2$ have exponential distributions, so that $h_1(x_1) = h_1$ and $h_2(x_2) = h_2$ are constants, we have

(8) $$h_{\underset{\sim}{X}}(\underset{\sim}{x}) = [1 - \{ \beta \exp(h_j\ x_j) - 1\}^{-1}]h_1$$

with $\beta = 1 - [\alpha\{1 - \exp(-h_{3-j}x_{3-j})\}]^{-1}$. Hence the joint distribution is multivariate IHR if $\alpha$ is positive, DHR if $\alpha$ is negative.

If, in (6), $X_1$ and $X_2$ each have Weibull distributions, with

$$G_{X_j}(x_j) = \exp(-x_j^{c_j}) \qquad (c_j > 0,\ x_j > 0;\ j = 1,2)$$

then

$$h_{\underset{\sim}{X}}(\underset{\sim}{x})_j = [1 - \{\beta \exp(x_j^{c_j}) - 1\}^{-1}]c_j x_j^{c_j - 1} \qquad (j = 1,2)$$

where

$$\beta = 1 + [\alpha\{1 - \exp(-x_{3-j}^{c_{3-j}})\}]^{-1} .$$

Hence

$$\partial h_{\underset{\sim}{X}}(\underset{\sim}{x})_j/\partial x_j = [\{1-(g-1)^{-1}\}(c_j-1) + (g-1)^{-2} g\ c_j\ \log(g/\beta)]c_j x_j^{c_j-2}$$

where $g = \beta \exp(x_j^{c_j})$ .

Since $c_j > 0$ and $x_j > 0$, the sign of $\partial h_{\underset{\sim}{X}}(\underset{\sim}{x})_j/\partial x_j$ is the same as that of $H_j(g) = (c_j-1)(g-1)(g-2) + c_j g \log(g/\beta)$.

Note that $g$ must have the same sign as $\alpha$. (If $\alpha < 0$, then as noted above $\alpha > -1$ and so $\alpha^{-1} < -1$ and *a fortiori* $1+[\alpha\{1-\exp(-x_{3-j}^{c_{3-j}})\}]^{-1} < 0$.)

Now our general results show that if $\alpha > 0 (< 0)$ and $c_j > 1$ ($<1$) for $j = 1,2$, then the joint distribution of $X_1$ and $X_2$ is multivariate IHR (DHR).

We therefore need consider only the cases $\alpha > 0$, $c_j < 1$ or $\alpha < 0$, $c_j > 1$. Consider first the case $\alpha > 0$, $c_j < 1$.

In this case $H_j(g) \gtrless 0$ according as

$$c_j g \log(g/\beta) \gtrless (1-c_j)(g-1)(g-2)$$

or

$$\log \beta \lesseqgtr \log g - \frac{(1-c_j)(g-1)(g-2)}{c_j g} = K_j(g), \text{ say .} \tag{9}$$

Now

$$K_j'(g) = dK_j(g)/dg = - b_j + g^{-1} + 2b_j g^{-2}$$

with $b_j = c_j^{-1} - 1 > 0$. This is a decreasing function of $g$.

We further note that since $g = \beta \exp(x_j^{c_j})$ we must have $g \geq \beta > 2$ (since $1 > \alpha > 0$).

If $K_j(\beta) < \log \beta$ and $K'_j(\beta) < 0$, then $K_j(\beta) < \log \beta$ for all $g > \beta$, implying $H_j(g) < 0$ so that $h_{\underset{\sim}{X}}(\underset{\sim}{x})_j$ would be an always decreasing function of $x_j$.

Noting that for $g > 2$, $K_j(g) < \log 2$ and also that for $c_j \leq \frac{1}{2}$

$$K_j'(2) = 1 - \tfrac{1}{2}c_j^{-1} \leq 0$$

we see that if $\beta > 2$ and $c_j \leq \frac{1}{2}$ then $h_X(x)_j$ is a decreasing function of $x_j$. Since the condition $\beta > 2$ is satisfied for all $x_{3-j}$ and $\alpha$ with with $0 < \alpha < 1$, we have the result:

*The joint distribution of $X_1$ and $X_2$ is vector multivariate DHR if $0 < \alpha < 1$ and $c_1, c_2 \leq \frac{1}{2}$.*

This may still be the case for larger values of $c_j (\frac{1}{2} < c_j < 1)$, provided

(10) $$\max_{g>\beta} K_j(g) < \min_{x_{3-j}} \log \beta \qquad (j=1,2) .$$

However, we now need limits on the values of $\alpha$. By direct calculation we find that if $\frac{1}{2} < c_j < 1$

(11) $$\max_{g\geq 2} K_j(g) = K_j(g_0) = \log \left\{\frac{1+(1+8b_j^2)^{\frac{1}{2}}}{2b_j}\right\} - 1+3b_j - 8b_j^2/\{1+(1+8b_j^2)^{\frac{1}{2}}\}$$

where $g_0 = \{1+(1+8b_j^2)^{\frac{1}{2}}\}/(2b_j)$.

Also $\min_{x_{3-j}} \log \beta = \log(1+\alpha^{-1})$ .

So condition (10) is satisfied if either

(a) $g_0 > 1+\alpha^{-1}$ and $K_j(g_0) < \log(1+\alpha^{-1})$

or (b) $g_0 < 1+\alpha^{-1}$.

(For the additional condition $K_j(1+\alpha^{-1}) < \log(1+\alpha^{-1})$ is always satisfied because $1 + \alpha^{-1} > 2$.)

Combining (a) and (b) we find that $h_{\underset{\sim}{\chi}}(\underset{\sim}{x})_j$ is a decreasing function of $x_j$, whatever the value of $x_{3-j}$, if

$$0 < \alpha < \max[(g_0-1)^{-1}, \{\exp(K_j(g_0))-1\}^{-1}] .$$

The table below gives values of $g_0$, $K_j(g_0)$ and the upper limit for $\alpha$ for a few values of $c_j$.

| $c_j$ | $b_j$ | $g_0$ | $K_j(g_0)$ | Upper limit for $\alpha$ |
|---|---|---|---|---|
| 0.6 | 2/3 | 2.3508 | 0.720 | 0.948 |
| 0.7 | 3/7 | 3.0000 | 0.813 | 0.797 |
| 0.8 | 1/4 | 4.4495 | 1.018 | 0.566 |
| 0.9 | 1/9 | 9.2170 | 1.506 | 0.285 |

The upper limit for $\alpha$ in each case is, in fact $\{\exp(K_j(g_0))-1\}^{-1}$. We have not established analytically that

$$(g_0-1)^{-1} < \{\exp(K_j(g_0)-1\}^{-1} .$$

We will treat the case $0 > \alpha > -1$ and $c_j > 1$ more briefly. In this case $0 > 1 - |\alpha|^{-1} \geq \beta \geq g$ and we find that

$$H_j(g) \gtrless 0 \text{ according as } \log|\beta| \gtrless K_j^*(g)$$

where

$$K_j^*(g) = \log |g| - b_j^*(|g|+1)(|g|+2)/|g|$$

with $b_j^* = 1 - c_j^{*-1}$, so that $0 < b_j^* < 1$.

This case differs from that just discussed in that $|\beta|$ can take values as near to zero as desired (previously $\beta > 2$). This means that for any (negative) value of $g$ (and any $b_j^*$) we can find a value of $\alpha$ $(-1 < \alpha < 0)$ such that $K_j^*(g) > \log|\beta|$ for sufficiently large $x_{3-j}$. Hence for no value of $b_j^*$ (and so of $c_j$) we can say that $h_{\underset{\sim}{X}}(\underset{\sim}{x})_j$ is an always increasing function of $x_j$ for all $0 > \alpha > -1$, whatever the value of $x_{3-j}$. (In the previous case, we could ensure that $h_{\underset{\sim}{X}}(\underset{\sim}{x})_j$ is a decreasing function of $x_j$ by taking $c_j \leq \frac{1}{2}$.)

On the other hand

$$K_j^*(\beta) < \log |\beta|$$

so that for any specified $\alpha$ and $x_{3-j}$, $h_{\underset{\sim}{X}}(\underset{\sim}{x})_j$ is an increasing function of $x_j$ for sufficiently small $x_j$. It will be an increasing function of $x_j$ for all $x_j$ if

$$\log |\beta| > \max_{g<\beta} K_j^*(g) = \begin{cases} K_j^*(g_0^*) & \text{if } |\beta| < g_0^* \\ K_j^*(\beta) & \text{if } |\beta| \geq g_0^* \end{cases}$$

where $g_0^* = \{1+(1+8b_j^{*2})^{\frac{1}{2}}\}/(2b_j^*)$.

It will be an always increasing function of $x_j$ (for given $\alpha$) for all $x_{3-j}$ if

$$\log(|\alpha|^{-1} - 1) > \max_{g<1-|\alpha|^{-1}} K_j^*(g) \quad (12)$$

Inequality (12) gives lower limits for $\alpha$.

Further examples and comparisons with other definitions of multivariate hazard rate, and of multivariate IHR and DHR, will be given in a second paper.

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