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SOME REMARKS ON THE EQUIVALENCE OF GAUSSIAN PROCESSES

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A. GUALTIEROTTI AND S. CAMBANIS[1]

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1. INTRODUCTION. Let $X = \{X_t, t\epsilon T\}$ and $Y = \{Y_t, t\epsilon T\}$ be real, zero-mean Gaussian processes with respective covariances R_X and R_Y, defined on a probability space (Ω,A,P), where T is an arbitrary index set. Denote by P_X and P_Y the probabilities induced on $(R^T,C(R^T))$ by P, X and Y respectively, where $C(R^T)$ is the σ-algebra generated by the cylinder sets of the set R^T of real functions on T. $L_2(X)$ and $L_2(Y)$ will denote the subspaces of $L_2(\Omega,A,P)$ generated by X and Y respectively, and H(X) and H(Y) the corresponding reproducing kernel Hilbert spaces (RKHS's). The associated canonical isometries will be denoted by U_X and U_Y respectively ($U_X X_t = R_X(t,\cdot)$, $t\epsilon T$, and similarly for U_Y). We say that the processes X and Y are equivalent, or that the probabilities P_X and P_Y are equivalent, $P_X \sim P_Y$, if P_X and P_Y are mutually absolutely continuous. The following properties are well known:

(i) $P_X \sim P_Y$ if and only if $Y_t = FX_t$, $t\epsilon T$, where F is an equivalence operator from $L_2(X)$ to $L_2(Y)$ (i.e., F has bounded inverse and $I-F^*F$ is Hilbert-Schmidt) [6]; or equivalently if and only if $Y_t = X_t - AX_t$, $t\epsilon T$, where A is a Hilbert-Schmidt operator in $L_2(X)$ and I-A has bounded inverse, and the equality is in law, i.e. $P_Y = P_{(I-A)X}$ [9].

(ii) If $P_X \sim P_Y$ then $sH(X) = sH(Y)$, where s indicates that what follows is considered as a set and not as a space [7, p. 181].

The first question considered in this note is the converse of (ii), i.e., if X and Y have the same RKHS's, under what additional condition on the RKHS's are they equivalent? The answer is given in Propositions 1 and 2 and results in a characterization of the norms in a RKHS corresponding to equivalent Gaussian processes.

The fact, mentioned in (i), that all Gaussian processes equivalent to X_t

are of the form $X_t - AX_t$, with A a Hilbert-Schmidt operator in $L_2(X)$, raises the problem of expressing AX_t in a more explicit way in terms of the process X. This is done in Propositions 4, 5 and 6.

2. THE RKHS OF EQUIVALENT GAUSSIAN PROCESSES. Here, and in the next section, we adopt the notation of the introduction.

PROPOSITION 1. $P_X \sim P_Y$ if and only if $sH(X) = sH(Y)$ and (a) the identity J on $sH(X) = sH(Y)$ is an equivalence operator from $H(X)$ to $H(Y)$, or (b) for every f in $sH(X)$

$$||f||^2_{H(Y)} = ||f||^2_{H(X)} + \langle f \otimes f, \Lambda \rangle_{H(X)\otimes H(X)}$$

for some $\Lambda \in H(X)\otimes H(X)$ which is symmetric and such that $-R_X << \Lambda$.

PROOF. Suppose first that $sH(X) = sH(Y)$ and J is an equivalence. For every $\xi \in L_2(Y)$ we have $<\xi, Y_t>_{L_2(Y)} = (U_Y\xi)(t) = (J^{-1}U_Y\xi)(t) =$ $= <U^*_X J^{-1} U_Y\xi, X_t>_{L_2(x)}$. Let $F^* = U^*_X J^{-1} U_Y$. Since J is an equivalence, so is J^{-1}, and since U_X, U_Y are unitary, F^* is an equivalence and so is F. It now follows from $<\xi, Y_t>_{L_2(Y)} = <F^*\xi, X_t>_{L_2(X)} = <\xi, FX_t>_{L_2(Y)}$, for all ξ in $L_2(Y)$, that $Y_t = FX_t$. Thus $P_X \sim P_Y$.

Conversely, suppose that $P_X \sim P_Y$. Then $Y_t = FX_t$ where F is an equivalence operator from $L_2(X)$ to $L_2(Y)$. For every f in $H(Y)$ we have $f(t) = <U^*_Y f, Y_t>_{L_2(Y)} = <F^* U^*_Y f, X_t>_{L_2(X)} = [U_X F^* U^*_Y f](t) = [JU_X F^* U^*_Y f](t)$. Thus $JU_X F^* U^*_Y = I_{H(Y)}$ and $J = U_Y (F^*)^{-1} U^*_X$. Since F is an equivalence, so is $(F^*)^{-1}$ and since U_X, U_Y are unitary, J is an equivalence.

Finally, (b) is equivalent to (a) as it follows from Property (i) and the fact that Hilbert-Schmidt operators on RKHS's have kernels in the direct product of the considered RKHS's [1].

The characterization of Proposition 1 is particularly useful if the elements in the common RKHS can be obtained in the way described in the following Proposition.

PROPOSITION 2. Suppose there exists a pair (H,L), where H is a Hilbert space and L a unitary map from H to $H(X)$. Then $P_X \sim P_Y$ if and only if $sH(X) = sH(Y)$ and for all h in H,

$$||Lh||^2_{H(Y)} = ||Lh||^2_{H(X)} + \langle LKh, Lh\rangle_{H(X)} \quad , \tag{1}$$

where K is a self-adjoint, Hilbert-Schmidt operator on H such that $-1 < \sigma(K)$.

REMARK 1. Condition (1) is equivalent to

(a) $||Lh||^2_{H(Y)} = ||h||^2_H + \langle Kh,h\rangle_H$, or

(b) $\langle Lh,Lh'\rangle_{H(Y)} = \langle Lh,LH\rangle_{H(X)} + \langle LKh,Lh'\rangle_{H(X)} = \langle h,h'\rangle_H + \langle Kh,h'\rangle_H$.

PROOF OF PROPOSITION 2. Suppose first that $P_X \sim P_Y$. Then J is an equivalence and J can be decomposed as $J = UW$, with $W = (J^*J)^{\frac{1}{2}}$ and U unitary. ($W: H(X) \to H(X)$ and $U: H(X) \to H(Y)$). Since W is onto, for every h in H there is a g in H such that $Lh = WLg$. Thus every h in H can be obtained as L^*WLg, for some g in H. Set $S = L^*WL$. Then $J = ULSL^*$ and thus for h in H, $||JLh||_{H(Y)} = ||ULSL^*Lh||_{H(Y)} = ||LSh||_{H(X)} = ||Sh||_H$. Now it follows from $S = L^*U^*JL$ that S is an equivalence, since it is obtained from an equivalence operator J by "unitary multiplication." Thus $I_H - S^*S$ is equal to a self-adjoint, Hilbert-Schmidt operator $-K$ that does not have -1 among its eigenvalues. Hence, as

$$||JLh||^2_{H(Y)} = ||Sh||^2_H = \langle (I+K)h,h\rangle_H = ||h||^2_H + \langle Kh,h\rangle_H \quad ,$$

and (1) follows from (a) of Remark 1.

Conversely, suppose that $sH(X) = sH(Y)$ and (1) holds. Define a unitary operator $T: H \to H(Y)$ by $T\{(I_H+K)^{\frac{1}{2}}h\} = JLh$. This definition makes sense since T is obviously onto and by (1), $||T\{(I_H+K)^{\frac{1}{2}}h\}||^2_{H(Y)} = ||JLh||^2_{H(Y)} = ||(I_H+K)^{\frac{1}{2}}h||^2_H$. But then $J = T(I_H+K)^{\frac{1}{2}}L^*$, where T and L^* are unitary and $(I_H+K)^{\frac{1}{2}}$ is an equivalence. Hence, J is an equivalence and $P_X \sim P_Y$.

REMARK 2. The existence of the assumed pair (H,L) in Proposition 2 is not as restrictive as it appears. In fact, whenever $H(X)$ (or equivalently $L_2(X)$) is separable, this assumption is satisfied and one can take H to be an L_2 space. This follows from Theorem 2 of [3]. Indeed, if $H(X)$ is separable, then, for an arbitrary measure space $(E,\mathcal{E},\mu)$ such that $L_2(\mu)$ is separable and infinite dimensional, we have for all t in T, $X_t = \int_E \phi_t(u)dZ(u)$, where $\phi_t \in L_2(\mu)$ and Z is an orthogonal random measure on $\mathcal{E}$ such that $L_2(Z) = L_2(X)$. This implies that there is a unitary map $A: L_2(\mu) \to L_2(X)$ such that $A\phi_t = X_t$, $t \in T$, and clearly $L: L_2(\mu) \to H(X)$ defined by $L = U_X A$ is unitary. The Hilbert space H is clearly non-unique. However in some specific cases, H can be chosen in a natural way. In fact, most of the known RKHS's are obtained in this way. We list some examples below.

EXAMPLES. 1) Let X have orthogonal increments on $[0,1]$ and start almost surely at the origin, with $R_X(s,t) = F(s \wedge t)$, F continuous. Then $H(X) = \{\int_0^t \phi(u)dF(u),\ \phi \in L_2(dF)\}$, $\langle \int_0^{\cdot} \phi dF, \int_0^{\cdot} \psi dF \rangle_{H(X)} = \langle \phi,\psi \rangle_{L_2(dF)}$, and $L: L_2(dF) \to H(X)$ defined by $[L\phi](t) = \int_0^t \phi dF$ is an isometry. This example includes the Wiener process $(F(u) = u)$.

2) Let X have the covariance $R_X(s,t) = F(s \wedge t)\, G(s \vee t)$, where F and G are continuous with bounded variation on $[0,1]$, F is strictly positive, except at the origin, and G is strictly positive. Suppose further that $H(u) = F(u)/G(u)$ is strictly increasing. Then $H(X) = \{G(t)\int_0^t \phi(u)dH(u),\ \phi \in L_2(dH)\}$ and $\langle G(\cdot)\int_0^{\cdot} \phi dH,\ G(\cdot)\int_0^{\cdot} \psi dH \rangle_{H(X)} = \langle \phi,\psi \rangle_{L_2[0,1]}$. Thus $L: L_2(dH) \to H(X)$ defined by $[L\phi](t) = G(t) \int_0^t \phi dH$ is unitary.

3) Let X be a linear operation on a stationary process with spectral measure μ. Then $R_X(s,t) = \int_{-\infty}^{\infty} \phi_s(\lambda)\overline{\phi_t}(\lambda)d\mu(\lambda)$ and if H is the closure in $L_2(\mu)$

of the linear span of $\{\phi_t, t\in T\}$ then $H(X) = \{\int_{-\infty}^{\infty} \phi_t\overline{\phi}d\mu,\ \phi\in H\}$ and $<\int\phi_.\overline{\phi}d\mu\ \ \int\phi_.\overline{\psi}d\mu>_{H(X)} =$ $= <\phi,\psi>_{L_2(\mu)}$. Thus $L : H \to H(X)$ defined by $[L\phi](t) = \int_{-\infty}^{\infty} \phi_t\overline{\phi}d\mu$ is unitary. This includes the case where X is stationary ($\phi_t(\lambda) = e^{it\lambda}$) and then $H = L_2(\mu)$ if $T = R$.

4) Let X be a linear operation on a harmonizable process with two-dimensional spectral measure r. Then $R_X(s,t) = \iint_{-\infty}^{\infty} \phi_s(u)\overline{\phi}_t(v)dr(u,v)$ and if H is the closure in the Hilbert space $\Lambda_2(r)$ [5] of the linear span of $\{\phi_t, t\in T\}$ then

$H(X) = \{\iint_{-\infty}^{\infty} \phi_t(u)\overline{\phi}(v)\ dr(u,v),\ \ \phi\in H\}$ and

$\left\langle\iint\phi_.\overline{\phi}dr, \iint\phi_.\overline{\psi}dr\right\rangle_{H(X)} = \left\langle\phi,\psi\right\rangle_{\Lambda_2(r)} = \iint_{-\infty}^{\infty} \phi(u)\overline{\psi}(v)dr(u,v)$. Thus $L: H \to H(X)$ defined by $[L\phi](t) = \iint_{-\infty}^{\infty} \phi_t(u)\overline{\phi}(v)dr(u,v)$ is unitary. This includes the case where X is harmonizable $(\phi_t(u) = e^{itu})$ and then $H = \Lambda_2(r)$ if $T = R$.

REMARK 3. The existence of the pair (H,L) as described in Proposition 2 is easily seen to be equivalent to the existence of a representation of the covariance R_X of the form $R_X(s,t) = <\phi_s,\phi_t>_H$ where $\{\phi_t, t\in T\} \subseteq H$.

Proposition 2 can be also expressed in terms of covariances and it then contains as particular cases the results of [10] and [8].

PROPOSITION 3. Suppose there exists a pair (H,L), where H is a Hilbert space and L a unitary map from H to $H(X)$. Let $\phi_t = L^*R_X(t,\cdot)$ or $R_X(s,t) = <\phi_s,\phi_t>_H$ (see Remark 3). Then $P_X \sim P_Y$ if and only if

(2) $$R_Y(s,t) = R_X(s,t) - <M\phi_s,\phi_t>_H$$

where M is self-adjoint, Hilbert-Schmidt and such that $\sigma(M) < 1$.

PROOF. Assume first that $P_X \sim P_Y$. Then $Y_t = FX_t$ where F is an equivalence operator from $L_2(X)$ to $L_2(Y)$. It follows that $Y_t = FU_X^*L\phi_t$ and $R_Y(s,t) = <L^*U_XF^*FU_X^*L\phi_s,\phi_t>_H$. Thus (2) is valid with $M = L^*U_X(I_{L_2(X)} - F^*F)U_X^*L$. Since $I_{L_2(X)} - F^*F$ is self-adjoint, Hilbert-Schmidt with $\sigma(I_{L_2(X)} - F^*F) < 1$, and U_X, L are unitary, it follows that M is self-adjoint Hilbert-Schmidt with

$\sigma(M) < 1$.

Conversely, assume that (2) is valid. Then $\phi_t = L^*U_X X_t$ yields $\langle M\phi_s,\phi_t\rangle_H = \langle U_X^* LML^* U_X X_s, X_t\rangle_{L_2(X)}$ and (2) is written $R_Y(s,t) =$ $= \langle (I_{L_2(X)} - U_X^* LML^* U_X)X_s, X_t\rangle_{L_2(X)}$. Since M is self-adjoint, Hilbert-Schmidt with $\sigma(M) < 1$, and U_X, L are unitary, it follows that $U_X^* LML^* U_X$ is self-adjoint, Hilbert-Schmidt. Also $I_{L_2(X)} - U_X^* LML^* U_X = U_X^* L(I_H - M)L^* U_X > 0$ and hence its square root F_0 is an equivalence. We then have $\langle Y_s, Y_t\rangle_{L_2(Y)} = R_Y(s,t) =$ $= \langle F_0 X_s, F_0 X_t\rangle_{L_2(X)}$ which implies that the map $F_0 X_t \to Y_t$ extends to a unitary operator U from $L_2(X)$ to $L_2(Y)$ and then $Y_t = FX_t$ where $F = UF_0$ is an equivalence. Thus $P_X \sim P_Y$.

REMARK 4. The relationship between the operators F, K and M of Property (i) and Propositions 2 and 3 respectively is as follows

$$F = U[U_X^* L(I_H + K)^{-1} L^* U_X]^{\frac{1}{2}} = U[U_X^* L(I_H - M)L^* U_X]^{\frac{1}{2}}$$

$$K = L^* U_X[(F^*F)^{-1} - I_{L_2(X)}]U_X^* L = (I_H - M)^{-1} - I_H$$

$$M = L^* U_X(I_{L_2(X)} - F^*F)U_X^* L = I_H - (I_H + K)^{-1} .$$

We also have

$$R_Y(s,t) = \langle \psi_s, \psi_t\rangle_H \quad \text{where} \quad \psi_t = (I_H - M)\phi_t .$$

PROOF. The expressions relating F and M are derived in the proof of Proposition 3. It then suffices to derive the relationship between K and M. Since $sH(X) = sH(Y)$ we have $R_Y(t,\cdot) \in H(X)$. Let $\psi_t = L^* R_Y(t,\cdot)$. Then $R_Y(s,t) = \langle \psi_s, \psi_t\rangle_H$. By Remark 1 we obtain $R_X(s,t) = \langle R_X(s,\cdot), R_Y(t,\cdot)\rangle_{H(Y)} =$ $= \langle JL\phi_s, JL\psi_t\rangle_{H(Y)} = \langle \phi_s, (I_H + K)\psi_t\rangle_H$. Since $R_X(s,t) = \langle \phi_s, \phi_t\rangle_H$ we have $\langle \phi_s, \phi_t\rangle_H = \langle \phi_s, (I_H + K)\psi_t\rangle_H$. Since $\{R_X(t,\cdot), t\in T\}$ is complete in $H(X)$ and L is unitary, $\{\phi_t, t\in T\}$ is complete in H and hence $\phi_t = (I_H + K)\psi_t$.

We now have $R_Y(s,t) = \langle R_Y(s,\cdot), R_X(t,\cdot)\rangle_{H(X)} = \langle L\psi_s, L\phi_t\rangle_{H(X)} = \langle \psi_s, \phi_t\rangle_H =$ $\langle (I_H + K)^{-1}\phi_s, \phi_t\rangle$ and since $R_X(s,t) = \langle \phi_s, \phi_t\rangle_H$ it follows from (2) by inspection that $M = I_H - (I_H + K)^{-1}$. Hence $K = (I_H - M)^{-1} - I_H$ and also $\psi_t = (I_H + K)^{-1}\phi_t = (I_H - M)\phi_t$.

3. REPRESENTATION OF EQUIVALENT GAUSSIAN PROCESSES. When a pair (H,L) as described in Propositions 2 and 3 exists, then one can obtain explicit representations of the process AX_t in Property (i). Here it is more appropriate to consider the unitary map $V: H \to L_2(X)$ related to L by $V = U_X^* L$.

PROPOSITION 4. Suppose that there exists a pair (H,V), where H is a Hilbert space and V a unitary map from H to $L_2(X)$. If A is a Hilbert-Schmidt operator in $L_2(X)$, then $AX_t = V\overline{A}h_t$, where $\overline{A}$ is a Hilbert-Schmidt operator in H and $h_t = V^* X_t$.

PROOF. If $\overline{A} = V^* AV$, then $\overline{A}$ is Hilbert-Schmidt in H and $AX_t = AVh_t = VV^* AVh_t = V\overline{A}h_t$.

EXAMPLES. 5) Let X be as in Example 1. Then $V: L_2(dF) \to L_2(X)$ is defined by $V\phi = \int_0^1 \phi(u)dX_u$. Consequently $h_t = I_t$, the indicator function of [0,t] and $AX_t = \int_0^1 [\overline{A}I_t](u)dX_u$. Since a Hilbert-Schmidt operator in $L_2(dF)$ is of integral type with kernel $\alpha(u,v)$ in $L_2(dF \times dF)$ we finally have

$$AX_t = \int_0^1 \int_0^t \alpha(u,v)dF(v)dX_u \quad .$$

This result is obtained for the Wiener process in [9].

6) Consider the case where the covariance of X has the representation $R_X(s,t) = \int_E \phi_s(u)\overline{\phi}_t(u)d\mu(u)$ with $(E,\mathcal{E},\mu)$ a finite (for simplicity) measure space and $\phi_t \in L_2(\mu)$. This includes both Examples 2 and 3. Then there is an orthogonal random measure Z on E such that $X_t = \int_E \phi_t(u)dZ(u)$ [5]. Let H be the closure in $L_2(\mu)$ of the linear span of $\{\phi_t, t \in T\}$. Then $V: H \to L_2(X)$ is defined by $V\phi = \int_E \phi(u)dZ(u)$. Consequently $h_t = \phi_t$ and $AX_t = \int_E [\overline{A}\phi_t](u)dZ(u)$. As in Example 5, $\overline{A}$ is of integral type with kernel $\alpha(u,v)$ in $L_2(\mu\times\mu)$ and finally we have

$$AX_t = \int_E\int_E \alpha(u,v)\overline{\phi}_t(v)d\mu(v)dZ(u) \quad .$$

In the case of Example 2, $H = L_2(dH)$ and $AX_t = G(t)\int_0^1\int_0^t \alpha(u,v)dH(v)dZ(u)$.

7) Consider the case where the covariance of X has the representation $R_X(s,t) = \iint_{EE} \phi_s(u)\overline{\phi}_t(v)dr(u,v)$ with $(E,\mathcal{E})$ a measurable space, r a two-dimensional spectral measure on $\mathcal{E}\times\mathcal{E}$ and ϕ_t in $\Lambda_2(r)$. Then there is a random measure Z on $\mathcal{E}$ such that $X_t = \int_E \phi_t(u)dZ(u)$ [5]. Let H be the closure in $\Lambda_2(r)$ of the linear span of $\{\phi_t, t\in T\}$. Then $V: H \to L_2(X)$ is defined by $V\phi = \int_E \phi(u)dZ(u)$. Consequently $h_t = \phi_t$ and $AX_t = \int_T[\overline{A}\phi_t](u)dZ(u)$, where $\overline{A}$ is a Hilbert-Schmidt operator in $\Lambda_2(r)$. However, no kernel representation of $\overline{A}$ seems to be available because Λ_2 is a more complicated space than L_2. Nevertheless, as follows from (i) of the Lemma at the end of this section, if E is an interval, $\overline{A}$ is the limit in the operator norm of a sequence of Hilbert-Schmidt operators $\{A_n\}_{n=1}^\infty$ in $L_2(\mu)$ with kernels $\{\alpha_n\}_{n=1}^\infty$ and we thus have

$$AX_t = \lim_{n\to\infty} \int_E\{\iint_{EE} \phi_t(v)\overline{\alpha}_n(w,u)dr(v,w)\}dZ(u)$$

where the limit is in $L_2(P)$. Consider now the important particular case where there is a measurable process $\{Z_u, u\in E\}$ and a measure μ on $\mathcal{E}$ equivalent to the Lebesgue measure such that $\int_E R_Z(u,u)d\mu(u) < \infty$ and for every $B\in\mathcal{E}$, $Z(B) = \int_B Z_u d\mu(u)$. Then $X_t = \int_E \phi_t(u)Z_u d\mu(u)$ and

$$AX_t = \lim_{n\to\infty} \int_E g_t^{(n)}(u)Z_u d\mu(u)$$

where $g_t^{(n)}(u) = \iint_{EE}\phi_t(v)\overline{\alpha}_n(w,u)R_Z(v,w)d\mu(v)\, d\mu(w)$ is in $L_2(\mu)$, the integral is defined almost surely, i.e., on the paths of Z, and the limit is in $L_2(P)$. As particular cases of this example we obtain the following representations.

PROPOSITION 5. Let X be mean square continuous, T an interval and A a Hilbert-Schmidt operator on $L_2(X)$. Then

$$AX_t = \int_T g_t(u)X_u d\mu(u) = \lim_{n\to\infty} \int_T g_t^{(n)}(u)X_u d\mu(u)$$

where the measure μ on the Borel sets of T satisfies (ii) of the Lemma, $g_t\in\Lambda_2(R_X\cdot\mu\times\mu)$, the first integral is defined in quadratic mean, $g_t^{(n)}\in L_2(\mu)$, the second integral is defined almost surely, i.e., on the paths of X, and the limit is in $L_2(P)$.

PROOF. X has a measurable modification which is henceforth considered. There exist finite measures μ, equivalent to the Lebesgue measure, and such that $\int_T R_X(t,t)d\mu(t) < \infty$ [2]. Let $dr(u,v) = R_X(u,v)d\mu(u)d\mu(v)$. Since X is mean square continuous, it has a representation $X_t = \int_T \phi_t(u)X_u d\mu(u)$ with $\{\phi_t, t\in T\} \subset \Lambda_2(r)$ [2]. The result then follows from the last case considered in Example 7.

PROPOSITION 6. Let X be mean square continuous on $[0,1]$, $X_0 = 0$ a.s., and R_X of bounded variation on $[0,1] \times [0,1]$. Let A be a Hilbert-Schmidt operator on $L_2(X)$. Then

$$AX_t = \int_0^1 g_t(u)dX_u = \lim_{n\to\infty} \int g_t^{(n)}(u)dX_u$$

where $g_t\in\Lambda_2(d^2R_X)$, μ corresponds to d^2R_X as in (i) of Lemma, $g_t^{(n)}\in L_2(\mu)$ are of the form $g_t^{(n)}(u) = \int_0^t\int_0^1 \overline{\alpha}_n(w,u)d^2R_X(v,w)$, and $\alpha_n\in L_2(\mu\times\mu)$.

PROOF. The proof is obvious from Example 7 and the observation that $X_t = \int_0^1 I_t(u)dX_u$, I_t the characteristic function of the interval $[0,t]$, i.e., $\phi_t = I_t$.

LEMMA. Let E be an interval, $\mathcal{E}$ its Borel sets, r a finite, two-dimensional spectral measure on $\mathcal{E}\times\mathcal{E}$, and K a Hilbert-Schmidt operator on $\Lambda_2(r)$. If

(i) μ is the finite measure defined on $\mathcal{E}$ by $\mu(B) = |r|(E\times B)$ for all $B\in\mathcal{E}$, or if

(ii) $dr(u,v) = R(u,v)d\mu(u)d\mu(v)$, where R is a covariance and μ a finite measure on $\mathcal{E}$, equivalent to the Lebesgue measure and such that $\int_E R(u,u)d\mu(u) < \infty$,

then $L_2(\mu) \subset \Lambda_2(r)$ and there is a sequence of Hilbert-Schmidt operators $\{K_n\}_{n=1}^\infty$ on $L_2(\mu)$ with kernels $\{k_n\}_{n=1}^\infty$, that are defined from $\Lambda_2(r)$ to $L_2(\mu)$ by $[K_n f](u) = \langle f(\cdot),k_n(\cdot,u)\rangle_{\Lambda_2(r)}$ and are such that $K_n \to K$ in the operator norm in $\Lambda_2(r)$.

PROOF. Both (i) and (ii) imply that $L_2(\mu) \subset \Lambda_2(r)$ and that there is a sequence

$\{f_n\}_{n=1}^{\infty}$ in $L_2(\mu)$ which is orthonormal and complete in $\Lambda_2(r)$. For (ii) this is shown in [2] and for (i) it is shown as Theorem 2 of [4].

In the sequel $<\cdot,\cdot>$ and $||\cdot||$ denote inner product and norm in $\Lambda_2(r)$. Since K is Hilbert-Schmidt we have $\sum_{n,m=1}^{\infty}|<Kf_n,f_m>|^2 < \infty$. For every $f \in \Lambda_2(r)$ we have $f = \sum_{n=1}^{\infty}<f,f_n>f_n$ and thus $Kf = \sum_{n=1}^{\infty}<f,f_n>Kf_n = \sum_{m=1}^{\infty}\{\sum_{n=1}^{\infty}<Kf_n,f_m><f,f_n>\}f_m$. Define $k_N(u,v) = \sum_{n,m=1}^{N}<Kf_n,f_m>f_n(u)f_m(v)$. Since k_N is in $L_2(\mu\times\mu)$, it defines a (finite rank) Hilbert-Schmidt operator K_N on $L_2(\mu)$. K_N is also defined from $\Lambda_2(r)$ to $L_2(\mu)$ by

$$[K_Nf](u) = <f(\cdot),k_N(\cdot,u)> = \sum_{n,m=1}^{N} <Kf_n,f_m><f,f_n>f_m .$$

Then

$$||Kf-K_Nf||^2 \le ||f||^2 \sum_{n,m=N+1}^{\infty} |< Kf_n,f_m>|^2$$

which implies that $K_N \to K$ in the operator norm in $\Lambda_2(r)$.

REFERENCES

1. N. Aronszajn, Theory of reproducing kernels, Trans. Amer. Math. Soc. 68 (1950), 337-404.
2. S. Cambanis, Representation of stochastic processes of second order and linear operations, J. Math. Anal. Appl. 41 (1973), 603-620.
3. S. Cambanis, The measurability of a stochastic process of second order and its linear space, Institute of Statistics Mimeo Series No. 832, University of North Carolina at Chapel Hill, 1972.
4. S. Cambanis and B. Liu, On harmonizable stochastic processes, Information and Control 17 (1970), 183-202.
5. H. Cramèr, A contribution to the theory of stochastic processes, Proc. Second Berkeley Symp. Math. Statist. Prob. 2 (1951), 329-339.
6. J. Feldman, Equivalence and perpendicularity of Gaussian processes, Pacific J. Math. 4 (1958), 699-708.
7. J. Neveu, Processus Aléatoires Gaussiens, Les Presses de l'Université de Montréal, 1968.
8. W.J. Park, On the equivalence of Gaussian processes with factorable covariance functions, Proc. Amer. Math. Soc. 32 (1972), 275-279.
9. H. Sato, On the equivalence of Gaussian measures, J. Math. Soc. Japan 19 (1967), 159-172.
10. L.A. Shepp, Radon-Nikodym derivatives of Gaussian measures, Ann. Math. Statist. 37 (1966), 321-354.

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