*Currently on leave at Cambridge University, England

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LOCATING THE MAXIMUM OF A STATIONARY NORMAL PROCESS†

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1. INTRODUCTION.

Considerable attention has been paid in recent years to questions surrounding the maximum value $M(0,T)$ of a stationary normal process $\xi(t)$ in the time interval $0 \le t \le T$. In particular the double exponential asymptotic form of $\Pr\{a_T(M(0,T) - b_T) \le x\}$ (as $T \to \infty$) has been obtained under a variety of conditions, (e.g. (Cramér 1965) (Pickands 1969) (Berman 1971)). (The exact distribution of $M(0,T)$ for *finite* T is unobtainable in general).

A problem suggested to me by D.R. Cox concerns the position $L = L_T$, say, of that point of $[0,T]$ where this maximum is attained, and, in particular, whether the distribution of L is uniform over this range. In this paper we show that the distribution of L is not, in general, uniform for finite values of T (and indeed will "usually" have jumps at $0,T$, but that it is asymptotically uniform as $T \to \infty$.

In section 2 we consider some aspects of the corresponding problem in discrete time, which give some feeling for the main results. Section 3 contains a discussion of the presence of jumps, and the symmetry of the distri-

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bution of L_T for finite T. In section 4 asymptotic uniformity of L_T is shown. This relies on a bivariate form of the asymptotic distribution of M(0,T) which - along with further aspects of this work - will be discussed in (Leadbetter 1973).

2. DISCRETE TIME CASE.

Let $\{\xi_n: n = 0, \pm 1 \ldots\}$ be a discete parameter stochastic process and $M_n = \max(\xi_1\ \xi_2 \ldots \xi_n)$. Let L_n be the "position of the maximum" i.e. a random variable taking the value k $(1 \le k \le n)$ if $\xi_k = M_n$. (To avoid ties, it will be convenient to assume that the joint distribution of $\xi_1 \ldots \xi_n$ is absolutely continuous.)

If the ξ_i are independent and identically distributed, it is immediate that L_n is uniformly distributed on the itegers $1 \ldots n$, $\Pr(L_n=k) = \frac{1}{n}$. That is, any of the ξ_k is equally likely to be the largest of $\xi_1 \ldots \xi_n$. A small opinion survey shows that it would be reasonable to conjecture that the same would be true if the ξ_i were not independent, but a stationary sequence.

This conjecture is, in fact, false. For example suppose ξ_j are all standard normal random variables such that $\text{cov}(\xi_i, \xi_j) = \rho$ for $|i-j| = 1$ and zero otherwise, where $|\rho| \le \frac{1}{2}$. (This may be achieved by forming the ξ's as "two-term moving averages" of a sequence of independent random variables.) Consider the simplest case, $n = 3$. Then

$$\Pr(L_3 = 1) = \Pr(\xi_1 \ge \xi_2,\ \xi_1 \ge \xi_3)$$
$$= \Pr(\eta_1 \ge 0,\ \eta_2 \ge 0)$$

where $\eta_1 = (\xi_1-\xi_2)/[\text{var}(\xi_1-\xi_2)]^{\frac{1}{2}}$, $\eta_2 = (\xi_1-\xi_3)/[\text{var}(\xi_1-\xi_3)]^{\frac{1}{2}}$ are standard normal random variables with correlation $\lambda_1 = \frac{1}{2}(1-\rho)^{-\frac{1}{2}}$. It is well known (and readily checked) that the probability that two standard normal random variables, with correlation λ, will both be positive is $\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\lambda$ and thus

$$\Pr(L_3 = 1) = \tfrac{1}{4} + \frac{1}{2\pi}\sin^{-1}\lambda_1 \; .$$

Similarly $\Pr(L_3 = 3) = \Pr(L_3 = 1)$, whereas

$$\Pr(L_3 = 2) = \tfrac{1}{4} + \frac{1}{2\pi}\sin^{-1}\lambda_2$$

where $\lambda_2 = (1-2\rho)/[2(1-\rho)]$. If $\rho > 0$, it follows that $\lambda_1 > \lambda_2$ and $P(L_3 = 1) = P(L_3 = 3) > P(L_3 = 2)$. (For $\rho < 0$ the inequality is reversed.) Thus even in the case $n = 3$, the position of the maximum is not uniform, even though the random variables concerned belong to a stationary sequence.

One can see an intuitive reason for this. For consider the above example, with general n. Then in "assessing the chances" that ξ_1 will be the largest, we notice that it is correlated with only one of its competitors (ξ_2). On the other hand $\xi_2 \ldots \xi_{n-1}$ are each correlated with two competitors. Of course the structure is more complicated than this since we are dealing with multidimensional, and not just bivariate distributions, but this makes it even more likely that the situation will change as we move away from the edges of the sequence. Thus we may call this an "edge effect" though it does not necessarily disappear for ξ_i near the middle of the sequence, especially if the correlation structure dies off slowly. On the other hand it does suggest that we may (a) have uniformity for L_n under a "circular" stationarity assumption and (b) in any case we may have asymptotic uniformity as n becomes large. We consider the first of these conjectures now, and the second (or rather its continuous time version) in section 4.

Specifically let $\xi_1 \ldots \xi_n$ be standard normal random variables with $\operatorname{cov}(\xi_i\, \xi_j) = \lambda_{ij}$. $\xi_1 \ldots \xi_n$ will be called "circularly" stationary if there exists r_k, $0 \le k \le n-1$ such that

$$\text{(a)} \quad \lambda_{ij} = r_{(i-j)} \qquad 1 \le i, j \le n$$

$$\text{(b)} \quad r_k = r_{n-k} \qquad 1 \le k \le n-1$$

This concept has been used for example in (Hannan 1960). Condition (a) is the usual weak stationarity requirement. (b) gives, in particular, $r_{n-1} = r_1$, so that we may regard ξ_n and ξ_1 as having the same correlation r_1 as if "n" and "1" were adjacent points. In other words we have stationarity when we regard the points $1 \ldots n$ as arranged on a circle.

<u>Theorem 2.1.</u> For circularly stationary normal random variables $\xi_1 \ldots \xi_n$, the position L_n of the maximum is uniformly distributed on $1, 2 \ldots n$.

<u>Proof.</u> Let $f(x_1 \ldots x_n)$ denote the joint density of $\xi_1 \ldots \xi_n$. From the circular stationarity it may be shown without difficulty, that, for all $x_1 \ldots x_n$,

$$f(x_1, x_2 \ldots x_n) = f(x_n, x_1, x_2 \ldots x_{n-1}) \tag{2.1}$$

The joint density $g(x_1, x_2, \ldots x_n)$ of the random variables $(\xi_2, \xi_3 \ldots, \xi_n, \xi_1)$ is $f(x_n, x_1, x_2 \ldots x_{n-1})$. Hence writing $A = \{(x_1, x_2 \ldots x_n) : x_i \le x_1 \ i = 2 \ldots n\}$ we have

$$\begin{aligned} \Pr\{L=2\} &= \int_A g(x_1, x_2 \ldots x_n)\, dx_1 \ldots dx_n \\ &= \int_A f(x_n, x_1, x_2 \ldots x_{n-1}) \\ &= \int_A f(x_1, x_2 \ldots x_n)\, dx_1 \ldots dx_n \end{aligned}$$

by (2.1). But this is just $\Pr\{L=1\}$, and, proceeding in this way, we have $\Pr\{L=1\} = \Pr\{L=2\} = \ldots = \frac{1}{n}$.

3. CONTINUOUS CASE. JUMPS AT 0,T AND SYMMETRY.

We consider now a (zero mean) stationary normal process $\xi(t)$ in continuous time, with zero, mean, unit variance, and having covariance function

$r(t)$ such that $\lambda_2 = -r''(0) < \infty$. Let L_T denote the point of the interval $[0,t]$ at which $\xi(t)$ (first) attains its maximum in that range. L_T is not, in general, uniformly distributed over $[0,T]$. However a simple case in which this *can* occur is that of a "periodic process"

$$\xi(t) = A\cos(\omega t + \phi)$$

where A, ϕ are independent random variables, A having a Rayleigh distribution and ϕ being uniform over $(-\pi,\pi)$, and ω is a fixed constant. ($\xi(t)$ is a stationary normal process.) As long as T is at least a period $(2\pi/\omega)$ of this waveform, L_T can be regarded as uniform over $(0,2\pi/\omega)$, and is uniform over $(0,T)$ if $T = 2\pi/\omega$.

On the other hand if $T < 2\pi/\omega$, then there is positive probability that $L_T = 0$ or T, viz

$$\Pr(L_T = 0) = \Pr(L_T = T) = \tfrac{1}{2}(1 - \frac{\omega T}{2\pi})$$

and

$$\Pr\{0 < L_T \le x\} = \frac{\omega x}{2\pi} \qquad (0 < x < T)$$

Thus the distribution of L, consists (for $T < 2\pi/\omega$) of equal jumps at $0,T$ and a uniform part in $0 < x < T$.

The existence of jumps of the distribution of L at 0 and T is not due to the special nature of this example. Indeed suppose that $\xi(t)$ is any stationary normal process with zero mean, and whose covariance function is four times differentiable at the origin. Let N be the number of crossings of zero by the derivative $\xi'(t)$ in $0 \le t \le T$, where T is chosen so that $EN < 1$. Then

$$\Pr\{\xi'(t)>0 \text{ in } (0,T) \text{ or } \xi'(t)<0 \text{ in } (0,T)\} = \Pr\{N = 0\}$$
$$\ge 1 - EN > 0$$

and hence $\Pr(L = 0) \ge \Pr\{\xi'(t) < 0 \text{ in } (0,T)\} > 0$ (since by symmetry this also equals $\Pr\{\xi'(t) > 0 \text{ in } (0,T)\}$. Thus the distribution of L has a jump at zero.

A stationary normal process $\xi(t)$ is "reversible" in the sense that $\eta(t) = \xi(-t)$ has the same finite dimensional distributions as $\xi(t)$. This property leads to *symmetry* of the distribution of L in $(0,T)$ (i.e. $\Pr(L \le x) = \Pr(L \ge T - x)$) and, in particular, the fact that the distribution of L has the same jump at T as it does at zero. This will be developed further in (Leadbetter 1973).

4. ASYMPTOTIC UNIFORMITY OF L_T.

In this section we again assume the conditions stated at the start of section 3 hold. Since the maximum of $\xi(t)$ in $0 \le t \le T$ occurs by time $\lambda T (0<\lambda<1)$ if and only if $M(0,\lambda T) \ge M(\lambda T,T)$, the distribution of L_T is given by

$$\Pr\{L_T \le \lambda T\} = \Pr\{M(0,\lambda T) \ge M(\lambda T,T)\}$$

The asymptotic behaviour of this distribution is considered in the following result

Theorem 4.1. For the stationary normal process considered, if $0 < \lambda < 1$

$$\Pr\{L_T \le \lambda T\} \to \lambda \text{ as } T \to \infty$$

provided either (i) $r(t) \log t \to 0$ or (ii) $\int_0^\infty r^2(t)\, dt < \infty$. Thus, in this precise sense L_T is "asymptotically uniform" over $[0,T]$.

Proof. It is known (Berman 1971) that if either (i) or (ii) holds (in addition to $\lambda_2 < \infty$ assumed) then

$$\Pr\{a_T[M(0,T) - b_T] \le x\} \to \exp(-e^{-x}) \text{ as } T \to \infty$$

where

$$a_T = (2 \log T)^{\frac{1}{2}}$$

$$b_T = (2 \log T)^{\frac{1}{2}} + \log (\lambda_2^{\frac{1}{2}}/2\pi) / (2 \log T)^{\frac{1}{2}}$$

It may be further shown, under the same conditions (Leadbetter 1973) that $\Pr\{X_T \le x,\ Y_T \le y\} \to \exp(-e^{-x}) \exp(-e^{-y})$ as $T \to \infty$ where $X_T = a_{\lambda T}[M(0,\lambda T) - b_{\lambda T}]$ $Y_T = a_{\mu T}[M(\lambda T,T) - b_{\mu T}]$ $(\mu = 1 - \lambda)$. Thus

$$\Pr\{L_T \le \lambda T\} = \Pr\{X_T - \frac{a_{\lambda T}}{a_{\mu T}} Y_T \ge a_{\lambda T}(b_{\mu T} - b_{\lambda T})\}$$

Now $a_{\lambda T}/a_{\mu T} \to 1$ as $T \to \infty$ and it is easily checked that $a_{\lambda T}(b_{\mu T} - b_{\lambda T}) \to \log(\mu/\lambda)$. An obvious weak convergence argument (the details of which will appear in somewhat more general terms in (Leadbetter 1973) then shows that

$$\Pr\{L_T \le \lambda T\} \to \Pr\{X - Y \ge \log(\mu/\lambda)\}$$

where X and Y are independent random variables with common distribution function $\exp(-e^{-x})$. This can be written as

$$\int_{-\infty}^{\infty}\{1 - \exp[-e^{-(y+\log(\mu/\lambda))}]\} \exp(-e^{-y})\, e^{-y}dy$$

by integrating over the appropriate part of the plane, and this reduces, by some amusing algebra, to the required value λ .

Finally we note that several authors have shown that the asymptotic distribution of $M(0,T)$ holds in certain circumstances even if $\lambda_2 = \infty$ (e.g. Pickands 1969).

The implications of this in the present context will be taken up in (Leadbetter 1973).

SUMMARY

The purpose of this paper is to describe certain results concerning the location L_T of the maximum of a stationary normal process in the time $0 \le t \le T$. It is shown that L_T does not, in general, have a uniform distribution on $0 \le t \le T$, but is asymptotically so under weak conditions as $T \to \infty$. The various features of the corresponding discete time problem are also discussed.

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