1. *Part I discusses the general linearly ordered case. (Simons 1972).*

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GENERALIZED CUMULATIVE DISTRIBUTION FUNCTIONS: II. THE σ - LOWER FINITE CASE[1]

by

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THE σ - LOWER FINITE CASE

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ABSTRACT

A mass $m(x) \geq 0$ is assigned to each point x of a partially ordered countable set X. It is further assumed that $M(x) = \Sigma_{y \leq x} m(y) < \infty$ for each $x \in X$. M is called a distribution function. For certain sets X, it is shown that M determines m. For others, M need not determine m uniquely. A theory is presented for σ - lower finite spaces (sets), which are defined in the paper. Such spaces are locally finite. I.e., each interval $[x,y] = \{z \in X : x \leq z \leq y\}$ has a finite number of points. Möbius functions, which have been defined for locally finite spaces, are used throughout. Distribution functions on a particular σ - lower finite space arise naturally from boundary crossing problems analyzed by Doob and Anderson. The theory is applied to this example and to another.

Key words and phrases: distribution function, Möbius function, partial ordering, boundary crossing

1. INTRODUCTION

Let X be a partially ordered countable set with each point $x \in X$ possessing a non-negative mass $m(x)$. We assume $M(x) = \Sigma_{y \leq x} m(y) < \infty, x \in X$, and refer to the function M as the (cumulative) *distribution function.* Sometimes, we shall require the total mass $m(X) = \Sigma_{x \in X} m(x)$ to be unity [3/]. We shall concern ourselves with the following questions:

(i) When does the distribution function determine the individual masses $m(x)$, $x \in X$?

(ii) How are they found when they are determined?

(iii) When is a function $M(x)$, $x \in X$, actually a distribution function?

When X is finite, one has an inversion formula expressed in terms of the *Möbius function* $\mu(x,y)$ of X:

(1) $\quad m(y) = \Sigma_{x \leq y} M(x)\mu(x,y), \; y \in X.$

Section 3 of Rota's (1964) fundamental paper on the theory of Möbius functions provides the relevant background. Thus, *the distribution function always determines the individual masses whenever* X *is finite.*

3/ The concept of a distribution function is fundamental in probability theory, an area where the requirement $m(X) = 1$ is most natural. However, we shall refrain from making this a requirement *at the outset* since its inclusion would tend to complicate the theory presented below.

We shall be concerned with the more difficult situation in which X is infinite. The full range of possibilities is much more than we can cope with at this early stage, and we shall be content with a modest beginning. We shall confine all of our attention to *locally finite* spaces since it is for such spaces that a Möbius function is defined. That is, for each pair of points $x, y \in X$, we shall insist that the interval $[x,y] = \{z: x \le z \le y\}$ be finite.

A rather uninteresting extension of the finite theory discussed in the second paragraph can be made when the number of summands in (1) is finite for each $y \in X$. Formula (1) still applies. We refer to such an X as *lower finite*. A lower finite space is necessarily locally finite.

An interesting example for which X is locally finite but not lower finite arises (apparently unnoticed) in a much cited paper by Doob (1949): Let X consist of a maximal point (0,1) and pairs of points $(n,1)$, $(n,2)$ for $n \ge 1$. $x = (n,j) < x' = (n',j')$ if and only if $n > n'$, $n' \ge 0$. See Figure I. Let $\{W(t), t \ge 0\}$ be a standard Wiener process (mean zero and variance t) and let U and L be two lines with U (the upper) having positive slope and intercept, and L (the lower) having negative slope and intercept. Let $M(0,1) = 1$ and, for $n \ge 1$, let $M(n,1)$ and $M(n,2)$ denote the probability that there exist n time $0 < t_1 < \cdots < t_n$ with $(t_1, W(t_1)), \cdots, (t_n, W(t_n))$ alternately in U and L beginning with U and L, respectively. (Anything may happen before t_1, between times, and after t_n.) Then $\{M(x), x \in X\}$ is a distribution function corresponding to individual masses such as $m(0,1) = P(W$ never touches U or $L)$,

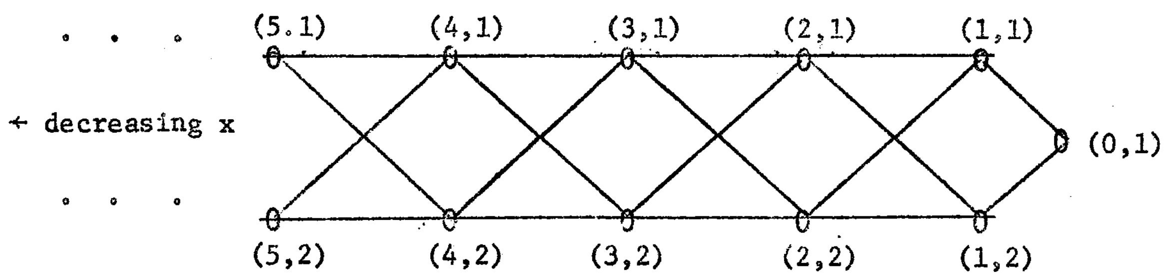
$m(1,1) = P$(W touches U but never touches L) and $m(2,2) = P$(W touches L before U then touches U then never touches L again).

Doob describes how to compute $M(x)$, $x \in X$. His interest is in expressing, in terms of these, a probability such as $m(0,1)$. In particular, he obtains a formula which, in our notation, becomes

(2) $m(0,1) = 1 - \Sigma_{n=1}^{\infty}(-1)^{n-1}\ \{M(n,1) + M(n,2)\}$.

Anderson (1960) obtains a formula for P(W touches U before L). (W does not need to touch L.) This can be expressed in terms of the individual masses as $\Sigma_{n=1}^{\infty} m(n,1)$. Anderson evaluates the probability as $M(1,1) - M(2,2) + M(3,1) - M(4,2) + \cdots$.

FIGURE I



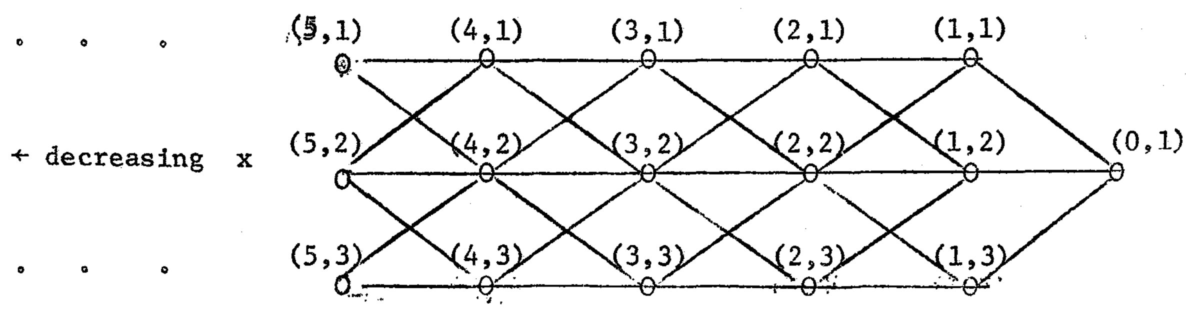
For the X of Figure I, we find that the distribution function always determines the individual masses. Besides equation (2), we have the related equations:

(3) $m(n,j) = M(n,j) - \Sigma_{k=n+1}^{\infty}(-1)^{k-1}\ \{M(k,1) + M(k,2)\}$, $n \geq 1$, $j=1,2$.

These equations can be checked by direct substitution.

Before we present some theory, we shall show that a locally finite space can have distribution functions which fail to determine the individual masses. Our example is only slightly more complicated than that of Figure 1. See Figure II. X consists of a maximal point (0,1) and triplets (n,1), (n,2), (n,3) for $n \geq 1$. $x=(n,j) < x' = (n',j')$ if and only if $n>n', n' \geq 0$.

FIGURE II



Example 1: $M(0,1)=1$ and $M(n,1)=M(n,2)=M(n,3)=2^{-n}, n \geq 1$. There are many possible values for the individual masses but all of the possibilities can be expressed as convex combinations of two extremal solutions:

Solution 1a: $m(0,1) = 1/2$; $m(n,1) = m(n,2) = m(n,3) = 2^{-n-1}$ for $n = 2,4,6 \ldots$; all other $m(x) = 0$.

Solution 1b: $m(n,1) = m(n,2) = m(n,3) = 2^{-n-1}$ for $n = 1,3,5, \ldots$; all other $m(x) = 0$.

Example 2: $M(0,1) = 1$ and $M(n,1) = M(n,2) = M(n,3) = 3^{-n}$, $n \geq 1$. There is only one solution: $m(0,1) = 2/5$ and $m(n,1) = m(n,2) = m(n,3) = (2/5) \cdot 3^{-n}$, $n \geq 1$.

These examples show that *the issue of uniqueness for the individual masses depends on the actual distribution function as well as on the structure of* X. We shall return to these examples later after we have some theory with which to justify our claims.

While the spaces described in figures I and II are not lower finite, they are what we shall refer to as σ - *lower finite*. That is there exists a sequence of partitions $X = A_k + B_k$ with $A_k > B_k$ (i.e., each point of A_k exceeds each point of B_k), $A_k \uparrow X$ as $k \to \infty$, and each A_k is lower finite (i.e., for each $y \in X$ and each $k \geq 1$, $\{x \in A_k: x \leq y\}$ is finite). A σ-lower finite space is necessarily locally finite. We shall give a definitive answer to questions (i), (ii) and (iii) for σ-lower finite spaces in Section 3. Although the σ-lower finite assumption is stronger than one might prefer, it permits a wide range of spaces. Departures from this assumption can be analyzed but the assumption permits us to develop a reasonably uncomplicated theory.

2. SOME PRELIMINARIES

Let X be locally finite. It will be recalled that the Möbius function can be defined recursively by

$$(4) \quad \mu(x,y) \begin{cases} = 1 & \text{for } x=y, \\ = -\Sigma_{x \leq z < y}\, \mu(x,z) & \text{for } x < y \\ = 0 & \text{for } x \not\leq y. \end{cases}$$

Another useful formula is

$$(5) \quad \mu(x,y) = -\Sigma_{x<z\leq y}\, \mu(z,y) \text{ for } x<y.$$

Let A' denote the complement of A for each subset A of X.

Suppose $X = A + B + C$ with $A > C > B$, where A, B or C may be the empty set ϕ. Define whenever the number of summands is finite:

$$\mu(x,A) = -\Sigma_{x\leq z<A}\,\mu(x,z) \quad , \quad x \in A' \,,$$

(6) $$\mu(B,y) = -\Sigma_{B<z\leq y}\,\mu(z,y) \quad , \quad y \in B' \,,$$

$$\mu(B,A) = -1 + \Sigma_{B<x,y<A}\,\mu(x,y).$$

It is easily checked that

(7) $$\mu(B,A) = -1 - \Sigma_{B<x<A}\,\mu(x,A) = -1 - \Sigma_{B<y<A}\,\mu(B,y).$$

It is most helpful to have an intuitive understanding of (6). $\mu(x,A)$, $\mu(B,y)$ and $\mu(B,A)$ are actual Möbius function values corresponding to various clustered versions of X. If one views A as a single point and the points of A' as individual entities, equation (4) with $y=A$ yields the definition of $\mu(x,A)$. In a similar manner, $\mu(B,y)$ arises from (5). $\mu(B,A)$ arises from viewing A and B as points and the points in between as individual entities.

Proposition 1. *Suppose* $X = A + B$ *where* A *and* B *are not empty and* $A > B$. *Then* A *and* B *have at least one minimal and one maximal point, respectively. Moreover,* $\mu(b,a_o) = \mu(b,A)$ *and* $\mu(b_o,a) = \mu(B,a)$ *for each* $a \in A$, $b \in B$, *minimal point* $a_o \in A$ *and maximal point* $b_o \in B$.

Proof. Let $a \in A$ and $b \in B$. $[b,a]$ is a finite interval and must contain a minimal point in A and maximal point in B. The equalities above easily follow from definitions. □

Proposition 2. *Suppose* $X = A + B$ *where* $A > B$. *Then*

(8) $\mu(b,a) = -\mu(b,A)\,\mu(B,a)$ *(with both factors well-defined) for each* $a \in A$ *and* $b \in B$.

Proof. Since $\mu(b_o, A) = \mu(B,a_o) = -1$ for each minimal point $a_o \in A$ and maximal point $b_o \in B$, the desired equality follows from Proposition 1 when a is a minimal point of A or b is a maximal point of B. The remainder of the proof uses induction based on the total number of elements in $[b,a]$. The induction step is:

$$\mu(b,a) = -\Sigma_{b<z\le a}\ \mu(z,a) = -\Sigma_{b<z<A}\,\mu(z,a) + \mu(B,a)$$

$$= \Sigma_{b<z<A}\,\mu(z,A)\,\mu(B,a) \quad + \quad \mu(B,a)$$

$$= (\Sigma_{b<z\le a_o}\ \mu(z,a_o))\ \ \mu(B,a)$$

$$= -\mu(b,a_o)\mu(B,a) = -\mu(b,A)\mu(B,a) \qquad \square$$

Proposition 3. Suppose $X = A + B$ *where* $A > B$ *and* $B \ne \phi$. *Let* x_o *be a minimal point of* A. *Further, let* x *and* y *be distinct points satisfying* $x \ne x_o$ *and* $y > x_o$. *Then*

$$\Sigma_{B<z\le y,\ z\ne x_o}\ \mu(x,z) = 0.$$

Proof. Using (4):

$$0 = \Sigma_{x\le z\le y}\,\mu(x,z) = \Sigma_{x\le z\le y,\,z\in B}\,\mu(x,z)$$

$$+ \Sigma_{x\le z\le y,\,z\in A}\,\mu(x,z) = \Sigma_{x\le z<x_o}\,\mu(x,z) + \Sigma_{B<z\le y}\,\mu(x,z)$$

$$= -\mu(x,x_o) + \Sigma_{B<z\le y}\,\mu(x,z) = \Sigma_{B<z\le y,\,z\ne x_o}\,\mu(x,z). \qquad \square$$

We describe now, in terms of the notation used in the definition, the three possible types of σ - lower finite spaces:

Type I : X is lower finite.

Type II : X is not lower finite and $\mu(B_{k+1}, A_k) = 0$ for infinitely many k.

Type III: X is not lower finite and $\mu(B_{k+1}, A_k) = 0$ for only finitely many k.

<u>Proposition 4</u>. *The distinction between types* II *and* III *is independent of the particular sequence of partitions* $X = A_k + B_k$, $k \geq 1$.

<u>Proof</u>. By viewing the points of B_{k+2} and the points of A_k as single points, it follows from Proposition 2 that $\mu(B_{k+2}, A_k)$ $= - \mu(B_{k+2}, A_{k+1}) \mu (B_{k+1}, A_k)$. In turn,

(9) $\mu(B_{k+\ell}, A_k) = \pm \Pi_{j=0}^{\ell-1} \mu(B_{k+j+1}, A_{k+j})$, $\ell \geq 2$. It follows that the type, II or III, is unaltered by adding or deleting partitions from a given sequence. □

3. SOME THEORY

We shall successively examine σ - lower finite spaces of types I, II and III.

We have already commented about type I spaces (lower finite spaces) in the introduction. They are so easily analyzed because each set $\{x: x \leq y\}$, $y \in X$, is finite and the restriction of a distribution function (on X) to $\{x:x \leq y\}$ is a distribution function on that finite space. Therefore, questions (i), (ii) and (iii) (appearing in the introduction) are easily answered with the use of (1).

Suppose X is a σ - lower finite space of type II. X has the following important property:

Proposition 5. *For a type* II σ - *lower finite space* X, *each set* $\{x:\mu(x,y) \neq 0\}$, $y \in X$, *is finite.*

Proof. In view of (9), we may assume, without sacrificing generality, that $\mu(B_{k+1}, A_k) = 0$ for $k \geq 1$. Suppose $y \in X$. Then, necessarily, $y \in A_k$ for some k. Since X is not lower finite, A_{k+1} must be a proper subset of X. Consequently, there exists a point $x_o \in B_{k+1}$. Since $[x_o, y]$ is a finite set, it suffices to show that $\mu(x,y) = 0$ for each $x \notin [x_o, y]$. We only need to consider $x \in B_{k+1}$. For such an x, we have, with successive applications of Proposition 2, $\mu(x,y) = -\mu(x,A_k)\mu(B_k, y) = \mu(x, A_{k+1})\mu(B_{k+1}, A_k)\mu(B_k,y) = 0$. □

The following theorem tells us that (1) holds for type II spaces:

Theorem 1. *For any locally finite space,* (1) *holds for a given* $y \in X$ *whenever the number of non-zero summands in* (1) *is finite.*

Proof. Under the assumption, $\Sigma_{x\leq y}\ \Sigma_{z\leq x}|m(z)\mu(x,y)| \leq M(y)\ \Sigma_{\{x:x\leq y,\ M(x)\neq 0\}}|\mu(x,y)| < \infty$. Thus, Fubini's theorem applies. Then, $\Sigma_{x\leq y} M(x)\ \mu(x,y) = \Sigma_{x\leq y}\ \Sigma_{z\leq x}\ m(z)\mu(x,y) = \Sigma_{z\leq y} m(z)\ \Sigma_{z<x\leq y}\mu(x,y) = m(y)$. □

While (1) can be used directly to answer question (iii) for type II spaces, it is easier to use the following theorem:

Theorem 2. *A function* M *on a type* II σ - *lower finite space* X *is a distribution function if and only if*

(a) *the sum in* (1) *is non-negative for each* $y \in X$, *and*

(b) *the set* $\{x: x \le y, |M(x)| \ge \epsilon\}$ *is finite for each* $\epsilon > 0$ *and* $y \in X$.

Proof. Assume (a) and (b), and fix y. Find an A_k containing y and an $x_k \le y$ which is a minimal point of A_k. Define m in terms of M by (1). Except for a finite number of $x \in X$, $\mu(x,z) = 0$ for all z in any given finite subset of X. Thus, we may interchange the order of summation below, and we have with the aid of Proposition 3:

$$\Sigma_{B_k < z \le y,\ z \ne x_k} m(z) = \Sigma_{x \le y} M(x) \Sigma_{B_k < z \le y,\ z \ne x_k} \mu(x,z)$$

$$= M(y)\mu(y,y) - M(x_k)\mu(x_k,x_k) = M(y) - M(x_k).$$

Letting $k \to \infty$ leads to $\Sigma_{z \le y} m(z) = M(y)$. Thus, M is a distribution function. Conversely, if M is a distribution function for the individual masses $m(x)$, $x \in X$, then (a) follows from Theorem 1 and (b) follows from the local finiteness of X. □

Suppose X is a σ - lower finite space of type III. Depending on the structure of X, there may be two distinct sets of individual masses with the same distribution function. This contrasts with type I and type II spaces where the answers to question (i) is "Always."

Proposition 4 permits us to assume that $A_1 \neq \phi$ and $\mu(B_k, A_1) \neq 0$ for each $k \geq 1$. For each $x \in X$, choose an arbitrary A_k containing x and define $\beta(x) = -\mu(B_k, x)/\mu(B_k, A_1)$.

Proposition 6. *β is well-define in the sense that the value of $\beta(x)$ does not depend on how one chooses k. $\beta(x) = \mu(B_1, x)$ for $x \in A_1$. β is not identically zero.*

Proof. For $x \in A_k$, $\mu(B_{k+1}, x) = -\mu(B_{k+1}, A_k)\mu(B_k, x)$ (c.f., (8)), and $\mu(B_{k+1}, A_1) = -\mu(B_{k+1}, A_k)\mu(B_k, A_1)$ (c.f., (8)). Thus $\beta(x)$ is well-defined. If $x \in A_1$, $\mu(B_1, A_1) = -1$ and $\beta(x) = \mu(B_1, x)$. Finally, if x_k is a minimal point of A_k, then $\mu(B_k, x_k) = -1$ and $\beta(x_k) = \mu(B_k, A_1)^{-1} \neq 0$. □

Let β^+ and β^- denote the positive and negative parts of β, respectively.

Theorem 3. *For each $y \in X$,*

(10) $$\Sigma_{x \leq y} \beta^+(x) = \Sigma_{x \leq y} \beta^-(x).$$

Thus, if $\Sigma_{x \leq y} |\beta(x)| < \infty$ for each $y \in X$, $\{\beta^+(x)\}$ and $\{\beta^-(x)\}$ represent two distinct sets of individual masses with the same distribution function.

Proof. Fix y and choose a minimal point x_k of A_k satisfying $x_k \leq y$, for each A_k containing y. Then $\Sigma_{B_k < z \leq y, z \neq x_k} \mu(B_k, z) = 0$ (c.f., Proposition 3), and hence,

(11) $$\Sigma_{B_k < z \leq y,\ z \neq x_k} \beta(z) = 0.$$

Then (10) follows by letting $k \to \infty$. □

A space X will be called a *determining space* if every distribution function on X corresponds to a unique set of individual masses.

Theorem 4. *A type III σ - lower finite space X is a determining space if and only if $\Sigma_{x \leq y_o} |\beta(x)| = \infty$ for some $y_o \in X$.* We shall defer the proof of the "if" part until later. The "only if" part is immediate from Theorem 3.

Define

$$\nu(x,y) = \begin{cases} \mu(x,y) & \text{for } x \in A_1, \\ \mu(x,y) + \mu(x,A_1)\beta(y) & \text{for } x \in B_1. \end{cases}$$

$\nu(x,y) = 0$ whenever $x \in B_k$ and $y \in A_k$ for some $k \geq 1$ (c.f.,(8)). Likewise, $\nu(x,y) = 0$ whenever $x \in A_1$ and $x \nleq y$ (c.f.,(4)). Thus, the sum $\Sigma_{x \in X} M(x)\, \nu(x,y)$ has at most a finite number of non-zero summands for each $y \in X$.

Proposition 7. *Let M be a distribution function corresponding to the individual masses $\{m(x)\}$ and let $\alpha = m(B_1)$. M and α together determine m. In particular,*

(12) $m(y) = \Sigma_{x \in X} M(x)\nu(x,y) + \alpha\, \beta(y), \quad y \in X.$

Proof. $\alpha < \infty$ since there exists a point $y_o \in A_1$ and $\alpha = m(B_1) \leq M(y_o) < \infty$. Fix $k \geq 1$ and view B_k as a point. Then (c.f.,(1)),

(13) $m(y) = \Sigma_{x \in A_k} M(x)\, \mu(x,y) + m(B_k)\, \mu(B_k, y), \quad y \in A_k.$

Next, view both $D = \{z \in A_1 : z \leq y_o\}$ and B_k as points. Then (c.f.,(1)),

$$M(y_o) - \alpha = m(D) = M(y_o)\cdot 1 + \Sigma_{x\in B_1-B_k} M(x)\mu(x,A_1) + m(B_k)\mu(B_k,A_1).$$

Consequently,

$$(14)\quad \alpha + \Sigma_{x\in B_1-B_k} M(x)\mu(x,A_1) + m(B_k)\mu(B_k,A_1) = 0.$$

Combining (13) and (14), we obtain for $y \in A_k$,

$$m(y) = \Sigma_{x\in A_k} M(x)\mu(x,y) + (\Sigma_{x\in B_1-B_k} M(x)\mu(x,A_1) + \alpha)\beta(y)$$

$$= \Sigma_{x\in X} M(x)\nu(x,y) + \alpha\,\beta(y). \qquad \square$$

Theorem 4 Proof ("if" part). Let $\{m_1(x)\}$ and $\{m_2(x)\}$ be distinct sets of individual masses which give rise to the distribution function M. From (12), we have $\Delta m(x) = \Delta\alpha\beta(x)$, where $\Delta m = m_2 - m_1$ and $\Delta\alpha = \Delta m(B_1)$. Since $\Delta m(x) \neq 0$ for some x, $\Delta\alpha \neq 0$. Thus for any $y \in X$,

$$\Sigma_{x\le y} |\beta(x)| = \Delta\alpha^{-1} \Sigma_{x\le y} |\Delta m(x)| \le 2\Delta\alpha^{-1} M(y) < \infty. \qquad \square$$

Theorem 5. *A function* M *on a type* III σ - *lower finite space is a distribution function if and only if*

(a) *there exists an* α *which makes the right hand side of* (12) *non-negative for each* $y \in X$, *and*

(b) *the set* $\{x : x\le y,\ |M(x)| \ge \epsilon\}$ *is finite for each* $\epsilon>0$ *and* $y \in X$.

Each such α *corresponds to a unique set of individual masses* $\{m(x)\}$ *defined by* (12), *and for this set,* $\alpha = m(B_1)$.

Proof. The proof of the "if" part parallels that for Theorem 2. But here one defines m by (12), using the α predicated in assumption (a), instead of (1). The complications brought in by β (both directly and indirectly through $\nu(x,y)$) are taken care of by using (11). One obtains $M(y) = \Sigma_{x\leq y}\, m(x)$, $y \in X$. If α were not equal to $m(B_1)$, Proposition 7 would be contradicted. For the converse, (a) is a consequence of Proposition 7 and (b) is due to the local finiteness of X. □

Let M be a distribution function. It is easy to see that the possible values for $\alpha = m(B_1)$ is an interval $[\alpha_{min}, \alpha_{max}]$, perhaps a point.

Proposition 8.

(15) $\alpha_{min} = \sup\{ -\Sigma_{x\in X}\, M(x)\nu(x,y)/\beta(y) \;:\; \beta(y) > 0 \}.$

(16) $\alpha_{max} = \inf\{ -\Sigma_{x\in X}\, M(x)\nu(x,y)/\beta(y) \;:\; \beta(y) < 0 \}.$

If $\Sigma_{x\leq z} |\beta(x)| = \infty$ *for some* z *(and necessarily for all* z), *then*

(17) $\alpha_{min} = \alpha_{max} = \lim_{k\to\infty}\{ \Sigma_{B_k<y<A_1} | M(x)\nu(x,y)| \,/\, \Sigma_{B_k<y<A_1} |\beta(y)|\}.$

If $m(B_k)\, |\mu(B_k,A_1)| \to 0$ *as* $k \to \infty$, *then*

(18) $\alpha_{min} = \alpha_{max} = -\lim_{k\to\infty} \Sigma_{B_k<x<A_1} M(x)\mu(x,A_1).$

<u>Proof</u>. (15) and (16) follow easily from Theorem 5. Suppose $\Sigma_{x \le z} |\beta(z)| = \infty$. We may suppose $z \in A_1$. Then from (12), we have for each k:

$$\left|\alpha \, \Sigma_{B_k < y < A_1} |\beta(y)| - \Sigma_{B_k < y < A_1} \left| \Sigma_{x \in X} M(x)\nu(x,y)\right|\right| \le \Sigma_{B_k < y < A_1} m(y) \le M(z) < \infty.$$

The first sum $\to \infty$ as $k \to \infty$, and (17) follows. (18) follows directly from (14). □

The condition for (18) is difficult to validate directly since one is not likely to know the value of $m(B_k)$ without knowing α. However, $m(B_k) \le M(x)$, $x \in A_k$, $k \ge 1$. So, an indirect validation is possible.

We now turn our attention toward the evaluation of the total mass $m(X)$. A requirement such as $m(X) = 1$ sometimes determines a unique choice for the individual masses when a distribution function, by itself, does not. The most direct formula, of course, is $m(X) = \Sigma_{x \in X} m(x)$. However, there are some advantages in securing formulas which primarily involve the values of the distribution function.

Let $X^* = X + \{x^*\}$ be an augmented space satisfying $x^* > X$. If $m(x^*) = 0$, then $M(x^*) = m(X)$. If X^* is locally finite, then $\mu^*(x,x^*) = \mu(x,\phi)$, $x \in X$, where μ^* denotes the extension of μ to X^* [4/].

4/ We shall superscript an extended function with an asterisk only when it aids clarity. Here, $\mu(x,\phi)$ has a different meaning from $\mu^*(x,\phi)$. The latter is zero for all $x \in X$.

<u>Proposition 9</u>. *Suppose* X^* *is a* σ*-lower finite space. Then* X *is a* σ*-lower finite space of the same type as* X^*. *If* X^* *is of type I or type II,*

(19) $m(X) = - \Sigma_{x\in X} M(x)\ \mu(x,\phi)$,

a sum with only a finite number of non-zero summands.

If X^* *is of type III, then* A_1 *is a finite set and*

(20) $m(X) = - \Sigma_{x\in A_1} M(x)\mu(x,\phi) - \alpha\ \mu(B_1,\phi)$,

where $\alpha = m(B_1)$.

<u>Proof</u>. Briefly, (20) follows from (c.f., (1)):

$$0 = m(x^*) = m(X)\mu^*(x^*,x^*) + \Sigma_{x\in A_1} M(x)\mu(x,\phi) + m(B_1)\mu(B_1,\phi).$$

(19) is shown similarly. The details are left to the reader. □

Proposition 9 is still usable when X^* is not σ-lower finite: Express X as $\{x_n, n \geq 1\}$. Then define $X_n = \{z\in X: z \leq x_j$ for some $j=1, \ldots, n\}$ and $X^*_n = X_n + \{x^*\}$, $n \geq 1$. If X is σ-lower finite, then so is each X^*_n (and of the same type as X). Furthermore, $m(X_n) \nearrow m(X)$ as $n\to\infty$.

<u>Theorem 6</u>. *Suppose* X *is a* σ*-lower finite space. If* X *is of type I or II, then*

(21) $m(X) = \lim_{n\to\infty} \Sigma_{x\in X}\{M(x)\ \ \Sigma_{y\in X_n}\ \mu(x,y)\}$

If X *is of type III,*

(22) $m(X) = \alpha + \lim_{n\to\infty} \Sigma_{x \in A_1} \{(M(x)-\alpha)\ \Sigma_{y \in X_n}\ \mu(x,y)\}$,

where $\alpha = m(B_1)$. *Each sum has a finite number of non-zero summands.*

Proof. The details easily follow from (6), (19), (20) and the foregoing discussion.

4. APPLICATIONS

We shall confine our applications to the two spaces illustrated in figures I and II. Both are σ - lower finite spaces of type III. The basic facts about these spaces are summarized in Table I. For Figure I, $\alpha_{min} = \alpha_{max} = \Sigma_{n=1}^{\infty}(-1)^{n-1}\{M(n,1) + M(n,2)\}$ (c.f.,(18)). This is consistent with (2). For Figure II, define $S_o = M(0,1)$ and, for $n \geq 1$, $S_n = \sum_{k=1}^{n} (-2)^{k-1}\{M(k,1)+M(k,2)+M(k,3)\} + (-2)^n \min(M(n,1),M(n,2),M(n,3))$. Then, $\alpha_{min} = \sup\{S_1,S_3,S_5, \ldots\}$ and $\alpha_{max} = \inf\{S_o,S_2,S_4, \ldots\}$ (c.f., (15) and (16)).

Finally, we turn our attention to examples 1 and 2. We see in Example 1 that the first and second solutions correspond to $m(x) = 1/2\ \beta^-(x)$ and $m(x) = 1/2\ \beta^+(x)$ $(x \in X)$, respectively. For the first solution $\alpha = \alpha_{min} = 1/2$, and for the second $\alpha = \alpha_{max} = 1$. For Example 2, (18) applies, and we obtain $\alpha = 3/5$. With this, the values of $m(x)$ follow immediately from (12).

TABLE I

X	Figure I	Figure II
A_n	$\{x : x > (n,1)\}$	$\{x : x > (n,1)\}$
$\mu((n,j),(n',j'))$	$(-1)^{n-n'}$ for $n' < n$ 1 for $n=n', j=j'$ 0 otherwise	$-(-2)^{n-n'-1}$ for $n' < n$ 1 for $n=n', j=j'$ 0 otherwise
$\mu(B_n,(n',j'))$	$(-1)^{n-n'}$ for $n' < n$	$-(-2)^{n-n'-1}$ for $n' < n$
$\mu((n,j), A_{n'})$	$(-1)^{n-n'+1}$ for $n' \leq n$	$-(-2)^{n-n'}$ for $n' \leq n$
$\mu(B_n, A_{n'})$	$(-1)^{n-n'+1}$ for $n' \leq n$	$-(-2)^{n-n'}$ for $n' \leq n$
$\beta(n,j)$	$(-1)^{n+1}$	$-(-2)^{-n}$
$\sum_{x \leq (0,1)} \|\beta(x)\|$	∞	4
$\sum_{x \leq (n,j)} \beta^{+}(x)$	∞	2^{1-n}
$\sum_{x \leq (n,j)} \beta^{-}(x)$	∞	2^{1-n}
$\nu((n,j),(n',j'))$	1 for $n=n'=0, j=j'=1$ $(-1)^{n-n'-1}$ for $n'>n\geq 1$ -1 for $n=n'\geq 1, j\neq j'$ 0 otherwise	1 for $n=n'=0, j=j'=1$ $(-2)^{n-n'-1}$ for $n'>n\geq 1$ $+\frac{1}{2}$ for $n'>n\geq 1$ $-\frac{1}{2}$ for $n=n'\geq 1, j\neq j'$ 0 otherwise

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