JACKKNIFING MAXIMUM LIKELIHOOD ESTIMATES FOR A SPECIAL CLASS OF DISTRIBUTIONS

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Summary

The object of this note is to show how series for the lower moments of a maximum likelihood estimate and its jackknifed form can be taken to a higher order than usual when the underlying distribution belongs to a subclass of the usual Koopman exponential family. The moments are available to order n^{-2} at present, but in this special case we extend them to cover the n^{-4} term for the maximum likelihood estimate and the n^{-3} term for the jackknife. The series expansions are then illustrated using a single-parameter special beta distribution. In this example it turns out that we do not usually need these extra terms for the first two moments, but that they greatly improve the third and fourth.

1. Introduction

In one of the illustrations used in Ferguson and Fryer (1973) moment calculations to the second order for the maximum likelihood estimate and its jackknifed form turned out to be particularly simple. This was largely because the second derivative of the log-likelihood function did not involve the basic random variable. For this type of distribution it seemed likely that we could extend the moment formulae for the two estimates to a higher order and so get a better approximation to the exact values of the moments. The object of this note is to report on our findings. We shall use exactly the same set-up here as we did in our previous work and the reader is referred to it for the details in order to save space.

Taking $\frac{\partial^2 \log f(x;\theta)}{\partial\theta}$ to be a function of θ alone implies that in the region of non-zero density $f(x;\theta)$ has the form

$$f(x;\theta) = \exp\{A(\theta) + \theta B(x) + C(x)\} \tag{1}$$

and so belongs to a sub-class of the usual Koopman exponential family. The special beta density that we used before and that we will be using here as an illustration is given by

$$f(x;\theta) = (\theta + 1)(\theta + 2)x^{\theta}(1 - x) \quad \text{for } 0 < x < 1 \text{ and } \theta > 1$$

and so can be written as

$$f(x;\theta) = \exp\{\log[(\theta + 1)(\theta + 2)] + \theta \log x + \log(1 - x)\} \tag{2}$$

in the appropriate region. We note that in this case $\frac{1}{n}\sum_i \log x_i = \log G$ say is a single sufficient statistic for θ and that the maximum likelihood estimate

$$\hat{\theta} = -\frac{[(2 + 3\log G) + [4 + (\log G)^2]^{\frac{1}{2}}]}{2\log G} \tag{3}$$

is also sufficient. On the face of it there is evidently a case to be made for using the maximum likelihood estimate in such cases, which justifies taking our moment expansions further. The other question we have to ask ourselves is whether there is any case to be made for jackknifing a sufficient statistic. The problem is that jackknifing will destroy the sufficiency property as a rule, and this is not a thing that we would relish doing in general. However, we feel that it may be a reasonable thing to do when we are unable to find a monotone function of a sufficient statistic with acceptable levels of bias and variance and there must surely be cases when this is so. This provides the motivation for extending the moments of the jackknifed maximum likelihood estimate to a higher order. The search for a suitable function of the sufficient statistic for this class of distributions is an interesting problem that is beyond the scope of this note. We shall concern ourselves here with the much simpler task of comparing the lower moments of $\hat{\theta}$, a sufficient statistic with known asymptotic properties with those of its jackknifed form $\hat{\theta}_p$.

2. Moment Formulae for $\hat{\theta}$ and $\hat{\theta}_p$.

Expansions for $(\hat{\theta}-\theta)$ and $(\hat{\theta}_p-\theta)$ given in Ferguson and Fryer (1973) allow us to calculate their first four (series) moments up to order n^{-2}. Beyond that the algebra is unthinkable in general, but additional terms are within reach for this special Koopman class. With considerable effort we have managed to extend the maximum likelihood results to order n^{-4}, and the moments of the jackknifed estimate generally to order n^{-3}(n^{-4} was too demanding). In order to do this we have to take our basic expansions for $\hat{\theta}$ and $\hat{\theta}_p$ to a higher order. Previously we used

$$(\hat{\theta}-\theta) = \frac{\alpha}{n} + \frac{\beta}{n^2} + \frac{\gamma}{n^3} + \frac{\delta}{n^4} + O\left(\frac{A^5}{n^5}\right) \tag{4}$$

but the n^{-4} results for both estimates call for the next four terms in the expansion (at most) leaving a remainder of order $\frac{A^9}{n^9}$. There are many expectations to be evaluated on the way to the final forms for the moments (for example, $E\left[A^2(\sum_j A_j^2)^2\right]$ and $E(A^8)$) and many inter-relationships to exploit (like $K = 3I^2 - (1,1,1,1)$) but to save space we will not go into the details. There is some additional notation needed though if we are to quote the final results in a compact form. In fact we need symbols for higher derivatives of log f and use M, N, P and Q to denote

$\frac{\partial^6 \log f}{\partial\theta^6}$, $\frac{\partial^7 \log f}{\partial\theta^7}$, $\frac{\partial^8 \log f}{\partial\theta^8}$ and $\frac{\partial^9 \log f}{\partial\theta^9}$ respectively. With this and previous notation the results for the maximum likelihood estimate, $\hat{\theta}$, reduce to

$$\underset{se}{E}(\hat{\theta} - \theta) = \frac{J}{2nI^2} + \frac{1}{n^2}\left\{\frac{L}{8I^3} + \frac{13JK}{12I^4} + \frac{11J^3}{8I^5}\right\}$$

$$+ \frac{1}{n^3}\left\{\frac{N}{48I^4} + \frac{1}{I^5}\left[\frac{17JM}{48} + \frac{11KL}{16}\right] + \frac{1}{I^6}\left[\frac{55JK^2}{12} + \frac{25J^2L}{8}\right] + \frac{135J^3K}{8I^7} + \frac{175J^5}{16I^8}\right\}$$

$$+ \frac{1}{n^4}\left\{\frac{Q}{384I^5} + \frac{1}{I^6}\left[\frac{7JP}{96} + \frac{19KN}{96} + \frac{307LM}{960}\right]\right.$$

$$+ \frac{1}{I^7}\left|\frac{1247JL^2}{384} + \frac{447K^2L}{96} + \frac{119JKM}{24} + \frac{617J^2N}{576}\right|$$

$$+ \frac{1}{I^8}\left[\frac{1365JK^3}{48} + \frac{34111J^2KL}{576} + \frac{2009J^3M}{192}\right] + \frac{1}{I^9}\left[\frac{29855J^3K^2}{144} + \frac{27559J^4L}{384}\right]$$

$$\left. + \frac{21805}{64}\frac{J^5K}{I^{10}} + \frac{19005}{128}\frac{J^7}{I^{11}}\right\} + 0\left(\frac{1}{n^5}\right) \qquad (5)$$

$$\underset{se}{E}\left[(\hat{\theta} - \theta)^2\right] = \frac{1}{nI} + \frac{1}{n^2I^4}\left[IK + \frac{11J^2}{4}\right]$$

$$+ \frac{1}{n^3}\left\{\frac{M}{4I^4} + \frac{1}{I^5}\left[\frac{85JL}{24} + \frac{31K^2}{12}\right] + \frac{95J^2K}{4I^6} + \frac{175J^4}{8I^7}\right\}$$

$$+ \frac{1}{n^4}\left\{\frac{P}{24I^5} + \frac{1}{I^6}\left[\frac{49JN}{48} + \frac{19KM}{8} + \frac{299L^2}{192}\right]\right.$$

$$+ \frac{1}{I^7}\left[\frac{583JKL}{12} + \frac{617J^2M}{48} + \frac{35K^3}{3}\right] + \frac{1}{I^8}\left[\frac{32095J^2K^2}{144} + \frac{9877J^3L}{96}\right] + \frac{26005J^4K}{48I^9}$$

$$\left. + \frac{19005J^6}{64I^{10}}\right\} + 0\left(\frac{1}{n^5}\right) \qquad (6)$$

$$E_{se}[(\hat{\theta} - \theta)^3] = \frac{7J}{2n^2I^3} + \frac{1}{n^3}\left\{\frac{15L}{8I^4} + \frac{79JK}{4I^5} + \frac{30J^3}{I^6}\right\}$$

$$+ \frac{1}{n^4}\left\{\frac{7N}{16I^5} + \frac{1}{I^6}\left[\frac{147JM}{16} + \frac{277KL}{16}\right] + \frac{1}{I^7}\left[\frac{1085JK^2}{8} + \frac{3013J^2L}{32}\right] + \frac{9205J^3K}{16I^8}\right.$$

$$\left. + \frac{13405J^5}{32I^9}\right\} + O(\frac{1}{n^5}) \qquad (7)$$

$$E_{se}[(\hat{\theta} - \theta)^4] = \frac{3}{n^2I^2} + \frac{1}{n^3}\left\{\frac{9K}{I^4} + \frac{65J^2}{2I^5}\right\}$$

$$+ \frac{1}{n^4}\left\{\frac{7M}{2I^5} + \frac{1}{I^6}\left[\frac{237JL}{4} + \frac{85K^2}{2}\right] + \frac{1385J^2K}{3I^7} + \frac{7805J^4}{16\ I^8}\right\} + O(\frac{1}{n^5}) \qquad (8)$$

The fact that these formulae were derived by both of us working independently almost certainly means that they are correct. However, we also checked them out using a simple example. We considered the problem of estimating θ from an $N(0,\frac{1}{\theta})$ distribution for which $\hat{\theta} = \frac{n}{\sum_i x_i^2}$ of course. From the theory of the χ^2 distribution we know that $E(\hat{\theta}) = \frac{n\theta}{(n-2)}$ exactly, and so from this we get

$$E_{se}(\hat{\theta} - \theta) = \frac{2\theta}{n} + \frac{4\theta}{n^2} + \frac{8\theta}{n^3} + \frac{16\theta}{n^4} + O(n^{-5})$$

Evaluating $E_{se}(\hat{\theta} - \theta)$ from (5) gives precisely the same answer, despite the fact that the individual elements like P and Q are non zero (in fact $P = \frac{-2520}{\theta^8}$ and $Q = \frac{20160}{\theta^9}$). The other three moment formulae check out in a similar way.

Summarising now the moment expansions for the jackknife when r is fixed we find that

$$E_{se}(\hat{\theta}_p - \theta) = -\left(\frac{r}{r-1}\right)\frac{1}{n^2}\left\{\frac{L}{8I^3} + \frac{13JK}{12I^4} + \frac{11J^3}{8I^5}\right\}$$

$$-\left[r\frac{(2r-1)}{(r-1)^2}\right]\frac{1}{n^3}\left\{\frac{N}{48I^4} + \frac{1}{I^5}\left[\frac{17JM}{48} + \frac{11KL}{16}\right]\right.$$

$$\left. + \frac{1}{I^6}\left[\frac{55JK^2}{12} + \frac{25J^2L}{8}\right] + \frac{135J^3K}{8I^7} + \frac{175J^5}{16I^8}\right\}$$

$$-\left[r\frac{(3r^2 - 3r+1)}{(r-1)^3}\right]\frac{1}{n^4}\left\{\frac{Q}{384I^5} + \frac{1}{I^6}\left[\frac{7JP}{96} + \frac{19KN}{96} + \frac{307LM}{960}\right]\right.$$

$$+ \frac{1}{I^7}\left[\frac{1247JL^2}{384} + \frac{447K^2L}{96} + \frac{119JKM}{24} + \frac{617J^2N}{576}\right]$$

$$+ \frac{1}{I^8}\left[\frac{1365JK^3}{48} + \frac{34111J^2KL}{576} + \frac{2009J^3M}{192}\right] + \frac{1}{I^9}\left[\frac{29855J^3K^2}{144} + \frac{27559J^4L}{384}\right] +$$

$$\left. + \frac{21805J^5K}{64\ I^{10}} + \frac{19005J^7}{128\ I^{11}}\right\} + 0\left(\frac{1}{n^5}\right) \qquad (9)$$

$$E_{se}\left[(\hat{\theta}_p - \theta)^2\right] = \frac{1}{nI} + \frac{1}{n^2}\left(\frac{r}{r-1}\right)\frac{J^2}{2I^4}.$$

$$-\frac{1}{n^3}\left(\frac{r}{r-1}\right)\left\{\frac{M}{4I^4} + \frac{1}{I^5}\left[\frac{13K^2}{6} + \frac{35JL}{12}\right] + \frac{203J^2K}{12I^6} + \frac{55J^4}{4I^7}\right\}$$

$$+ \frac{1}{n^3}\frac{r}{(r-1)^2}\left\{\frac{JL}{2I^5} + \frac{4J^2K}{I^6} + \frac{9J^4}{2I^7}\right\}$$

$$+ \frac{r(r^2-2)}{n^3(r-1)^3}\left\{\frac{K^2}{6I^5} + \frac{J^2K}{I^6} + \frac{3J^4}{2I^7}\right\} + 0\left(\frac{1}{n^4}\right) \qquad (10)$$

$$E_{se}\left[(\hat{\theta}_p - \theta)^3\right] = \frac{2J}{n^2I^3} + \frac{r(r-2)}{n^3(r-1)^2}\ \frac{J^3}{I^6}$$

$$-\frac{1}{n^3}\left(\frac{r}{r-1}\right)\left\{\frac{3L}{8I^4} + \frac{JK}{4I^5} - \frac{15J^3}{8I^6}\right\} + 0\left(\frac{1}{n^4}\right) \qquad (11)$$

$$E_{se}\left[(\hat{\theta}_p - \theta)^4\right] = \frac{3}{n^2I^2} + \frac{3K}{n^3I^4} + \frac{3(5r-4)J^2}{n^3(r-1)I^5} + 0\left(\frac{1}{n^4}\right) \qquad (12)$$

Again, both of us produced these results working independently.

Note the relationship between the bias expansions for $\hat{\theta}$ and $\hat{\theta}_p$. The coefficient of n^{-r} in the bias of $\hat{\theta}_p$ is the same as that of n^{-r} in the bias of $\hat{\theta}$ except that it is multiplied by a function of r. For example in the case of n^{-4} the function of r is simply $-\frac{r(3r^2 - 3r + 1)}{(r-1)^3}$. There are no such obvious relationships for the other three moments. What can be said of them is that to order n^{-3} all of the distinct elements in each series results for $\hat{\theta}$ (like $\frac{J^2}{n^2 I^4}$ and $\frac{J^2 K}{n^3 I^6}$) are present in the corresponding results for $\hat{\theta}_p$ but take different coefficients. When r is set equal to n and n is not too small, the <u>absolute</u> values of these coefficients for $\hat{\theta}_p$ are invariably smaller than those for $\hat{\theta}$, often very much smaller, which augers well for the second and third moments of $\hat{\theta}_p$, at least.

In our earlier work we noted that whether we fixed r or s for $\hat{\theta}_p$ did make a difference to the final bias and variance formulae to order n^{-2}, though the actual value that we used for fixed s did not. However, the value used for fixed s does appear in the formulae for $\hat{\theta}_p$ when the order is raised to n^{-3} and the density belongs to this special Koopman class (presumably in general too). On retracing the algebraic steps using fixed s instead of fixed r, we find for example that

$$E_{se}(\hat{\theta}_p - \theta)^2 = \frac{1}{nI} + \frac{J^2}{2n^2 I^4} - \frac{1}{n^3}\left\{\frac{1}{I^4}\left(\frac{M}{4} - \frac{sJ^2}{2}\right) + \frac{1}{I^5}\left(2K^2 + \frac{35JL}{12}\right) + \frac{191J^2K}{12I^6} + \frac{49J^4}{4I^7}\right\} \qquad (13)$$

which means incidentally that s = 1 minimises the mean squared error to order n^{-3}. However, comparing this result with that for fixed r at (10) we find that there is no need for a separate derivation. The reader will note that if we substitute $\frac{n}{s}$ for r in (10) and retain terms of up to n^{-3} taking s to be fixed then we get the result for fixed s at (13), and this is what we would have hoped for. All of the other formulae for fixed s can be deduced in the same way.

Finally we have some extended series results for the moments of two of the standard estimates of the variances of $\hat{\theta}$ and $\hat{\theta}_p$, which we have previously denoted by $\hat{V}_1$ and $\hat{V}_2$. (Tukey's estimate S_T^2 proved much less tractable so we abandoned it). To order n^{-4} we have concluded that in this special Koopman class

$$E_{se}(\hat{V}_1) = \frac{1}{nI} + \frac{1}{n^2}\left\{\frac{K}{2I^3} + \frac{3J^2}{2I^4}\right\} + \frac{1}{n^3}\left\{\frac{M}{8I^4} + \frac{1}{I^5}\left(\frac{41JL}{24} + \frac{5K^2}{4}\right)\right.$$
$$\left. + \frac{275J^2K}{24I^6} + \frac{85J^4}{8I^7}\right\} + \frac{1}{n^4}\left\{\frac{P}{48I^5} + \frac{1}{I^6}\left(\frac{JN}{2} + \frac{9KM}{8} + \frac{35L^2}{48}\right)\right.$$
$$+ \frac{1}{I^7}\left(\frac{49J^2M}{8} + \frac{545JKL}{24} + \frac{65K^3}{12}\right) + \frac{1}{I^8}\left(\frac{2485J^2K^2}{24} + \frac{385J^3L}{8}\right) + \frac{4025J^4K}{16I^9}$$
$$\left. + \frac{2205J^6}{16\ I^{10}}\right\} + 0(\frac{1}{n^5}) \qquad (14)$$

with

$$E_{se}(\hat{V}_1 - \frac{1}{nI})^2 = \frac{J^2}{n^3I^5} + \frac{1}{n^4}\left\{\frac{1}{I^6}\left(\frac{3K^2}{4} + JL\right) + \frac{27J^2K}{2I^7} + \frac{75J^4}{4I^8}\right\} + 0(\frac{1}{n^5}) \qquad (15)$$

The formula for $E_{se}(\hat{V}_2)$ when r is fixed on the other hand has more complicated coefficients which generally depend on r. We think that there is very little point in condensing the following form for it any further:

$$E_{se}(\hat{V}_2) = \frac{1}{nI} + \frac{1}{n^2}\left\{\frac{K}{2I^3} + \frac{J^2}{I^4}\right\} + \frac{1}{n^3}\left\{\frac{M}{8I^4} + \frac{1}{I^5}\left(\frac{4JL}{3} + \frac{3K^2}{4}\right)\right.$$
$$\left. + \frac{13J^2K}{2I^6} + \frac{5J^4}{I^7}\right\} - \frac{r}{(r-1)n^3}\left\{\frac{JL}{8I^5} + \frac{5J^2K}{6I^6} + \frac{7J^4}{8I^7}\right\}$$
$$+ \frac{1}{n^4}\left\{\frac{P}{48I^5} + \frac{1}{I^6}\left(\frac{5JN}{12} + \frac{3KM}{4} + \frac{5L^2}{12}\right) + \frac{1}{I^7}\left(\frac{85J^2M}{24} + \frac{71JKL}{6} + \frac{21K^3}{8}\right)\right.$$
$$\left. + \frac{1}{I^8}(42J^2K^2 + 20J^3L) + \frac{161J^4K}{2I^9} + \frac{35J^6}{I^{10}}\right\}$$
$$- \frac{r}{(r-1)n^4}\left\{\frac{1}{I^6}\left(\frac{KM}{8} + \frac{L^2}{16}\right) + \frac{1}{I^7}\left(\frac{13K^3}{12} + \frac{15JKL}{8} + \frac{J^2M}{4}\right)\right.$$

$$+ \frac{1}{I^8}\left(\frac{87J^2K^2}{8} + \frac{143J^3L}{48}\right) + \frac{133J^4K}{6I^9} + \frac{95J^6}{8I^{10}} \Big\}$$

$$+ \frac{r(r^2-2)}{n^4(r-1)^3}\left\{\frac{K^3}{12I^7} + \frac{2J^2K^2}{3I^8} + \frac{7J^4K}{4I^9} + \frac{3J^6}{2I^{10}}\right\}$$

$$- \frac{r}{n^4(r-1)^2}\Big\{\frac{JN}{16I^6} + \frac{1}{I^7}\left(\frac{17J^2M}{16} + \frac{29JKL}{16}\right) + \frac{1}{I^8}\Big(\frac{47J^2K^2}{4}$$

$$+ \frac{71J^3L}{8}\Big) + \frac{355J^4K}{8I^9} + \frac{453J^6}{16I^{10}}\Big\}$$

$$- \frac{r(r-2)}{n^4(r-1)^2}\Big\{\frac{JN}{24I^6} + \frac{1}{I^7}\left(\frac{17J^2M}{24} + \frac{11JKL}{8}\right) + \frac{1}{I^8}\left(\frac{55J^2K^2}{6} + \frac{73J^2L}{12}\right)$$

$$+ \frac{131J^4K}{4I^9} + \frac{167J^6}{8I^{10}}\Big\} + \frac{(5r-4)}{n^4(r-1)}\Big\{\frac{J^2M}{8I^7} + \frac{1}{I^8}\left(J^3L + \frac{3J^2K^2}{4}\right)$$

$$+ \frac{9J^4K}{2I^9} + \frac{3J^6}{I^{10}}\Big\} + 0\left(\frac{1}{n^5}\right) \qquad (16)$$

In this case

$$E_{se}(\hat{V}_2 - \frac{1}{nI})^2 = \frac{J^2}{n^3I^5} + \frac{r\;\;J^4}{2n^4(r-1)\;I^8} + \frac{1}{n^4}\Big\{\frac{1}{I^6}\left(\frac{3K^2}{4} + JL\right)$$

$$+ \frac{11J^2K}{I^7} + \frac{13J^4}{I^8}\Big\} + 0\left(\frac{1}{n^5}\right) \qquad (17)$$

3. An Illustration

First we look at the series for bias in $\hat{\theta}$ and $\hat{\theta}_p$ when the sample is drawn from our special beta distribution. The value of the coefficient of n^{-i} in the bias of $\hat{\theta}$, α_i say, is set out in table 1 for each $i \leq 4$ and $\theta = 0, 1 \ldots 4$. These coefficients are not large, damp out quickly as we raise the value of i, and increase in size with the value of θ. In most cases the ratio $\frac{\alpha_{i+1}}{\alpha_i} \simeq \frac{1}{2}$. Plotting the function $\alpha_i(\theta)$ for several values of θ shows that it is very nearly linear for each value of i. ($\alpha_1(\theta) = \frac{\theta}{2} + \frac{3}{4}$ can actually be proven algebraically for large values of θ). For fixed r the corresponding coefficients for the jackknife, $\{\alpha_i^*\}$, are given by $\alpha_1^* = 0$, $\alpha_2^* = -\frac{r\alpha_2}{(r-1)}$, $\alpha_3^* = -\frac{r(2r-1)\alpha_3}{(r-1)^2}$ and

$$\alpha_4^* = -\frac{r(3r^2 - 3r + 1)\alpha_4}{(r-1)^3},$$ and all of them are decreasing

functions of r. Evidently when r is large $\alpha_2^* \simeq -\alpha_2$, $\alpha_3^* \simeq -2\alpha_3$ and $\alpha_4^* \simeq -3\alpha_4$. Some numerical values for the bias in $\hat{\theta}$ and $\hat{\theta}_p$ are given in table 2 for selected values of θ and n. In each case we fix r = n for $\hat{\theta}_p$ and this is optimal. They show that only when n is very small would we need to go beyond the approximation to order n^{-2} (previous simulations support the levels of bias indicated by the levels in the Table), and that $\hat{\theta}_p$ is virtually unbiased for $n \geq 10$ as we have noted before. The series results for fixed s = 1 are very similar to those for fixed r = n, and optimal for choice of s.

Moving on now to the second order moments, we find the value of the coefficient β_m of n^{-m} in $V_{se}(\hat{\theta})$ and γ_m the corresponding coefficient for $E_{se}(\hat{\theta} - \hat{\theta})^2$ set out in table 3 for $m \leq 4$ and selected values of θ. These coefficients (which appear to be roughly quadratic in θ for each m) certainly increase with m but not dramatically so, and this leads us to expect fairly stable series for second moments unless n is minute (or θ massive). Table 4 which contains some series calculations shows this conjecture to be true. There is little difference between the second and fourth order results when $n \geq 10$, and the second order refinement for $\hat{\theta}$ is scarcely needed for n > 50. The results for $\hat{\theta}_p$ with fixed r = n show a similar trend and here the Cramer-Rao bound is approached even faster. Jackknifing evidently leads to useful gains in mean squared error here. especially when n is relatively small, as we have remarked before. Note that like the bias the size of the second order moments increase with θ. Again series results for fixed s = 1 are optimal for choice of s and very similar to those for fixed r = n. Simulated values that we have outlined in our earlier work seem to be very much in line with values from these higher order series even when n is small.

The series coefficients for the third order moments are much larger than those for the bias and variance and furthermore increase much faster, which suggests less stable expansions. However, the addition of the n^{-3} term to the n^{-2} term in $E_{se}(\hat{\theta} - \theta)^3$ brings the series results much closer to the simulated values (described in our earlier work) and the addition of the n^{-4} term closer still as we can see from table 5. From this table (where we have fixed r = n) it is equally clear that the addition of the term of order n^{-3} effectively closes the gap between the simulated value for $E(\hat{\theta}_p - \theta)^3$ and the series result to order n^{-2}. This reinforces out previous conclusion that $E(\hat{\theta} - \theta)^3 \approx 2E(\hat{\theta}_p - \theta)^3$. Again, fixing r = n minimises the third order moment and gives very similar answers to those for fixed s = 1.

Just as we might expect the series approximations for the fourth order moment are less satisfactory than those for the third, and this can be seen in table 6 where we have fixed r = n. Adding the terms of order n^{-3} and n^{-4} to that of order n^{-2} certainly successively narrows the gap between $E_{se}(\hat{\theta} - \theta)^4$ and the simulated value, but it looks as if the term in n^{-5} will be needed to bridge the remainder for the lower values of n. It seems also that at least one further term in $E_{se}(\hat{\theta}_p - \theta)^4$ is necessary to match it with its simulated result, but judging by previous patterns it may be that the term in n^{-5} is less critical than it is for $\hat{\theta}$.

Summarising then, this special beta distribution is a case where the bias and mean squared error of $\hat{\theta}$ and $\hat{\theta}_p$ can often be adequately approximated by the series formula to order n^{-2}. When n is very small the addition of the term of order n^{-3} will usually give an acceptable approximation to the exact value of the moment. The third order moment is often considerably improved by the addition of the term in n^{-3}, though for very small samples the term in n^{-4} may also be needed. As for the fourth moment, the approximation to order n^{-3} will usually be inadequate unless θ is small and n large. The addition of the n^{-4} term will always help but it seems that for $\hat{\theta}$ at least the n^{-5} term is needed in the more extreme cases. Whether the improvement that the jackknife brings in mean squared error and other moments is worth more than the sufficiency that it destroys the reader will have already decided for her(him)self.

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References

Ferguson, R.A. and Fryer, J.G. (1973), Jackknifing maximum likelihood estimates in the case of a single parameter. North Carolina Institute of Statistics Mimeo Series No.

Table 2

Some Numerical Comparisons of the Bias in $\hat{\theta}$ and $\hat{\theta}_p$

Value of θ	Sample Size (n)	Bias to order n^{-1}		Bias to order n^{-2}		Bias to order n^{-3}		Bias to order n^{-4}	
		$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}_p$
0	6	0.1200	0	0.1298	- 0.0117	0.1305	- 0.0138	0.1306	- 0.0141
	10	0.0720	-	0.0755	- 0.0039	0.0757	- 0.0043	0.0757	- 0.0043
	20	0.0360	-	0.0369	- 0.0009	0.0369	- 0.0010	0.0369	- 0.0010
	50	0.0144	-	0.0145	- 0.0001	0.0145	- 0.0001	0.0145	- 0.0001
2	6	0.2912	-	0.3150	- 0.0286	0.3170	- 0.0339	0.3172	- 0.0347
	10	0.1747	-	0.1833	- 0.0095	0.1837	- 0.0106	0.1838	- 0.0106
	20	0.0874	-	0.0895	- 0.0023	0.0896	- 0.0024	0.0896	- 0.0024
	50	0.0349	-	0.0353	- 0.0004	0.0353	- 0.0004	0.0353	- 0.0004
4	6	0.4582	-	0.4961	- 0.0453	0.4993	- 0.0539	0.4995	- 0.0550
	10	0.2749	-	0.2886	- 0.0152	0.2892	- 0.0168	0.2893	- 0.0169
	20	0.1375	-	0.1409	- 0.0036	0.1410	- 0.0038	0.1410	- 0.0038
	50	0.0550	-	0.0555	- 0.0006	0.0555	- 0.0006	0.0555	- 0.0006

Table 3

The coefficients of n^{-r} in $E_{se}(\hat{\theta} - \theta)^2$ and $V_{se}(\hat{\theta})$

Value of θ	Coefficient						
	β_1 and γ_1	β_2	γ_2	β_3	γ_3	β_4	γ_4
0	0.8000	1.9200	2.4384	2.6851	3.1910	3.2240	3.5880
1	2.7692	5.9040	7.4481	8.1788	9.6766	9.7396	10.8695
2	5.7600	11.8923	14.9450	16.4332	19.4309	19.4821	21.7414
3	9.7561	19.8862	24.9427	27.4352	32.4335	32.4777	36.2392
4	14.7541	29.8827	37.4412	41.1861	48.6848	48.7251	54.3628

Note The β's are for $V_{se}(\hat{\theta})$ and the γ's for $E_{se}(\hat{\theta} - \theta)^2$

Table 4

The Second Order Moments of $\hat{\theta}$ and $\hat{\theta}_p$ to Various Orders

Value of θ	Sample Size (n)	Variance to n^{-1}	Variance to n^{-2}		Variance to n^{-3}		M.S.E. to n^{-2}	M.S.E. to n^{-3}	M.S.E. to n^{-3}	Variance to n^{-4}
		$\hat{\theta}$ and $\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}$	$\hat{\theta}$	$\hat{\theta}$
0	6	0.1333	0.1867	0.1679	0.1991	0.1790	0.2011	0.2158	0.2186	0.2016
	10	0.0800	0.0992	0.0915	0.1019	0.0935	0.1044	0.1076	0.1079	0.1022
	20	0.0400	0.0448	0.0427	0.0451	0.0429	0.0461	0.0465	0.0465	0.0452
	50	0.0160	0.0168	0.0164	0.0168	0.0164	0.0170	0.0170	0.0170	0.0168
2	6	0.9600	1.2903	1.1635	1.3664	1.2172	1.3751	1.4651	1.4819	1.3815
	10	0.5760	0.6949	0.6438	0.7114	0.6531	0.7255	0.7449	0.7471	0.7133
	20	0.2880	0.3177	0.3041	0.3198	0.3050	0.3254	0.3278	0.3279	0.3199
	50	0.1152	0.1200	0.1177	0.1201	0.1177	0.1212	0.1213	0.1213	0.1201
4	6	2.4590	3.2891	2.9629	3.4798	3.0936	3.4990	3.7244	3.7664	3.5174
	10	1.4754	1.7742	1.6434	1.8154	1.6658	1.8498	1.8985	1.9039	1.8203
	20	0.7377	0.8124	0.7775	0.8176	0.7798	0.8313	0.8374	0.8377	0.8179
	50	0.2951	0.3070	0.3013	0.3074	0.3014	0.3101	0.3104	0.3105	0.3074

Table 5

The Third Moments about θ to Different Orders

Value of θ	Sample Size (n)	To Order n^{-2}		To Order n^{-3}		To Order n^{-4}	Simulated Values	
		$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}$	$\hat{\theta}_p$
0	6	0.1120	0.0640	0.1850	0.0938	0.2125	0.2194	0.0967
	10	0.0403	0.0230	0.0561	0.0293	0.0597	0.0580	0.0292
	20	0.0101	0.0058	0.0120	0.0065	0.0123	0.0124	0.0068
	50	0.0016	0.0009	0.0017	0.0010	0.0017	0.0017	0.0010
2	6	1.9569	1.1182	3.0968	1.5625	3.5169	3.6324	1.6674
	10	0.7045	0.4026	0.9507	0.4956	1.0051	1.0983	0.5663
	20	0.1761	0.1006	0.2069	0.1120	0.2103	0.2249	0.1231
	50	0.0282	0.0161	0.0301	0.0168	0.0302	0.0331	0.0195
4	6	7.8872	4.5070	12.4280	6.2637	14.0953	14.7205	6.6860
	10	2.8394	1.6225	3.8202	1.9903	4.0363	4.0975	2.1100
	20	0.7099	0.4056	0.8325	0.4505	0.8460	0.9355	0.5393
	50	0.1136	0.0649	0.1214	0.0677	0.1218	0.1164	0.0631

Table 6

Fourth Moments about θ to Various Orders

Value of θ	Sample Size (n)	To Order n^{-2}	To Order n^{-3}		To Order n^{-4}	Simulated Values	
		$\hat{\theta}$ and $\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}$	$\hat{\theta}$	$\hat{\theta}_p$
0	6	0.0533	0.1941	0.1369	0.3105	0.4250	0.2602
	10	0.0192	0.0496	0.0368	0.0647	0.0671	0.0452
	20	0.0048	0.0086	0.0070	0.0095	0.0098	0.0075
	50	0.0008	0.0010	0.0009	0.0010	0.0010	0.0009
2	6	2.7648	8.8752	6.3537	13.4974	15.9551	9.1698
	10	0.9953	2.3152	1.7518	2.9142	3.5496	2.3980
	20	0.2488	0.4138	0.3418	0.4513	0.5072	0.3966
	50	0.0398	0.0504	0.0457	0.0513	0.0550	0.0492
4	6	18.1403	57.1623	40.9915	86.3987	107.9776	63.5952
	10	6.5305	14.9593	11.3474	18.7483	21.4185	14.9475
	20	1.6326	2.6862	2.2250	2.9230	3.3721	2.6946
	50	0.2612	0.3287	0.2988	0.3347	0.3317	0.3004