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A STOCHASTIC MODEL FOR HUMAN FERTILITY, I

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1. Introduction and Summary

Fertility as the positive element in the vital process is largely responsible for the current rapid acceleration in the growth of human populations. Implementation of family planning programs for the purpose of altering fertility behavior is becoming a popular course of population policy. Awareness of this situation has led to the frequent utilization of the tools of mathematics and statistics to investigate the fertility process. A number of mathematical models has been proposed for this purpose. The main interest in these models is twofold: the first is to develop techniques, as well as schemes of data collection, necessary to detect and evaluate the significance of changes that take place in fertility; the second is to enhance the understanding of the fertility process by identifying the nature of the interactions of various factors and fertility.

The nature of human fertility suggests four basic variables that should be considered for inclusion in the fertility models:

1. The effective reproductive period, i.e., the age span during which reproduction is possible is the first variable considered. We will assume this period begins at a woman's age of marriage or her age of menarche whichever occurs last, and ends at her age of death, sterility, divorce, widowhood, or menopause whichever occurs first. The finite nature of the effective reproductive period implies that the observed fertility behavior is a truncated sample of the behavior that would have been observed if that period were infinite.

2. Fecundability, as originally defined by Gini, (1924) is also considered. This concept of fecundability is hereafter referred to as conditional fecundability. Conditional fecundability is defined as the probability of conception per unit time conditional on a woman being susceptible, and at the risk of conception, at the beginning of that time unit. The fact that being susceptible at any point of time is a random event, gives rise to another version of

fecundability hereafter referred to as unconditional fecundability. Unconditional fecundability is defined as the probability of conception per time unit.

3. The third variable considered is the various outcomes of pregnancy which depend upon the probability of a conception ending in a given outcome.

4. The duration of nonsusceptible periods, associated with various pregnancy outcomes, during which another conception is impossible is the last variable considered for inclusion in fertility models. This duration consists of two adjacent periods--a pregnancy period and a postpartum nonsusceptible period. The length of these periods depend on the pregnancy outcome.

Several models have been proposed to study the mode of change in these variables and its relation to fertility performance. These models can be dichotomized according to whether or not they conform to the definition of a renewal process. The major portion of the literature treats human fertility as a renewal process (e.g. Henry 1953, Singh 1963, 1964, Perrin & Sheps 1964) since the theory of renewal processes is well developed and the underlying mathematical analysis is more tractable. However, the incongruity of the assumptions underlying human reproduction with those of a renewal process introduces inadequacies that will make the analysis inconclusive. To treat human reproduction as a renewal process implies that the basic variables underlying the process are considered independent of age, parity and marriage duration. Further, since results for instantaneous time in a renewal process are not available in simple forms, treating human reproduction as a renewal process usually assumes that the reproductive period is sufficiently long such that a state of equilibrium can be achieved.

In the present paper, it is recognized that age and parity are the most important variables associated with the basic variables underlying fertility. Human fertility is thus described as a continuous time (age) discrete state (parity) stochastic process. Such a description is considered in relation to two types of marriage cohorts. The first is a time-age marriage cohort, referring to a group of women who marry at the same age at the same time point. The second is a time marriage cohort, referring to a group of women who marry at the same time. The relationship between a time marriage cohort and a time-age marriage cohort which is derived from the age distribution at marriage will be utilized in subsequent work in studying the effect on fertility of changes in the pattern of input to the marriage system.

2. Definitions and Assumptions: In this study, we shall follow the convenient practice of assigning a couple's reproductive performance to its female number. In this section we adopt the following formal definitions and assumptions as the basis for the remainder of this work. So far as possible, all definitions are consistent with their usual meanings prevailing in current literature.

DEFINITIONS

2.1 *FERTILITY* is the actual reproductive performance of an individual or a group of individuals.

2.2 *FECUNDITY* is the capacity of an individual to participate in the reproductive process. The lack of this capacity is called *infecundity*.

2.3 The *REPRODUCTIVE PERIOD* is the age span during which a woman can normally reproduce.

2.4 *TERMINAL EVENTS* are those which cause a woman to become incapable of participating in the reproductive process. These include sterility, divorce and widowhood.

2.5 *CONCEPTION* is the fertilization of an ovum *and* implantation of the resulting zygote in the uterine wall.

2.6 The *EFFECTIVE REPRODUCTIVE PERIOD* is the portion of the reproductive period which begins with marriage and ends in a terminal event or death. This is the period during which a woman can conceive.

2.7 *CONDITIONAL FECUNDABILITY* is the probability that a non-contracepting woman will conceive over a time interval given that she is fecund at the beginning of that time interval. This probability is usually referred to in the literature as *natural fecundability*.

2.8 *UNCONDITIONAL FECUNDABILITY* is the probability that a non-contracepting woman will conceive over a time interval. This is a new version of fecundability introduced in this study and is the product of natural fecundability and the probability of being fecund at the beginning of a time interval.

2.9 *LIVE BIRTH* is a confinement at which at least one live-born infant is delivered.

2.10 *FETAL LOSS* refers to a pregnancy terminating in any outcome other than a live birth.

2.11 The *NONSUSCEPTIBLE PERIOD* is the total time period during which ovulation is suppressed following a conception. This is a period of temporary infecundity.

2.12 *PARITY* is the number of live births a woman has had.

2.13 *TIME-MARRIAGE COHORT* is a group of women who marry at the same point in time.

2.14 *TIME-AGE MARRIAGE COHORT* is a group of women who marry at the same age and at the same point in time.

ASSUMPTIONS

2.15 The fertility histories of different members of a group are mutually independent.

2.16 Fecundability is a function of age and parity.

2.17 The length of nonsusceptible periods associated with a conception is a function of parity only and not of age. Although age is thought to have an effect on the length of these periods, this effect is ignored for simplicity. This action appears to be justified by available results obtained by simulation (Venkatacharya, 1969) indicating that this variable (length of nonsusceptible periods) is less important than other variables in determining the reproductive performance.

2.18 For any conception leading to a live birth, the associated gestation period is assumed constant.

2.19 Any conception ends in either a live birth or a fetal loss. No attempt is made to consider the different outcomes of a fetal loss.

2.20 The incidence of fetal loss is a function of age and parity.

2.21 Terminal events are treated as a function of age.

2.22 Multiple births are treated as one birth.

2.23 Fertility performance can change with calendar time. The specific pattern of a woman's reproductive performance is related to the calendar date of marriage.

3. The Stochastic Model. We shall, regrettably introduce in this section a good deal of notation. For this we now apologize. Let the random variables X_0 and T denote the age and calendar time respectively at which an individual or a group of individuals marries. The probabilities discussed in this section shall be conditioned upon the events $[X_0 = x_0]$ and $[T = \tau]$. For simplicity, explicit reference to such dependence shall, for the most part, be suppressed. Bearing this in mind, we shall let $p_{i,j}(x,y)$ denote the conditional probability that a woman is at parity j at age y given that she is at parity i at age x. In this section we desire to relate this probability to quantities which are either estimable, measurable, or for which a reasonable mathematical form may be assumed. Three basic probabilities are assumed to be quantities in this category:

3.1 $\rho_j(x)\Delta x + o(\Delta x)$ is the conditional fecundability at age x. This is the conditional probability that a woman will conceive exactly once over the interval $(x,x+\Delta x)$ given that she is fecund at age x and that she is at parity j at age x. Of course, $1-\rho_j(x)\Delta x + o(\Delta x)$ is the conditional probability that she will not conceive and $o(\Delta x)$, that she will conceive more than once.

3.2 $\theta_j(x)$ is the conditional probability that a conception at age x will end in a fetal loss given that the parity at age x is j. For convenience, we assume $\theta_j(x)$ is a continuous function of x.

3.3 $Q_j(x)$ is the conditional probability that a woman is alive, at the risk of, and susceptible to conception at age x given that her parity at age x is j.

We first examine the structure of $Q_j(x)$ in Theorem 3.1. To do this effectively we introduce 4 random variables.

3.4. $$\alpha(x) = \begin{cases} 1 & \text{If a woman is alive at age } x \\ 0 & \text{otherwise.} \end{cases}$$

3.5. $$\beta(x) = \begin{cases} 1 & \text{If a woman is in the effective reproductive state at age } x \\ 0 & \text{otherwise.} \end{cases}$$

3.6. $$\gamma_j(x) = \begin{cases} 1 & \text{If a woman whose parity is } j \text{ is passing through a nonsusceptible period at age } x \text{ following a conception given that the conception leading to the } j{+}1\text{st live birth does not occur before age } x. \\ 0 & \text{otherwise.} \end{cases}$$

3.7. $J(x)$ = the parity at age x

<u>Theorem 3.1</u>: $$Q_j(x) = Q^{(1)}(x) \times Q^{(2)}(x) \times Q_j^{(3)}(x)$$

where

(3.1.i) $$Q^{(1)}(x) = \Pr[\alpha(x) = 1]$$ (can be estimated from a life table)

and

(3.1.ii) $$Q^{(2)}(x) = \Pr[\beta(x)=1 \mid \alpha(x)=1] = q_0 e^{-\int_{x_0}^{x} \mu(t)dt}$$

with q_0 the probability of being in the effective reproductive state at age x_0 and $\mu(t)\Delta t + o(\Delta t)$ the probability that a woman in the effective reproductive state at age t will experience a terminal event (other than death) over $(t, t+\Delta t)$. Here we also assume $Q^{(2)}(x)$ is differentiable, and finally,

(3.1.iii) $$Q_j^{(3)}(x) = \Pr[\gamma_j(x)=0 \mid \alpha(x)=1, \beta(x)=1, J(x)=j]$$

satisfies the differential equation

$$\frac{dQ_j^{(3)}(x)}{dx} = -Q_j^{(3)}(x)\rho_j(x) + Q_j^{(3)}(x-c_{2j})\theta_j(x-c_{2j})\rho_j(x-c_{2j}).$$

Here, of course, we assume $Q_j^{(3)}(x)$ is differentiable and we let c_{2j} be the length of the nonsusceptible period associated with fetal loss while at parity j.

Proof: In terms of the random variables, α, β, γ_j and J,

$$Q_j(x) = \Pr[\alpha(x)=1,\ \beta(x)=1,\ \gamma_j(x)=0|J(x)=j].$$

Hence, that $Q_j(x) = Q^{(1)}(x) \times Q^{(2)}(x) \times Q_j^{(3)}(x)$ is immediate. To prove that $Q^{(2)}(x) = e^{-\int_{x_0}^{x} \mu(t)dt}$, we note that $1 - \mu(t)\Delta t + o(\Delta t)$ is the probability a woman in the effective reproductive state at age t will *not* leave over the interval $(t, t+\Delta t)$. Hence a woman is in the effective reproductive state at age $t+\Delta t$ if she is in the effective reproductive state at age t and she does not leave it over $(t, t+\Delta t)$. That is,

$$Q^{(2)}(t+\Delta t) = Q^{(2)}(t)(1-\mu(t)\Delta t + o(\Delta t)) .$$

Rewriting,

$$\frac{Q^2(t+\Delta t) - Q^{(2)}(t)}{\Delta t} = -Q^{(2)}(t)\mu(t) + \frac{o(\Delta t)}{\Delta t} .$$

Letting $\Delta t \to 0$ yields $dQ^{(2)}(t)/dt = -Q^{(2)}(t)\mu(t)$. Separating variables and integrating,

$$\log Q^{(2)}(x) - \log Q^{(2)}(x_0) = -\int_{x_0}^{x} \mu(t)dt .$$

The initial condition $Q^{(2)}(x_0) = q_0$, yields (3.1.ii).

Finally, noting that $Q_j^{(3)}(x) = Pr[\gamma_j(x)=0|\alpha(x)=1, \beta(x)=1, J(x)=j]$, we obtain

$$Q_j^{(3)}(x+\Delta x) = Pr[\gamma_j(x+\Delta x) = 0|\alpha(x+\Delta x)=1, \beta(x+\Delta x)=1, J(x+\Delta x)=j]$$

(3.1.1)
$$= Pr[\gamma_j(x+\Delta x)=0, \gamma_j(x)=1|\alpha(x+\Delta x)=1, \beta(x+\Delta x)=1, J(x+\Delta x)=j]$$
$$+ Pr[\gamma_j(x+\Delta x)=0, \gamma_j(x)=1|\alpha(x+\Delta x)=1, \beta(x+\Delta x)=1, J(x+\Delta x)=j] .$$

Now the first term on the right-hand side of (3.1.1) is

$$Pr[\gamma_j(x+\Delta x)=0|\gamma_j(x)=0, \alpha(x)=1, \beta(x)=1, J(x)=j]$$

(3.1.2)
$$\times Pr[\gamma_j(x)=0|\alpha(x)=1, \beta(x)=1, J(x)=j]$$
$$= Q_j(x)(1 - \rho_j(x)\Delta x + o(\Delta x)) .$$

The event, $[\gamma_j(x+\Delta x)=0, \gamma_j(x)=1|\alpha(x+\Delta x)=1, \beta(x+\Delta x)=1, J(x+ x)=j]$, appearing in the second term on the right-hand side of (3.1.1) can occur only if a woman was passing through a nonsusceptible period at age x occuring after a fetal loss and then moves to a susceptible state at age $x+\Delta x$. This can occur only when a woman is susceptible at age $x-c_{2j}$, then has a conception ending in fetal loss over the interval $(x-c_{2j}, x-c_{2j} + \Delta x)$. Thus this second term on the R.H.S. of (3.1.1) is

$$Pr[\gamma_j(x-c_{2j})=0|\alpha(x-c_{2j})=1, \beta(x-c_{2j})=1, J(x-c_{2j})=j]$$

$$\times \; Pr[\text{a conception ending in fetal loss over } (x-c_{2j},x-c_{2j}+\Delta x)|J(x-c_{2j})=j] .$$

Symbolically, this can be written

(3.1.3)
$$Q_j^{(3)}(x-c_{2j})[\rho_j(x-c_{2j})\Delta x + o(\Delta x)]\theta_j(x-c_{2j}) .$$

Combining (3.1.2) and (3.1.3) yields

$$Q_j^{(3)}(x+\Delta x) = Q_j^{(3)}(x)[1-\rho_j(x)\Delta x] + Q_j^{(3)}(x-c_{2j})[\rho_j(x-c_{2j})\theta_j(x-c_{2j})\Delta x] + o(\Delta x).$$

Subtracting $Q_j^{(3)}(x)$ from both sides and then dividing through by Δx,

$$\frac{Q_j^{(3)}(x+\Delta x)-Q_j^{(3)}(x)}{\Delta x} = Q_j^{(3)}(x)\rho_j(x) + Q_j^{(3)}(x-c_{2j})\rho_j(x-c_{2j})\theta_j(x-c_{2j}) + \frac{o(\Delta x)}{\Delta x} .$$

Letting $\Delta x \to 0$, we obtain the differential equation in (3.1.iii). Note we may use initial conditions $Q_0^{(3)}(x_0) = 1$ and $Q_j^{(3)}(x_0) = 0$, $j \geq 1$ to determine the solution to this system of differential equations.

We may now relate the quantities in 3.1 and 3.3 to the unconditional fecundability mentioned in 2.8.

3.8. $R_j(x)\Delta x + o(\Delta x)$ is the unconditional fecundability at age x when the parity is j. This is the probability that a woman will conceive exactly once over $(x, x+\Delta x)$ given that she is at parity j at age x. $1-R_j(x)\Delta x + o(\Delta x)$ is the probability she will not conceive and $o(\Delta x)$ the probability of more than one conception.

<u>Lemma 3.2.</u> $$R_j(x) = \rho_j(x)Q_j(x) .$$

<u>Proof</u>: $R_j(x)\Delta x + o(\Delta x) = \Pr[\text{exactly one conception}|J(x) = j]$.

$\rho_j(x)\Delta x + o(\Delta x) = \Pr[\text{exactly one conception}|J(x) = j, \alpha(x) = 1, \beta(x) = 1, \gamma_j(x) = 0]$.

$Q_j(x) = \Pr[\alpha(x) = 1, \beta(x) = 1, \gamma_j(x) = 0, J(x) = j]$.

Therefore,

$$R_j(x)\Delta x + o(\Delta x) = [\rho_j(x)\Delta x + o(\Delta x)] \times Q_j(x) .$$

Dividing through by Δx and then letting $\Delta x \to 0$ yields

$$R_j(x) = \rho_j(x)Q_j(x)$$

which is the desired result.

We now relate the structure of fetal losses to the quantities $R_j(x)$ and $\theta_j(x)$. Let $Z(x,y)$ be the number of consecutive fetal losses over the age interval (x,y). We may define

3.9. $$h_{j,i}(x,y) = Pr[Z(x,y) = i,\ J(y+g) = j \mid J(x+g) = j] .$$

Note that $h_{j,i}(x,y)$ is the probability that a woman will experience i consecutive fetal losses over the interval (x,y) and not yet experience the conception leading to the $(j+1)$st live birth by age y given that she had not yet experienced it by age x. The quantity g is the length of the gestation period which we assume constant. It is somewhat more convenient to express results in terms of the mathematically equivalent probability generating function.

3.10. $$H_j(x,y,s) = \sum_{i=0}^{\infty} h_{j,i}(x,y)s^i$$

Theorem 3.3: If $h_{j,i}(x,y)$ is a differentiable function of y, if $R_j(x)$ and $\theta_j(x)$ are continuous functions of x and if $h_{j,i}(x,y)$ has the limiting properties

(3.3.i) $$h_{j,0}(x,x) = 1 \text{ and } h_{j,i}(x,x) = 0 \text{ for } i \geq 1,$$

then

$$H_j(x,y,s) = \exp\left[\int_x^y \{s\theta_j(t)-1\}R_j(t)dt\right],\ x < y,\ j \geq 0.$$

Proof: We construct a system of differential-difference equations in the usual manner.

(3.3.1) $$h_{j,0}(x,y+\Delta y) = h_{j,0}(x,y)[1-R_j(y)\Delta y + o(\Delta y)]$$

and

$$(3.3.2) \qquad h_{j,i}(x,y+\Delta y) = h_{j,i}(x,y)[1-R_j(y)\Delta y + o(\Delta y)] + h_{j,i-1}(x,y)[R_j(y)\Delta y + o(\Delta y)]\theta_j(y) + o(\Delta y) .$$

The logic behind the construction is relatively simple. The R.H.S. of (3.3.1) and the first term on the R.H.S. of (3.3.2) correspond to the situation where a woman already has experienced i fetal loss over (x,y) and does not conceive again over $(y,y+\Delta y)$. The second term on the R.H.S. of (3.3.2) corresponds to the case that she has experienced $i-1$ fetal losses over (x,y) and another over $(y,y+\Delta y)$. The $o(\Delta x)$ corresponds to fewer than $i-1$ fetal losses over (x,y) and hence more than 1 fetal loss over $(y,y+\Delta y)$.

Subtracting $h_{j,i}(x,y)$ from both sides of (3.3.1) and (3.3.2), dividing through by Δy and passing to the limit as $\Delta y \to 0$, we obtain

$$(3.3.3) \qquad \frac{\partial h_{j,0}(x,y)}{\partial y} = -R_j(y)h_{j,0}(x,y)$$

and

$$(3.3.4) \qquad \frac{\partial h_{j,i}(x,y)}{\partial y} = -R_j(y)h_{j,i}(x,y) + \theta_j(y)R_j(y)h_{j,i-1}(x,y) \qquad i \geq 1 .$$

Multiplying (3.3.4) by s^i and then summing with (3.3.3) yields

$$(3.3.5) \qquad \sum_{i=0}^{\infty} \frac{\partial h_{j,i}(x,y)}{\partial y} s^i = -R_j(y) \sum_{i=0}^{\infty} h_{j,i}(x,y)s^i + s\theta_j(y)R_j(y) \sum_{i=1}^{\infty} h_{j,i-1}(x,y)s^{i-1} .$$

Now since $|h_{j,i}(x,y)s^i| \leq |s^i|$, the infinite series $\sum_{i=0}^{\infty} h_{j,i}(x,y)s^i$ converges uniformly in y for $|s| < 1$. Hence the L.H.S. of (3.3.5) converges uniformly converges uniformly in y for $|s| < 1$, so that

$$\frac{\partial}{\partial y} \sum_{i=0}^{\infty} h_{j,i}(x,y) s^i = \sum_{i=0}^{\infty} \frac{\partial h_{j,i}(x,y)}{\partial y} s^i$$

Combining (3.3.5), (3.3.6) and (3.10), we have

$$\frac{\partial H_j(x,y,s)}{\partial y} = R_j(y)[s\theta_j(y)-1]H_j(x,y,s).$$

With the initial condition, $H_j(x,x,s) = 1$ (from 3.3.i), a unique solution exists if the function $R_j(y)\{s\theta_j(y)-1\}$ is continuous for all y (see Bellman and Cooke 1963, p. 29). This solution is found, by separation of variables,to be:

$$H_j(x,y,s) = \exp\left[\int_x^y \{s\theta_j(t)-1\}R_j(t)dt\right] .$$

We now relate the fetal loss distribution to the parity structure in the following two theorems.

3.11 $$p_{i,j}(x,y) = \Pr[J(y)=j \mid J(x)=i] .$$

That is, $p_{i,j}(x,y)$ is the probability that a woman at parity i at age x will move to parity j by age y.

<u>Theorem 3.4</u>: Under the hypothesis of Theorem 3.3,

$$p_{j,j}(x,y) = \exp\left[\int_{x-g}^{y-g} \{\theta_j(t)-1\}R_j(t)dt\right],\ x < y,\ j \geq 0.$$

<u>Proof</u>: We first note that $H_j(x,y,s)$ is analytic in s on $|s| < 1$. Hence there is a unique analytic extension to $|s| = 1$ and this is given by $H_j(x,y,1) = \sum_{i=0}^{\infty} h_{j,i}(x,y)$. Now the sequence of events, $[Z(x,y) = i]$ form a

disjoint sequence of events so that

$$\sum_{i=0}^{\infty} h_{j,i}(x,y) = \Pr[\{\bigcup_{i=0}^{\infty} [Z(x,y)=i]\} \cap [J(y+g)=j] \,|\, J(x+g)=j]$$

$$= \Pr[J(y+g)=j \,|\, J(x+g)=j] \ .$$

That is,

(3.4.1) $$H_j(x,y,1) = \Pr[J(y+g)=j \,|\, J(x+g)=j] \ .$$

By definition then

$$p_{j,j}(x,y) = H_j(x-g,\ y-g,\ 1) \ ,$$

so that by Theorem 3.3

$$p_{j,j}(x,y) = \exp\left[\int_{x-g}^{y-g} \{\theta_j(t)-1\}R_j(t)dt\right] \ .$$

<u>Theorem 3.5</u>: Under the conditions of Theorem 3.3,

$$p_{i,j}(x,x+1) = \begin{cases} \exp\left[\int_{x-g}^{x+1-g} \{\theta_i(t)-1\}R_i(t)dt\right] & \text{for } j = i \\ 1-q(x,x+1) - \exp\left[\int_{x-g}^{x+1-g} \{\theta_i(t)-1\}R_i(t)dt\right] & \text{for } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$

where $q(x,x+1)$ is the probability that a woman who is alive at the beginning of month x will die during the month $(x,x+1)$.

<u>Proof</u>: Let D be the event that a woman alive at age x dies by age $x+1$.

Let A be the event that a woman alive at age x is still alive at age $x+1$.

Let P_i be the event that a woman at age x is still at parity i at age $x+1$.

Let P_{i+1} be the event that a woman at parity i at age x is at parity $i+1$ at age $x+1$.

By assumption 2.22 the event $D \cup \{A \cap P_i\} \cup \{A \cap P_{i+1}\}$ has probability one. Hence

$$\Pr(A \cap P_{i+1}) = 1 - \Pr(D) - \Pr(A \cap P_i)..$$

By the previous theorem

$$\Pr(A \cap P_i) = \exp\left[\int_{x-g}^{x+1-g} \{\theta_i(t)-1\}R_i(t)dt\right]$$

We shall make use of this result in our discussion in section 5, where age increases are considered to occur in discrete increments.

4. <u>The Truncation Effect</u>. The non-negative quantity, $\{1-\theta_j(x)\}R_j(x)$, can be given the interpretation as the *force of a j-th live birth.* It is clear that any realistic model of fertility would require this force to approach 0 as the age x became large. The rate at which this quantity approaches 0 becomes critical as the following theorem demonstrates.

<u>Theorem 4.1</u>: Under the assumptions of Theorem 3.3 if

$$\int_{x_0}^{\infty} \{1-\theta_j(t)\}R_j(t)dt = \infty$$

then the parity j is a transient state. Moreover if $\int_{x_0}^{\infty} \{\theta_j(t)-1\}R_j(t)dt < \infty$, then a woman achieving parity j at age x_j will remain in parity j with probability $\exp[\int_{x_j}^{\infty} \{\theta_j(t)-1\}R_j(t)dt]$.

Proof: Parity j will be transient if $p_{j,j}(x_j,\infty) = 0$. Since $\{1-\theta_j(t)\}R_j(t)$ is a continuous function, over any finite interval, it is bounded. Hence $\int_{x_0}^{x_j}\{1-\theta_j(t)\}R_j(t)dt < \infty$ implying $\int_{x_j}^{\infty}\{\theta_j(t)-1\}R_j(t)dt = -\infty$. That is $p_{j,j}(x_j,\infty) = 0$. The probability of remaining in parity j is $p_{j,j}(x_j,\infty) = \exp[\int_{x_j}^{\infty}\{\theta_j(t)-1\}R_j(t)dt]$.

Notice, in particular, that if the force of a j-th birth is bounded below by a constant, as for example with models based in renewal theory, the parity j is transient.

In general, a good procedure will be to construct our model so that the force of a j-th live birth is 0 for all ages greater than some fixed age, say w, which is then taken to be the end of the effective reproductive period. Theorem 4.2 gives a measure of disparity due to this "truncation effect."

Theorem 4.2: Let Y_{j+1} denote the age at which conception ending in the (j+1)st live birth occurs. Then

4.2.i Assuming an infinite reproductive period,

$$\Pr[Y_{j+1} > y | J(x+g) = j] = H_j(x,y,1)$$

4.2.ii Assuming a finite reproductive period and considering only those women who will have at least a (j+1)st live birth before the end of that period,

$$\Pr[Y_{j+1} > y | J(x+g) \geq j+1,\ J(x+g) = j] = \frac{H_j(x,y,1)[1-H_j(y,w,1)]}{1-H_j(x,w,1)}, \quad x < y < w.$$

Proof: Noticing that the events $[Y_{j+1}>y]$ and $[J(y+g)=j]$ are equivalent, we have by (3.4.1)

$$\Pr[Y_{j+1}>y \mid J(x+g)=j] = \Pr[J(y+g)=j \mid J(x+g)=j] = H_j(x,y,1) .$$

Consider now 4.2.ii.

Let $\Pr[Y_{j+1}>y \mid J(w+g)\geq j+1,\ J(x+g)=j] = p$, say. Then

$$p = \Pr[J(y+g)=j \mid J(w+g)\geq j+1,\ J(x+g)=j] .$$

By the definition of conditional probability,

$$p = \frac{\Pr[J(y+g)=j,\ J(w+g)\geq j+1 \mid J(x+g)=j]}{P[J(w+g)\geq j+1 \mid J(x+g)=j]} .$$

By Bayes' theorem,

$$p = \frac{\Pr[J(w+g)\geq j+1 \mid J(y+g)=j,\ J(x+g)=j] \times \Pr[J(y+g)=j \mid J(x+g)=j]}{\Pr[J(w+g)\geq j+1 \mid J(x+g)=j]} .$$

Since $\Pr(A|B) = 1 - \Pr(A^c|B)$,

$$p = \frac{\Pr[J(y+g)=j \mid J(x+g)=j]\{1-\Pr[J(w+g)<j+1 \mid J(y+g)=j,\ J(x+g)=j]\}}{1 - \Pr[J(w+g)<j+1 \mid J(x+g)=j]} .$$

Since parity is known to be j at x+g and less than j+1 at w+g, it must be j at w+g also. Thus

$$p = \frac{H_j(x,y,1)\{1-\Pr[J(w+g)=j \mid J(y+g)=j,\ J(x+g)=j]\}}{1 - H_j(x,w,1)} .$$

Now consider

$$\Pr[J(w+g)=j \mid J(y+g)=j,\ J(x+g)=j]$$

$$= \frac{\Pr[J(w+g)=j,\ J(y+g)=j \mid J(x+g)=j]}{\Pr[J(y+g)=j \mid J(x+g)=j]}$$

But $J(w+g)=j$ and $J(x+g)=j$ implies $J(y+g)=j$, $x < y < w$.

$$\Pr[J(w+g)=j \mid J(y+g)=j,\ J(x+y)=j]$$

$$= \frac{\Pr[J(w+g)=j \mid J(x+g)=j]}{\Pr[J(y+g)=j \mid J(x+g)=j]} = \frac{H_j(x,w,1)}{H_j(x,y,1)} .$$

Now since $H_j(x,y,1) = \exp\left[\int_x^y \{\theta_j(t)-1\}R_j(t)dt\right]$,

$$\frac{H_j(x,w,1)}{H_j(x,y,1)} = \exp\left[\int_x^w \{\theta_j(t)-1\}R_j(t)dt - \int_x^y \{\theta_j(t)-1\}R_j(t)dt\right]$$

$$= \exp\left[\int_y^w \{\theta_j(t)-1\}R_j(t)dt\right] = H_j(y,w,1).$$

This completes the proof of Theorem 4.2.

The implication of Theorem 4.2 is then that the ratio $\frac{1-H_j(y,w,1)}{1-H_j(x,w,1)}$ is an adjustment factor for converting the results of a model with no upper bound on the effective reproductive period to one with such a bound .

5. Age-Parity Distribution in Time-Age Marriage Cohorts. The objective of this section is to use the theory developed in Section 3 to derive the moments of the age parity distribution in a time-age marriage cohort. To do this we will introduce explicit notation indicating the dependence of $p_{i,j}(x,y)$ on the events $[X_0 = x_0]$ and $[T = \tau]$. We shall use the notation, $p_{i,j}(x,y \mid x_0,\tau)$, but we shall feel free to supress the x_0,τ condition in proofs where explicit notation is not used.

The method suggested for the derivation of the moments of the age-parity distribution consists of dividing the reproductive period into a finite number of equally spaced age points and then developing a recurrence relation between the parity distribution at any two successive age points. The division of the reproductive period into a finite number of age points is a rather subjective matter and depends on the purpose of the study. Here the reproductive period is divided into monthly intervals which is appropriate since the reproductive process has a cycle with a monthly nature. We now introduce several necessary random variables.

5.1 $n_j(x|x_0,\tau)$ is the random number of women with parity j at the beginning of month x out of all those who marry at x_0 and at time τ.

5.2 $n_{i,j}(x|x_0,\tau)$ is the random number of women married at age x_0 at time τ who move from parity i to parity j over the month $(x,x+1)$.

In both 5.1 and 5.2 we shall suppress the x_0,τ condition in proofs not explicitly requiring that notation.

Lemma 5.1:

$$n_0(x+1|x_0,\tau) = n_{0,0}(x|x_0,\tau)$$

and

$$n_j(x+1|x_0,\tau) = n_{j-1,j}(x|x_0,\tau) + n_{j,j}(x|x_0,\tau)$$

Proof: Immediate from Definitions 5.1 and 5.2 and Assumption 2.22 that multiple births increase parity by one.

Lemma 5.2:

5.2.i $$E\{n_{i,j}(x|x_0,\tau)\} = p_{i,j}(x,x+1|x_0,\tau)E\{n_i(x|x_0,\tau)\} .$$

(Note that under Assumption 2.22, this is zero unless j = i or i+1.)

5.2.ii $\mathrm{Cov}\{n_{i,j}(x|x_0,\tau),\ n_{\ell,k}(x|x_0\tau)\} =$

$$\begin{cases} p_{i,j}(x,x+1|x_0,\tau)p_{\ell,k}(x,x+1|x_0,\tau)\mathrm{Cov}\{n_i(x|x_0,\tau),n_\ell(x|x_0,\tau)\} & i \neq \ell \\ \{\delta_{jk}p_{i,j}(x,x+1|x_0,\tau)-p_{i,j}(x,x+1|x_0,\tau)p_{i,k}(x,x+1|x_0,\tau)\} \\ \qquad E\{n_i(x|x_0,\tau)\} & i = \ell \end{cases}$$

(Note that this is zero unless j=i or i+1 and k=ℓ or ℓ+1.)

where $\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$.

Proof: Two facts are used in this proof:

5.2.1 $$E(X) = E\{E(X|Y)\}$$

and

5.2.2 Given $n_i(x)$ and assuming $n_{i,d}(x)$ is the number of women who die over $(x,x+1)$, the joint distribution of the random variables $n_{i,i}(x)$, $n_{i,i+1}(x)$ and $n_{i,d}(x)$ is trinomial with probabilities $p_{i,i}(x,x+1)$, $p_{i,i+1}(x,x+1)$ and $1-p_{i,i}(x,x+1) - p_{i,i+1}(x,x+1)$.

To prove 5.2.i, by the properties of the trinomial distribution

$$E\{n_{i,j}(x)|n_i(x)\} = n_i(x)p_{i,j}(x,x+1) \ .$$

Taking expectations on both sides,

$$E\{n_{i,j}(x)\} = p_{i,j}(x,x+1)E\{n_i(x)\} \ .$$

For 5.2.ii and when $i \neq \ell$, it similarly follows that

$$E\{n_{i,j}(x)n_{\ell,k}(x)|n_i(x),n_\ell(x)\} = n_i(x)p_{i,j}(x,x+1)n_\ell(x)p_{\ell,k}(x,x+1).$$

Thus

$$E\{n_{i,j}(x),n_{\ell,k}(x)\} = p_{i,j}(x,x+1)p_{\ell,k}(x,x+1)E\{n_i(x)n_\ell(x)\}$$

so that

$$\mathrm{Cov}\{n_{i,j}(x),n_{\ell,k}(x)\} = p_{i,j}(x,x+1)p_{\ell,k}(x,x+1)\mathrm{Cov}\{n_i(x),n_\ell(x)\}.$$

When $i = \ell$, the joint probability distribution of $n_{i,j}(x)$ and $n_{i,k}(x)$ conditioned on $n_i(x)$ is given by

$$f(n_{i,j}(x),n_{i,k}(x)|n_i(x)) =$$

$$\frac{n_i(x)!}{n_{i,j}(x)!n_{i,k}(x)!\{n_i(x)-n_{i,j}(x)-n_{i,k}\}!}\left[p_{i,j}(x,x+1)\right]^{n_{i,j}(x)}\left[p_{i,k}(x,x+1)\right]^{n_{i,k}(x)} \times$$

$$\left[1-p_{i,j}(x,x+1)-p_{i,k}(x,x+1)\right]^{n_i(x)-n_{i,j}(x)-n_{i,k}(x)} .$$

It then follows that

$$\mathrm{Cov}\{n_{i,j}(x),n_{i,k}(x)|n_i(x)\} =$$

$$\begin{cases} -n_i(x)p_{i,j}(x,x+1)p_{i,k}(x,x+1) & j \neq k \\ n_i(x)p_{i,j}(x,x+1)\{1-p_{i,j}(x,x+1)\} & j = k \end{cases}$$

Using δ_{jk} we can rewrite as

$$\mathrm{Cov}\{n_{i,j}(x),n_{i,k}(x)|n_i(x)\} =$$

$$\{\delta_{jk}p_{i,j}(x,x+1) - p_{i,j}(x,x+1)p_{i,k}(x,x+1)\}n_i(x).$$

Taking expectations on both sides furnishes the required result.

<u>Theorem 5.3</u>: (Expected Age-Parity Distribution)

5.3.i $E[n_0(x+1|x_0,\tau)|N^{\tau}_{x_0}] = p_{0,0}(x,x+1|x_0,\tau)E\{n_0(x|x_0,\tau)\}$

and

5.3.ii $E[n_j(x+1|x_0,\tau)|N^{\tau}_{x_0}] = p_{j-1,j}(x,x+1|x_0,\tau)E\{n_{j-1}(x|x_0,\tau)\} +$

$p_{j,j}(x,x+1|x_0,\tau)E\{n_j(x|x_0,\tau)\}$, $j \geq 1$

with the initial conditions

$$E[n_j(x_0|x_0,\tau)|N^{\tau}_{x_0}] = \begin{cases} N^{\tau}_{x_0} & j = 0 \\ 0 & j \neq 0 , \end{cases}$$

where $N^{\tau}_{x_0}$ is the initial size of a cohort marrying at age x_0 at calendar time τ .

Proof: The proof follows by taking expectations on both sides of the expressions in Lemma 5.1 and then applying 5.2.i.

Corollary 5.4:

5.4.i $\quad p_0(x+1|x_0\tau) = p_{0,0}(x,x+1|x_0,\tau)p_0(x|x_0,\tau)$

and

5.4.ii $\quad p_j(x+1|x_0,\tau) = p_{j-1,j}(x,x+1|x_0,\tau)p_{j-1}(x|x_0,\tau) +$

$$p_{j,j}(x,x+1|x_0,\tau)p_j(x|x_0,\tau) \qquad j \geq 1$$

with the initial condition

$$p_j(x_0|x_0,\tau) = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}$$

where $p_j(x|x_0,\tau)$ is the probability that a woman marrying at age x_0 at time τ will have parity j at age x and is defined by the relationship: $E\{n_j(x|x_0,\tau)\} = E\{N^{\tau}_{x_0}\}p_j(x|x_0,\tau)$.

Proof: The proof follows from Theorem 5.3 by replacing $E\{n_j(x|x_0,\tau)|N^{\tau}_{x_0}\}$ with $N^{\tau}_{x_0}p_j(x|x_0,\tau)$ and then taking expectations over $N^{\tau}_{x_0}$.

Corollary 5.5: Let X_j be the age at delivery of the j-th live birth, then

$$Pr\{x<X_j<x+1|x_0\} = p_{j-1}(x|x_0,\tau)p_{j-1,j}(x,x+1|x_0,\tau).$$

Proof: Obvious.

We note here that the probabilities $p_j(x|x_0,\tau)$ and $p_{j-1,j}(x,x+1|x_0,\tau)$ can be evaluated with and without the truncation effect mentioned in Section 4. Consequently the discrete approximation of the probability distribution of X_j, derived in Corollary 5.5 can be calculated with and without this effect, hence also the moments of X_j.

Theorem 5.6: (Covariance Matrix of Age-Parity Distribution).

$$\mathrm{Cov}\{n_j(x+1|x_0,\tau),\ n_k(x+1|x_0,\tau)\} =$$
$$\mathrm{Cov}\{n_{j-1,j}(x|x_0,\tau),\ n_{k-1,k}(x|x_0,\tau)\} + \mathrm{Cov}\{n_{j-1,j}(x|x_0\tau),\ n_{k,k}(x|x_0,\tau)\} +$$
$$\mathrm{Cov}\{n_{j,j}(x|x_0,\tau),\ n_{k-1,k}(x|x_0,\tau)\} + \mathrm{Cov}\{n_{j,j}(x|x_0,\tau),\ n_{k,k}(x|x_0,\tau)\}$$

with initial conditions $\mathrm{Cov}\{n_i(x_0|x_0,\tau),\ n_k(x_0|x_0,\tau)\} = 0 \quad i,k \geq 0.$

Proof: Obvious.

With respect to initial conditions, the recurrence relations displayed in Theorems 5.3 and 5.6 can be completely solved. The same procedure can be used to find the higher moments of the age-parity distribution, if required.

6. Age-Parity Distribution in Time Marriage Cohorts. A time marriage cohort is a weighted sum of time-age marriage cohorts where the weights are derived from the probability distribution of age at marriage at the corresponding point in time. The results of Section 5 can be used to derive the moments of the parity distribution in time marriage cohorts. The derivation of these moments is done under the assumptions of 5. We give the following definition.

6.1 $$n_j(x+1|\tau) = \sum_{x_0=1}^{x} n_j(x+1|x_0,\tau)$$

where $n_j(x+1|\tau)$ is the random number of women with parity j at the beginning of month $x+1$ out of those who marry at calendar time τ.

Lemma 6.1: Assuming that the Markov property holds for the random variables X_j, $j \geq 0$, $p_{i,j}(x,x+1|x_0,\tau)$ is independent of x_0.

Proof: $p_{i,j}(x,x+1|x_0,\tau) = P[X_{j+1} > x+1|X_{i+1} > x, X_0 = x_0, T = \tau]$

$= P[X_{j+1} > x+1|X_{i+1} > x, T = \tau]$

by the Markov assumption.

Theorem 6.2: (Expected Age-Parity Distribution). Under the Markov assumption,

6.2.i $\quad E\{n_0(x+1|\tau)\} = p_{0,0}(x,x+1|\tau)E\{n_0(x|\tau)\} + M^{\tau}(x)$

and

6.2.ii $\quad E\{n_j(x+1|\tau)\} = p_{j-1,j}(x,x+1|\tau)E\{n_{j-1}(x|\tau)\}$

$+ p_{j,j}(x,x+1|\tau)E\{n_j(x|\tau)\} \quad j \geq 1,$

where

$p_{i,j}(x,x+1|\tau)$ is the probability for a woman marrying at calendar time τ to move from parity i to parity j over the month $(x,x+1)$ given that the parity is i at x. According to Lemma 6.1, this probability is independent of the age at marriage x_0, and

$M^{\tau}(x)$ is the expected number of women in the age interval $(x,x+1)$ at calendar time τ out of all those women who marry at time τ.

Proof: Definition 6.1 can be rewritten as

$$n_j(x+1|\tau) = \sum_{x_0=1}^{x-1} n_j(x+1|x_0,\tau) + n_j(x+1|x,\tau) .$$

Taking expectations on both sides,

$$E\{n_j(x+1|\tau)\} = \sum_{x_0=1}^{x-1} E\{n_j(x+1|x_0,\tau)\} + E\{n_j(x+1|x,\tau)\} .$$

Applying Theorem 5.3,

$$E\{n_0(x+1|\tau)\} = \sum_{x_0=1}^{x-1} p_{0,0}(x,x+1|x_0,\tau)E\{n_0(x|x_0,\tau)\} + E\{n_0(x+1|x,\tau)\}$$

and

$$E\{n_j(x+1|\tau)\} = \sum_{x_0=1}^{x-1} [p_{j-1,j}(x,x+1|x_0,\tau)E\{n_{j-1}(x|x_0,\tau)\} + p_{j,j}(x,x+1|x_0,\tau)E\{n_j(x|x_0,\tau)\}] + E\{n_j(x+1|x,\tau)\}, \quad j \geq 1 .$$

But the quantity $E\{n_j(x+1|x,\tau)\}$ is the expected number of women with parity j at the beginning of month x+1 out of those who marry over the age month (x,x+1) at calendar time τ. The nature of the fertility process implies this quantity is zero for $j \neq 0$. For $j = 0$, this quantity is the expected number of women marrying over the age month (x,x+1) at calendar time τ.

That is,

$$E\{n_j(x+1|x,\tau)\} = \begin{cases} M^\tau(x) & j = 0 \\ 0 & j \neq 0 \end{cases} .$$

We also recall from Lemma 6.1 that the probabilities $p_{i,j}(x,x+1|x_0,\tau)$ are independent of x_0 and so can be written as $p_{i,j}(x,x+1|\tau)$.

Thus

$$E\{n_0(x+1|\tau)\} = p_{0,0}(x,x+1|\tau) \sum_{x_0=1}^{x-1} E\{n_0(x|x_0,\tau)\} + M^\tau(x)$$

and

$$E\{n_j(x+1|\tau)\} = p_{j-1,j}(x,x+1|\tau) \sum_{x_0=1}^{x-1} E\{n_{j-1}(x|x_0,\tau)\} + p_{j,j}(x,x+1|\tau) \sum_{x_0=1}^{x-1} E\{n_j(x|x_0,\tau)\} \quad j \geq 1.$$

With appropriate initial conditions, the system of equations, 6.2.i and 6.2.ii, can be completely solved. The higher moments of $n_j(x|\tau)$ can also be derived in a similar manner.

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