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ON SYMMETRICALLY DISTRIBUTED RANDOM MEASURES[1]

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13. ABSTRACT

A random measure ξ defined on some measurable space $(S,\mathcal{S})$ is said to be symmetrically distributed with respect to some fixed measure ω on S, if the distribution of $(\xi A_1,\ldots,\xi A_k)$ for $k\in N$ and disjoint $A_1,\ldots,A_k\in\mathcal{S}$ only depends on $(\omega A_1,\ldots,\omega A_k)$. The first purpose of the present paper is to extend to such random measures (and then even improve) the results on convergence in distribution and almost surely, previously given for random processes on the line with interchangeable increments, and further to give a new proof of the basic canonical representation. The second purpose is to extend a well-known theorem of Slivnyak by proving that the symmetrically distributed random measures may be characterized by a simple invariance property of the corresponding Palm distributions.

KEY WORDS: Random measures and point processes, symmetric distributions, interchangeable increments, weak and strong convergence, invariance of Palm distributions.

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1. <u>Introduction</u>. Let S be a locally compact second countable Hausdorff space and let $\mathcal{B}$ be the ring of bounded Borel sets in S. Write $M(S)$ for the space of Radon measures on $(S,\mathcal{B})$, endowed with the vague or weak topology [7], and let $N(S)$ be the sub-space of Z_+-valued measures. Given any fixed $\omega \in M(S)$, we say that a random measure or point process ξ on S (i.e. a random element in $M(S)$ or $N(S)$ respectively [2,5]) is *symmetrically distributed with respect to* ω [6], if for $k \in N$ and disjoint $A_1,\ldots,A_k \in \mathcal{B}$ the distribution of $(\xi A_1,\ldots,\xi A_k)$ only depends on $(\omega A_1,\ldots,\omega A_k)$. As shown in [5], a simple point process ξ is symmetrically distributed with respect to some diffuse (non-atomic) measure ω iff ξ is a mixed Poisson or sample process. In case of random measures and diffuse ω with $\omega S<\infty$, a canonical representation was given in [6] in terms of a random variable $\alpha \geq 0$, prescribing the total diffuse mass of ξ, and a (canonical) point process β on $R_+' = (0,\infty)$, whose atom *positions* prescribe the atom *sizes* of ξ, cf. (1.2).

In the particular case when S is a real interval and ω is Lebesgue measure, ξ is seen to be symmetrically distributed with respect to ω iff the corresponding cumulative random process has interchangeable increments, so in this case the theory of [7] and [8] applies. However, a direct extension of the results there to more general spaces is certainly not easy (cf. [6]). Furthermore, certain substantial improvements are obtainable in the present case, similar to those attained in [5], Theorem 3.1., when specializing the central limit theorem to non-negative random variables. Finally, the present more general framework suggests some interesting generalizations which seem rather artificial on the line. For these three reasons, the theory of weak and strong convergence deserves its own treatment for random measures being given here in Section 2, along with a new proof of the canonical representation in [6].

In Section 3, we extend a result by Slivnyak [13], Papangelou [10] and myself ([5], Theorem 5.3) by showing that the class of symmetrically distributed random measures may be characterized by a simple invariance property of the corresponding Palm distributions [4,5]. Furthermore, it will be shown that a natural strengthening of this invariance condition will essentially delimit the class of symmetrically distributed random measures with symmetrically distributed canonical point processes, (containing in particular all homogeneous (with respect to ω) additive [5] random measures). Extensions of Slivnyak's theorem in an entirely different direction have been given by Kerstan, Matthes and others (see [9,4]).

Paralleling the exposition in [7], we shall now introduce four types of symmetrically distributed random measures on S. Type I random measures are by definition of the form

$$\xi = \sum_{j=1}^{k} \eta_j \delta_{t_j} \quad , \tag{1.1}$$

where $k \in N$, $t_1,\ldots,t_k \in S$, and the η_j are interchangeable random variables in R_+ with canonical point process $\pi = \sum_j \delta_{\eta_j}$ [7]. Here $\delta_s \in N(S)$ is the measure with a unit atom at $s \in S$. Type II random measures are also given by (1.1), but with $k = \infty$. We then suppose that $\{t_j\}$ has no limit point in S and that $\eta_1,\eta_2,\ldots$ are interchangeable random variables in R_+ with canonical random measure μ [7]. In both cases we put $\nu = \sum_j \delta_{t_j}$. Type III random measures are given by

$$\xi = \alpha\omega + \sum_{j=1}^{\infty} \beta_j \delta_{\tau_j} \tag{1.2}$$

for arbitrary $\omega \in M(S)$ with $\omega S = 1$, independent random elements $\tau_1,\tau_2,\ldots$ in S with common distribution ω, and random variables $\alpha \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$ independent of $\{\tau_j\}$ with $\sum_j \beta_j < \infty$. The associated canonical point process on R'_+ is $\beta = \sum_j \delta_{\beta_j}$. Finally, a Type IV random measure is by definition conditionally homogeneous with respect to ω , additive and infinitely divisible with L-transform (L = Laplace) given, for measurable[2] $f: S \to R_+$, by

$$- \log E(e^{-\xi f}|\gamma,\lambda) = \gamma\omega f + \int_S \int_{R'_+} (1-e^{-xf(s)})\lambda(dx)\omega(ds) \quad \text{a.s.},$$

(cf. Lemma 3.1 in [5]). Here $\omega \in M(S)$, $\gamma \geq 0$ is a random variable and λ is a random measure on R'_+ with $\int \frac{x\lambda(dx)}{1+x} < \infty$ a.s. The canonical quantities (ν,π), (ν,μ), (ω,α,β) and (ω,γ,λ), which clearly determine the distributions of the corresponding ξ , will always be defined as above, with the same affixes as ξ if any.

For convenience, we introduce some further notation. For any function f and measure m on S, we write $(fm)(dx) = f(x)m(dx)$ and $mf \equiv \int_S f(x)m(dx)$, and for measures m on R_+ we define $m^k(dx) \equiv x^k m(dx)$, $k \in N$. The letter g is reserved for the function $g(x) \equiv (1+x)^{-1}$. Equality and convergence in distribution [2] will be denoted by $\overset{d}{=}$ and $\overset{d}{\to}$ respectively. We further write $\overset{v}{\to}$ and $\overset{w}{\to}$ for vague and weak convergence of measures [7], and to distinguish between the corresponding notions of convergence in distribution, we use the symbols $\overset{vd}{\longrightarrow}$ and $\overset{wd}{\longrightarrow}$. For α,β,γ, and λ as above, we often abbreviate $\alpha\delta_0+\beta^1$ by B and $\gamma\delta_0+\lambda^1$ by Λ . (Note that the present use of α, B and Λ differs from that in [7,8].) Finally, $|A|$ denotes the diameter of $A \in \mathcal{B}$ in any fixed metric generating the toplogy of S.

2. Convergence and Related Topics.

In the following theorem we give extensions and partial improvements of the criteria for convergence in distribution, given for random processes with interchangeable increments in [7], Theorems 2.2, 2.3, 4.1, 3.2, 4.2 and 3.3.

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[2] This obvious attribute will be suppressed in the sequel.

Theorem 1. *Assuming* ξ *to be of Type III, we have* $\xi_n \xrightarrow{wd} \xi$ *if, for* ξ_n *of Type*

$$\text{I:} \quad \nu_n S \to \infty , \quad \nu_n/\nu_n S \overset{w}{\to} \omega \quad \pi_n^1 \xrightarrow{wd} B , \tag{2.1}$$

$$\text{III:} \quad \omega_n \overset{w}{\to} \omega , \quad B_n \xrightarrow{wd} B , \tag{2.2}$$

while assuming ξ *to be of Type IV with* $\omega S = \infty$*, we have* $\xi_n \xrightarrow{vd} \xi$ *if, for some* $\{r_n\}$ *and/or* $\{c_n\}$ *and for* ξ_n *of Type*

$$\text{I:} \quad \left\{ \begin{matrix} r_n \to \infty \\ c_n = r_n/\nu_n S \to 0, \end{matrix} \right\} \quad \nu_n/r_n \overset{v}{\to} \omega , \quad c_n g \pi_n^1 \xrightarrow{wd} g\Lambda , \tag{2.3}$$

$$\text{II:} \quad r_n \to \infty , \quad \nu_n/r_n \overset{v}{\to} \omega , \quad r_n g \mu_n^1 \xrightarrow{wd} g\Lambda , \tag{2.4}$$

$$\text{III:} \quad c_n \to 0 , \quad \omega_n/c_n \overset{v}{\to} \omega , \quad c_n g B_n \xrightarrow{wd} g\Lambda , \tag{2.5}$$

$$\text{IV:} \quad \omega_n/c_n \overset{v}{\to} \omega , \quad c_n g \Lambda_n \xrightarrow{wd} g\Lambda . \tag{2.6}$$

Conversely, if the ξ_n *are of Type I or III and* $\xi_n \xrightarrow{wd}$ *some* ξ *without fixed atoms and with* $P\{\xi \neq 0\} > 0$*, then* ξ *is of Type III and (2.1) or (2.2) holds respectively, while if the* ξ_n *are of Type I, II, III or IV and* $\xi_n \xrightarrow{vd}$ *some* ξ *without fixed atoms and with* $P\{\xi S = \infty\} > 0$*, then* ξ *is of Type IV and (2.3), (2.4), (2.5) or (2.6) holds respectively for some* $\{r_n\}$ *and/or* $\{c_n\}$.

Note that pleasant criteria for convergence towards mixed Poisson and sample processes may be obtained by specializing to point processes. In particular, Theorem 3 of Benczur [1] follows by combination with Theorem 5.2 in [5].

Proof. The sufficiency part will only be proved in the case I → III, the remaining cases being similar, so suppose that (2.1) holds. For $n \in N$, let $\eta_{n1}, \ldots, \eta_{nk_n}$ be the atom positions of π_n taken in random order, and let us first assume the π_n to be non-random. From (2.1) we get $\pi_n^1 R \to BR$, so in particular the η_{nj} are uniformly bounded, and we may thus conclude that $\pi_n^2 \xrightarrow{w} \beta^2$ (cf. Theorem 5.2 in [2]). By Theorem 5.1 in [2], these results extend to random π_n, in the sense that

$$(\pi_n^1 R,\ \pi_n^2) \xrightarrow{wd} (BR, \beta^2) \quad \text{in } R_+ \times M(R_+). \tag{2.7}$$

Now define the random processes X_n in $D[0,1]$ by

$$X_n(t) = \sum_{j \le k_n t} \eta_{nj}\ , \quad t \in [0,1], \quad n \in N,$$

and let X be a random process in $D[0,1]$ with $X(0) \equiv 0$ and with interchangeable increments, possessing canonical random elements BR, 0, β, in the sense of [7]. By Theorem 2.2 in [7], (2.7) yields $X_n \xrightarrow{d} X$ in the Skorohod J_1 topology [2]. Considering arbitrary $m \in N$ and $A_1, \ldots, A_m \in \mathcal{B}$ with $A_1 \subset \ldots \subset A_m$ and $\omega \partial A_j = 0$, $j = 1, \ldots, m$, we get by (2.1) $\nu_n A_j / k_n \to \omega A_j$, $j = 1, \ldots, m$, so

$$(X_n(\nu_n A_1/k_n), \ldots, X_n(\nu_n A_m/k_n)) \xrightarrow{d} (X(\omega A_1), \ldots, X(\omega A_m))$$

in R_+^m by Theorem 5.5 in [2], since X has no fixed jumps [7]. But by interchangeability, this is equivalent to

$$(\xi_n A_1, \ldots, \xi_n A_m) \xrightarrow{d} (\xi A_1, \ldots, \xi A_m). \tag{2.8}$$

Taking differences, it is seen that (2.8) remains true even without the restriction $A_1 \subset \ldots \subset A_m$, and since the class $\mathcal{U} = \{A \in \mathcal{B}: \omega \partial A = 0\}$ is clearly a DC-ring in the sense of [5], satisfying $\xi \partial A = 0$ a.s. for any $A \in \mathcal{U}$, Theorem 1.1 in [5] yields $\xi_n \xrightarrow{vd} \xi$. Finally by (2.7),

$$\xi_n S = \pi_n^1 R \xrightarrow{d} BR = \xi S,$$

and $\xi_n \xrightarrow{wd} \xi$ follows as asserted.

Conversely, suppose that the ξ_n are of Type I and that $\xi_n \xrightarrow{wd}$ some ζ with

$$\xi\{s\} = 0 \quad \text{a.s.}, \; s\in S; \quad P\{\xi\neq 0\} > 0 \tag{2.9}$$

Then $\pi_n^1 R = \xi_n S \xrightarrow{d} \xi S < \infty$ a.s., so $\{(\pi_n^1 R, \pi_n^1)\}$ is vaguely tight in $R_+ \times M(R_+)$, and hence by Theorem 5.1 in [2] and the point process nature of the π_n, $\{\pi_n^1\}$ is even weakly tight. Furthermore, $\{\nu_n/\nu_n S\}$ is automatically weakly relatively compact in $M(\overline{S})$, (the bar for one-point compactification). It follows that any sequence $N' \subset N$ must contain some sub-sequence N'' satisfying the conditions

$$\pi_n^1 \xrightarrow{wd} \text{some } B \text{ in } M(R_+), \quad n\in N'', \tag{2.10}$$

$$\nu_n/\nu_n S \xrightarrow{w} \text{some } \omega \text{ in } M(\overline{S}), \; n\in N''. \tag{2.11}$$

To prove that

$$\nu_n S \to \infty \qquad n\in N'', \tag{2.12}$$

suppose on the contrary that $\sup\{\nu_n S\colon n\in N'''\} < \infty$ for some sequence $N''' \subset N''$. If the set of all ν_n-atoms corresponding to n in N''' had no limit point in S, we would get $\xi_n \xrightarrow{vd} 0 \overset{d}{=} \xi$ contrary to (2.9), so we may assume the existence of some converging sequence $\{s_n\} \subset S$ with $\nu_n\{s_n\} \geq 1$, $n\in N'''$. But since $\xi_n\{s_n\} \xrightarrow{d} 0$ by (2.9), we may conclude from interchangeability that $\xi_n S \xrightarrow{d} 0 = \xi S$, again contradicting (2.9). This proves (2.12). By the sufficiency of (2.1), we now obtain from (2.10) - (2.12)

$$\xi_n \xrightarrow{wd} \text{some } \zeta \text{ in } M(\overline{S}), \quad n\in N'', \tag{2.13}$$

where ζ is a random measure in S of Type III and with canonical quantities ω,α,β. But (2.13) implies both $\xi_n S \xrightarrow{d} \xi S \overset{d}{=} \zeta S$ and $\xi_n S = \xi_n \overline{S} \xrightarrow{d} \zeta S$, and hence by combination $\zeta S \overset{d}{=} \zeta\overline{S}$, so e.g. by considering the expectation of $e^{-\zeta S} - e^{-\zeta\overline{S}} \geq 0$, we get $\zeta(\overline{S}\backslash S) = 0$ a.s., and finally $\omega(\overline{S}\backslash S) = 0$. It follows that (2.11) is also true in $M(S)$, and that ξ ($\overset{d}{=} \zeta$ on S) is of Type III with the same canonical quantities. Furthermore, ω must be diffuse by (2.9), and so ω,α and β are a.s. uniquely determined by the diffuse component and the atom sizes and positions of ξ. The proof of (2.1) may now be completed by applying Theorem 2.3 in [2]. A similar argument proves the necessity of (2.2).

We next consider the necessity of (2.4) when the ξ_n are of Type II and $\xi_n \xrightarrow{vd}$ some ξ with the stated properties. Choosing compact sets $C_j \uparrow S$ with $C_j \subset C_{j+1}^o$, $P\{\xi C_j>0\} > 0$ and $\xi\partial C_j = 0$ a.s., $j\in N$, we get $\xi_n \xrightarrow{wd} \xi$ in each $M(C_j)$, $j\in N$, so by the necessity of (2.1), there exist diffuse measures $\omega_1,\omega_2,\ldots\in M(S)$ such that

$$\nu_n/\nu_n C_j \xrightarrow{w} \omega_j \quad \text{in } M(C_j), \quad j\in N.$$

Furthermore, $r_n = \nu_n C_1 \to \infty$, and the restriction of ξ to C_j is symmetrically distributed with respect to ω_j. In particular then $r_n/\nu_n C_j \to \omega_j C_1 \neq 0$, $j\in N$, so

$$\nu_n/r_n = (\nu_n/\nu_n C_j)(\nu_n C_j/r_n) \xrightarrow{w} \omega_j/\omega_j C_1 \quad \text{in } M(C_j) \quad j\in N.$$

Thus the measures $\omega_j/\omega_j C_1$, $j\in N$, are all restrictions of some common diffuse measure $\omega\in M(S)$ such that $\nu_n/r_n \xrightarrow{v} \omega$ in $M(S)$. Moreover, ξ is symmetrically distributed with respect to ω, so we must have $\omega S = \infty$ since otherwise $\xi S < \infty$ a.s. Now let $A\in\mathcal{B}$ be such that $\omega A > 1$ and $\omega\partial A = 0$. Then $\nu_n A/r_n \to \omega A > 1$, so for large n

$$\sum_{j\le r_n} \eta_{nj} \le \sum_{j\le \nu_n A} \eta_{nj} \stackrel{d}{=} \xi_n A \xrightarrow{d} \xi A \tag{2.14}$$

by interchangeability. This proves tightness in R_+ of the sequence of leftmost members in (2.14), and also, by interchangeability, of the sequence

$$\Big(\sum_{j\le r_n} \eta_{nj}\ ,\ \sum_{j\le 2r_n} \eta_{nj},\ \ldots\Big)\ ,\quad n\in N \tag{2.15}$$

of random elements in R_+^∞. Hence, given any sequence $N'\subset N$, there exists some sub-sequence $N''\subset N'$ for which (2.15) converges in distribution, and by Theorem

1.3 in [7] we get $\mu_n^{*r_n} \to$ some μ, $n\in N''$. In the particular case of non-random μ_n, it follows by Theorem 3.1 in [5] that $r_n g\mu_n^1 \xrightarrow{w}$ some $g\Lambda$, $n\in N''$, and this extends to general μ_n by randomization, (cf. the proof of Theorem 3.2 in [7]). Hence (2.4) holds for $n\in N''$, and in particular it follows by the sufficiency part that ξ is of Type IV with canonical quantities ω,γ,λ. Proceeding as in the proof of Theorem 3.1 in [7], it is seen that γ and λ are unique, and so (2.4) holds for $n\in N$ by Theorem 2.3 in [2]. The necessity of (2.6) may be proved by similar arguments.

We finally consider the necessity of (2.3) when the ξ_n are of Type I and $\xi_n \xrightarrow{vd}$ some ξ with the stated properties. Proceeding as above, we get $r_n = \nu_n C_1 \to \infty$, and further $\nu_n/r_n \xrightarrow{v}$ some ω, where ω is diffuse with $\omega S = \infty$ and such that ξ is symmetrically distributed with respect to ω. Moreover,

$$\liminf_{n\to\infty} c_n^{-1} = \liminf_{n\to\infty} \nu_n S/r_n \geq \omega S = \infty.$$

If N' is an arbitrary sub-sequence of N, it follows as before that (2.15) converges as $n \to \infty$ through some $N''\subset N'$, and by comparison of Theorems 3.2 and 4.1 in [7], it is seen that this remains true with the η_{nj} of (2.15) replaced by some η'_{nj} which for fixed $n\in N''$ are interchangeable random variables with canonical random measure $\mu_n = \pi_n/\nu_n S = c_n\pi_n/r_n$. Hence

$$r_n g\mu_n^1 = c_n g\pi_n^1 \xrightarrow{wd} \text{some } g\Lambda \text{ in } \mathcal{M}(R_+),\ n\in N''.$$

The remainder of the proof is similar to that of (2.4). A similar argument proves the necessity of (2.5).

We shall now show how the canonical representations of symmetrically distributed random measures given in [6] for bounded ω, may be deduced from Theorem 1. (See also Theorem 5.1 in [5] and Theorems 2.1 and 3.1 in [7].)

Corollary 1. *Let* ξ *be a random measure on* S *and let* $\omega \in M(S)$ *be diffuse. Then* ξ *is symmetrically distributed with respect to* ω *iff* ξ *is of Type III or IV (depending on whether* $\omega S < \infty$ *or* $\omega S = \infty$*).*

Proof. Assume that $P\{\xi \neq 0\} > 0$. We first consider bounded S, in which case we may suppose that $\omega S = 1$. For each $n \in N$, divide S into finitely many disjoint sets $I_{nj} \in \mathcal{B}$ with $\omega I_{nj} > 0$ and $\omega \partial I_{nj} = 0$, $j=1,\ldots,k_n$, and such that $\{I_{n+1,j}\}$ is a refinement of $\{I_{nj}\}$ for each n and $\max_j |I_{nj}| \to 0$. Put $r_n = n/(\min \omega I_{nj})$ and choose $\nu_n = \sum_k \delta_{t_{nk}} \in N(S)$ with $\nu_n I_{nj} \equiv [r_n \omega I_{nj}]$. An easy calculation yields

$$1 - \frac{1}{n} < \frac{\nu_n I_{nj}}{r_n \omega I_{nj}} \leq 1, \quad j=1,\ldots,k_n, \quad n \in N,$$

which clearly remains true with I_{nj} replaced by any non-empty union U of sets among $I_{n1},\ldots,I_{nk_n}$. For such U,

$$\left|\frac{\nu_n U}{\nu_n S} - \omega U\right| \leq 2\left|\frac{\nu_n U - \nu_n S \omega U}{r_n \omega U}\right| \leq 2\left|\frac{\nu_n U}{r_n \omega U} - 1\right| + 2\left|\frac{\nu_n S}{r_n \omega S} - 1\right| < \frac{4}{n}, \quad (2.16)$$

and in particular, $\nu_n/\nu_n S \xrightarrow{w} \omega$. For $n \in N$, we next divide S into disjoint sets $A_{nj} \in \mathcal{B}$ with $\omega A_{nj} = (\nu_n S)^{-1}$, $j=1,\ldots,\nu_n S$, and put $\xi_n = \sum_j \xi A_{nj} \delta_{t_{nj}}$. Then the ξ_n are of Type I, and $\xi_n \xrightarrow{wd} \xi$ follows from (2.16) by Theorem 1.1 in [5] and the easily verified fact that, for disjoint $U_{n1},\ldots,U_{nk}$, $n \in Z_+$, $(\omega U_{n1},\ldots,\omega U_{nk}) \to (\omega U_{01},\ldots,\omega U_{0k})$ implies $(\xi U_{n1},\ldots,\xi U_{nk}) \xrightarrow{d} (\xi U_{01},\ldots,\xi U_{0k})$. (Use the symmetry of ξ and the fact that $U_n \downarrow \emptyset$ implies $\xi U_n \to 0$ a.s.) We may now conclude from the converse part of Theorem 1 that ξ is of Type III. In the case of unbounded S, we may again use the converse part of Theorem 1, now with the ξ_n chosen as restrictions of ξ to some suitable sequence of bounded sets.

For the applicability when $\omega S = \infty$, note that by Fatou's lemma, for disjoint $A_1, A_2, \ldots \in \mathcal{B}$ with $\omega A_k > 1$ and for sufficiently small $\varepsilon > 0$,

$$P\{\xi S = \infty\} \geq P\{\xi A_k > \varepsilon \text{ i.o.}\} \geq \limsup_{k \to \infty} P\{\xi A_k > \varepsilon\} > 0.$$

In [7], Theorem 5.3, it was shown that the distribution of any random process with interchangeable increments is uniquely determined by that of its restriction to any fixed sub-interval. Obviously, this result generalizes to the case of random measures, yielding alternative convergence criteria in Theorem 1. In the particular case of non-random canonical quantities (allowing interpretations in terms of sampling from finite populations), we may obtain still simpler determining (and therefore also convergence determining) classes ([2], p. 15). In fact, symmetrization in Lemma 11.2 by Rosén [12] (cf. Theorem 12.1 in [12] and Theorems 4-5 in [3]) yields the

Proposition 1. *Let* ξ *be a random measure on* S *of Type III, and suppose that* B *is non-random. If* ω *is known, then the distribution of* ξ *is uniquely determined by that of* ξA *for any fixed* $A \in \mathcal{B}$ *with* $0 < \omega A < \omega S$.

Note that the corresponding statement for Type IV random measures is also true. For simple point processes, we need not even assume B (or Λ) to be non-random ([5], Theorem 5.2).

We conclude this section by considering extensions and partial improvements of the variational and ergodic results given for random processes with interchangeable increments in [8], Theorems 5.1, 6.2, 6.3 and 6.4. Clearly, Π_n should now denote a partition of S or of some $S_n \in \mathcal{B}$ into disjoint measurable sets $A_{n1}, \ldots, A_{nk}$. For any $m \in M(S)$, we write $\Pi_n m = \sum_k \delta_{mA_{nj}} \in N(R_+)$.

In analogy with [8], we further define $|\Pi_n|_\infty = \max_j \omega A_{nj}$, $|\Pi_n|_2^2 = \sum_j (\omega A_{nj})^2 =$ $(\Pi_n\omega)^2 R$, and we say that $\{\Pi_n\}$ is *nested* if it proceeds by successive refinements. For Type IV random measures with $\omega S = \infty$, μ_p denotes the canonical random measure corresponding to a partition of S into sets of ω-measure $p > 0$, while B_n corresponds to the restriction of ξ to $S_n \in \mathcal{B}$. Using these notations, we may state the strong counterpart of Theorem 1 as follows.

<u>Theorem 2</u>. *If* ξ *is a Type III random measure and if* $\Pi_1, \Pi_2, \ldots$ *are partitions of* S *which are either nested with* $|\Pi_n|_\infty \to 0$ *or satisfy* $\sum_n |\Pi_n|_2^2 < \infty$, *then*

$$(\Pi_n\xi)^1 \overset{w}{\to} B \text{ a.s. in } M(R_+) \ . \tag{2.17}$$

On the other hand, if ξ *is of Type IV with* $\omega S = \infty$ *then*

$$g\mu_p^1/p \overset{w}{\to} g\Lambda \text{ as } p \to 0, \text{ a.s. in } M(R_+). \tag{2.18}$$

Furthermore, for $S_1 \subset S_2 \subset \ldots \in \mathcal{B}$ *with* $\omega S_n \to \infty$, *and for any* $f\colon R_+ \to R_+$,

$$fB_n/\omega S_n \overset{w}{\to} f\Lambda \text{ in } M(R_+), \text{ a.s. within } \{\Lambda f < \infty\}. \tag{2.19}$$

For such $\{S_n\}$, *let the* Π_n *be partitions of* S_n, $n \in N$, *satisfying*

$$|\Pi_n|_2^2/\omega S_n \to 0, \quad \sum_n |\Pi_n|_2^2/(\omega S_n)^2 < \infty \ .$$

Then

$$g(\Pi_n\xi)^1/\omega S_n \overset{w}{\to} g\Lambda \text{ a.s. in } M(R_+) \ . \tag{2.20}$$

It should be noticed that the monotonity of $\{S_n\}$ is essential for the truth of the statements involving (2.19) and (2.20). Similarly, the nestedness of $\{\Pi_n\}$ is essential for the truth of (2.17) in the case $|\Pi_n|_\infty \to 0$. However, the nestedness can be dispensed with if we assume ω to be diffuse and change the definition of $|\Pi_n|_\infty$ to $|\Pi_n|_\infty = \max_j |A_{nj}|$.

Proof. To prove the first assertion, we may clearly assume that B is non-random and that $\omega S = 1$. The probability that two particular atoms lie in the same set of Π_n is then $|\Pi_n|_2^2 \le |\Pi_n|_\infty$. If the Π_n are nested, then this event is non-increasing in n, so it can a.s. only occur finitely often provided $|\Pi_n|_\infty \to 0$. If $\sum_n |\Pi_n|_2^2 < \infty$, the same statement follows by the Borel-Cantelli lemma. The extension to any finite set of atoms being immediate, it follows easily that $\Pi_n \xi \overset{v}{\to} \beta$ a.s. in $N(R_+')$. Since $(\Pi_n \xi)^1 R = BR$ holds identically, this completes the proof of (2.17). The assertion involving (2.18) is essentially equivalent to the converse part of Theorem 3.1 in [5]. The remainder of the proof is easy, given the exposition in [8].

3. Invariance of Palm Distributions.

In this section we shall show that the defining property of symmetrically distributed random measures is closely related to some other symmetry properties, expressible in terms of Palm distributions. Just as for point processes in [5], the latter are defined for arbitrary random measures ξ with $E\xi \in M(S)$ as the distributions of random measures ξ_s, $s \in S$, satisfying, for any $f: M(S) \to R_+$,

$$Ef(\xi_s) = \frac{Ef(\xi)\xi(ds)}{E\xi(ds)}, \quad s \in S \quad \text{a.e. } E\xi. \tag{3.1}$$

(Cf. [4] for existence. For the related concept of Campbell measure, see [9].) We refer to [5] for the notion of mixed sample processes and for the abbreviation $MS(\omega, \phi)$. A measure is said to be *degenerate* if all its mass is concentrated to one single point.

Theorem 3. *Let* ξ *be a random measure on* S *with* $E\xi \in M(S)$. *Then* $(\xi_s\{s\}, \xi_s - \xi_s\{s\}\delta_s) \overset{d}{=}$ *some* (η, ζ) *independently of* s *a.e.* $E\xi$, *iff* $E\xi$ *is diffuse (except possibly for a.s. degenerate* ξ*) and* ξ *is symmetrically distributed* *with respect to* $E\xi$. *In this case,* (η,ζ) *is also symmetrically distributed*[3] *with respect to* $E\xi$, *and furthermore,* η *is independent of* ζ *iff either* (i) ξ *is of Type III with* $\alpha = 0$ *a.s. and* β *a mixed sample process, or* (ii) ξ *is of Type IV with* $\Lambda = \rho M$ *a.s. for some random variable* ρ *and some non-random* $M \in M(R_+)$.

For the particular case of point processes, a slightly stronger assertion was proved in [5], Theorem 5.3, (see also [10]). A (very) special case of the situation in (ii) was discussed by Port and Stone in [11], Example 2.

As will be seen from the proof, the distributions of ξ and (η,ζ) are related, in case of symmetry, by either or both of the relations

$$Ef(\eta,\hat{B}) = \frac{1}{EBR} E \int_R f(x, B-x\delta_x) B(dx) , \qquad (3.2)$$

$$Ef(\eta,\hat{\Lambda}) = \frac{1}{E\Lambda R} E \int_R f(x,\Lambda)\Lambda(dx) , \qquad (3.3)$$

for arbitrary $f: R_+ \times M(R_+) \to R_+$, where $\hat{B} = \hat{\alpha}\delta_0 + \hat{\beta}^1$ and $\hat{\Lambda}$ denote the canonical random measures of ζ. If $\alpha = 0$ and $\beta \overset{d}{=} MS(E\beta,\phi)$, then η has distribution EB/EBR, while $\hat{\alpha} = 0$ and $\beta \overset{d}{=} MS(E\beta,-\phi')$. On the other hand, if $\Lambda = \rho M$ for some random variable $\rho \geq 0$ with L-transform ϕ and some probability measure M on R_+, then η has distribution M while $\hat{\Lambda} = \hat{\rho}M$ for some random variable $\hat{\rho} \geq 0$ with L-transform $-\phi'$.

We finally remark that the cases (i) and (ii) where η and ζ are independent are not so far apart as they may appear. In fact, as may be seen from Theorem 1 (or rather from its proof), the random measures satisfying (i) or (ii)

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[3] This means of course that the distribution of $(\eta,\zeta A_1,\ldots,\zeta A_k)$ only depends on $(E\xi A_1,\ldots,E\xi A_k)$.

constitute the closure with respect to convergence in distribution in the vague topology of the class of random measures satisfying (i).

For the proof of Theorem 3, we need a lemma of some independent interest.

Lemma 1. *Let* ξ *be a random measure on* S *with* $\xi S < \infty$ *a.s. and let* τ *be a random element in* S *which for given* $\xi \neq 0$ *has conditional distribution* $\xi/\xi S$. *Then* τ *is conditionally independent of* $(\tilde{\eta},\tilde{\zeta}) = (\xi\{\tau\}, \xi - \xi\{\tau\}\delta_\tau)$, *given that* $\xi \neq 0$, *iff* ξ *is symmetrically distributed with respect to some diffuse (except possibly for a.s. degenerate* ξ*)* $\omega \in M(S)$. *In this case,* $(\tilde{\eta},\tilde{\zeta})$ *is conditionally symmetrically distributed with respect to* ω, *and the canonical random measure* $\tilde{B}$ *of* $\tilde{\zeta}$ *satisfies, for* $f: R_+ \times M(R_+) \to R_+$,

$$E[f(\tilde{\eta},\tilde{B})\,|\,\xi \neq 0] = E[\frac{1}{BR}\int_R f(x, B-x\delta_x)B(dx)\,|\,B \neq 0]. \tag{3.4}$$

Note that, in the particular case of point processes, τ is one of the atom positions chosen at random. Clearly the conditional distributions of ξ, given $\tau = s$, $s \in S$ a.e. $P\tau^{-1}$, here play the role of the Palm distributions in Theorem 3. Though perhaps at least as natural from the point of view of applications, they do not behave quite as well mathematically. A related type of competitors to the Palm distributions was considered by Slivnyak [13] for stationary point processes.

Proof of Lemma 1. Suppose that τ is conditionally independent of $(\tilde{\eta},\tilde{\zeta})$ with distribution ω, given that $\xi \neq 0$. Let α be the total diffuse mass and $\beta_1 \geq \beta_2 \geq \ldots$ the atom sizes of ξ, and put $\beta = \sum_j \delta_{\beta_j}$, $B = \alpha\delta_0 + \beta^1$. Since $(\tilde{\eta},B)$ depends measurably on $(\tilde{\eta},\tilde{\zeta})$, τ is even conditionally independent of $(\tilde{\eta},\tilde{\zeta})$ with distribution ω, given any $(\tilde{\eta},B) \neq 0$, so we may consider $\tilde{\eta}$ and B as fixed. Now define $\tau_1,\tau_2,\ldots$ as the atom positions corresponding to $\beta_1,\beta_2,\ldots,$ taken in random order within sets of equal β_j. Assuming $\beta_1 =\ldots= \beta_n > \beta_{n+1}$ and choosing $\tilde{\eta} = \beta_1$, we may clearly identify τ with either of $\tau_1,\ldots,\tau_n$, so it follows by induction that $\tau_1,\ldots,\tau_n$ are independent of $\xi - \sum_{j=1}^{n} \beta_j \delta_{\tau_j}$ and mutually independent with distribution ω. Since the same argument applies to any choice of $\tilde{\eta} > 0$, the last statement remains true for arbitrary n. If $\alpha > 0$, we next put $\tilde{\eta} = 0$. Then the conditional distribution of τ, given ξ, coincides with the normalized diffuse component $\tilde{\xi}$ of ξ, and this remains conditionally true, given $(\tilde{\eta},\tilde{\zeta})$ since $\tilde{\xi}$ depends measurably on $\tilde{\zeta}$, so we get $\tilde{\xi} = \omega$ a.s. For given B, ξ is therefore conditionally symmetrically distributed with respect to ω, and this clearly remains true unconditionally. If $\omega\{s\} > 0$ for some $s\in S$, then $\alpha = 0$ a.s. since $\alpha\omega$ is the diffuse component of ξ. Furthermore,

$$(\omega\{s\})^2 P\{\beta_2 > 0\} \le P\{\tau_1 = \tau_2,\ \beta_2 > 0\} \le P\{\beta_1 \ge \beta_1 + \beta_2,\ \beta_2 > 0\} = 0,$$

so $\beta_2 = 0$ a.s., and hence $\xi = \beta_1\delta_{\tau_1}$.

Conversely, if ξ is symmetrically distributed with respect to some diffuse $\omega\in M(S)$, it follows by Corollary 1 that, for given $(\tilde{\eta},B) \neq 0$, τ is conditionally independent of $(\tilde{\eta},\tilde{\zeta})$ with distribution ω, and this will clearly remain true unconditionally. Since, for given $B \neq 0$, $\tilde{\eta}$ has conditional distribution B/BR, (3.4) follows easily by the same corollary.

Proof of Theorem 3. Since forming the Palm distributions and restricting to a sub-space are interchangeable operations, we may assume that $\xi S < \infty$ a.s. when proving the first assertion. Let τ be defined as in Lemma 1 and consider any $f\colon M(S) \times S \to R_+$, $u\in R_+$ and $s\in S$. By Proposition 2 in [4], which clearly carries over to arbitrary random measures,

$$E[f(\xi,\tau);\ \xi S\in du,\ \tau\in ds] = E[f(\xi,s)P\{\tau\in ds|\xi\};\ \xi S\in du]$$

$$= E[f(\xi,s)\ \frac{\xi(ds)}{u};\ \xi S\in du] = \frac{1}{u}\ E[f(\xi_s,s);\ \xi_s S\in du]E\xi(ds),$$

so by the chain rule for Radon-Nikodym derivatives, for $(u,s) \in R_+ \times S$ a.e. $E[\xi(ds);\xi S\in du]$,

$$E[f(\xi,\tau)|\xi S = u,\ \tau = s] = \frac{E[f(\xi,\tau);\ \xi S\in du,\ \tau\in ds]}{P\{\xi S\in du,\ \tau\in ds\}} = \frac{E[f(\xi_s,s);\ \xi_s S\in du]}{P\{\xi_s S\in du\}}$$

$$= E[f(\xi_s,s)|\xi_s S = u],$$

(cf. Lemma 5.1 in [5]). In particular, this yields a.e., for $f\colon R_+ \times M(S) \to R_+$,

$$E[f(\xi\{\tau\},\ \xi - \xi\{\tau\}\delta_\tau)|\xi S = u,\tau = s] = E[f(\xi_s\{s\},\xi_s-\xi_s\{s\}\delta_s)|\xi_s S = u]. \quad (3.5)$$

Now if the distribution of $(\xi_s\{s\},\ \xi_s-\xi_s\{s\}\delta_s)$ is a.e. independent of s, then so is the right hand side of (3.5), and it follows that $(\xi\{\tau\},\xi-\xi\{\tau\}\delta_\tau)$ is conditionally independent of τ, given $\xi S = u$. Since conditioning on $\xi S = u$ will not change the definition of τ in terms of ξ, Lemma 1 applies, and so we may conclude that ξ is conditionally symmetrically distributed with respect to some normalized diffuse (except possibly for a.s. degenerate ξ) $\omega_u \in M(S)$. But for any $A\in\mathcal{B}$ and $u\in R_+$, by the assumed invariance,

$$E[\xi A;\xi S\in du] = \int_A P\{\xi_s S\in du\}E\xi(ds) = P\{\xi_s S\in du\}E\xi A\ ,$$

so by the chain rule

$$\omega_u A = \frac{1}{u}\ E[\xi A|\xi S = u] = \frac{E[\xi A;\ \xi S\in du]}{E[\xi S;\ \xi S\in du]}\ \frac{E[\xi S;\ \xi S\in du]}{uP\{\xi S\in du\}} = \frac{E\xi A}{E\xi S}\ ,$$

and we get $\omega_u = E\xi/E\xi S$, independently of $u \geq 0$ a.e. $P(\xi S)^{-1}$, proving that ξ is even unconditionally symmetrically distributed.

Conversely, suppose that ξ is symmetrically distributed with canonical quantities ω and B, where ω is diffuse and $0 < EBR < \infty$. For $f: R_+ \to R_+$, we get

$$E[\xi(ds)f(\xi S)] = E[E(\xi(ds)|BR)f(BR)] = E[BRf(BR)]\omega(ds),$$

so by (3.1),

$$Ef(\xi_s S) = \frac{E[BRf(BR)]}{EBR}, \quad s\in S \text{ a.e. } E\xi, \tag{3.6}$$

and in particular, the distribution of $\xi_s S$ is a.e. independent of s. Furthermore, by (3.5) and Lemma 1, the conditional distribution of $(\xi_s\{s\}, \xi_s-\xi_s\{s\}\delta_s)$, given $\xi_s S$, is a.e. independent of s, so by (3.6), the same thing is true for the unconditional distribution. From (3.5) it is also seen that $(\eta,\zeta) \overset{d}{=} (\xi_s\{s\},\xi_s-\xi_s\{s\}\delta_s)$ is symmetrically distributed with respect to ω, and by combination of (3.4) and (3.5), we get for $f: R_+ \times \mathcal{M}(R_+) \to R_+$

$$E[f(\eta,\hat{B})|\xi_s S = u] = \frac{1}{BR} E[\int_R f(x,B-x\delta_x)B(dx)|BR = u],$$
$$u > 0 \text{ a.e. } P(\xi_s S)^{-1},$$

which yields (3.2) when inserted in (3.6).

To prove (3.3) when ξ is symmetrically distributed with canonical ω and Λ, $\omega S = \infty$, let B and $\hat{B}$ correspond to the restrictions of ξ and ζ respectively to some $A\in\mathcal{B}$ with $\omega A = t > 0$. If Λ is non-random, we have $\alpha = t\gamma$ while β is a Poisson process with intensity $t\lambda$ [7], so by (3.1) and the remark following Theorem 5.3 in [5], we get for $f: R_+ \times R_+ \times \mathcal{N}(R_+') \to R_+$

$$E\int_R f(x,\alpha,\beta-\delta_x)B(dx) = E\int_R f(x,\alpha,\beta-\delta_x)x\beta(dx) + E[\alpha f(0,\alpha,\beta)]$$
$$= \int_R Ef(x,\alpha,\beta_x-\delta_x)xt\lambda(dx) + E[t\gamma f(0,\alpha,\beta)]$$
$$=t\int_R Ef(x,\alpha,\beta)x\lambda(dx) + tE[\gamma f(0,\alpha,\beta)] = tE\int_R f(x,\alpha,\beta)\Lambda(dx),$$

and this result extends by conditioning to arbitrary Λ. Since $EBR = tE\Lambda R$, we get by (3.2) for $f: R_+ \times M(R_+) \to R_+$

$$Ef(\eta,\hat{B}/t) = \frac{1}{E\Lambda R} E \int_R f(x,B/t)\Lambda(dx). \tag{3.7}$$

Letting $A \uparrow S$ we get by Theorem 2 $B/t \xrightarrow{v} \Lambda$ and $\hat{B}/t \xrightarrow{v} \hat{\Lambda}$ a.s. in $M(R_+)$, so for bounded and continuous f, (3.3) follows from (3.7) by repeated dominated convergence, and we may extend (3.3) successively, first by monotone convergence to indicators of open sets (cf. Theorem 1.2 in [2]), then by Dynkin's theorem to arbitrary indicators, and finally by linearity and monotone convergence to arbitrary f.

Now suppose that η and ζ are independent. We may assume that $0 < E\xi S < \infty$, since otherwise the proof may be reduced to this case by restricting ξ to bounded sub-spaces. By (3.1) and (3.2), we get for any $t \geq 0$ and $f: M(R_+) \to R_+$

$$Ee^{-\eta t}Ef(\hat{B}) = E[e^{-\eta t}f(\hat{B})] = \frac{1}{EBR} E \int_R e^{-xt}f(B-x\delta_x)B(dx)$$

$$= \frac{1}{EBR} \int_R e^{-xt}Ef(B_x-x\delta_x)EB(dx) ,$$

so if $0 < Ef(B) < \infty$, we obtain

$$Ee^{-\eta t} = \int_R e^{-xt} \frac{Ef(B_x-x\delta_x)}{Ef(\hat{B})} \frac{EB(dx)}{EBR} .$$

Comparing this with the formula obtained for $f \equiv 1$, it follows by the uniqueness theorem for L-transforms that

$$Ef(B_x-x\delta_x) = Ef(\hat{B}) , \quad x \geq 0 \text{ a.e. } EB.$$

Turning to an $f\colon R_+ \times N(R_+') \to R_+$ and using (3.1), we thus obtain for $x > 0$ a.e. $E\beta$

$$Ef(\hat{\alpha},\hat{\beta}) = \frac{E[B(dx)f(\alpha,\beta-\delta_x)]}{EB(dx)} = \frac{E[\beta(dx)f(\alpha,\beta-\delta_x)]}{E\beta(dx)} = Ef(\alpha_x,\beta_x-\delta_x), \qquad (3.8)$$

(in an obvious notation), and also, provided $E\alpha > 0$,

$$Ef(\hat{\alpha},\hat{\beta}) = \frac{E[\alpha f(\alpha,\beta)]}{E\alpha} \quad . \qquad (3.9)$$

From now on, we may assume that $E\beta R < \infty$, since otherwise we may consider the restrictions of β to compact sub-intervals of R_+'. Proceeding as in the proof of Theorem 5.3 in [5], we may conclude from (3.8) that, for given $\nu = \beta R$, β is conditionally a sample process independent of α with intensity $\nu E\beta/E\nu$. At this stage, we may assume that $E\alpha$ and $E\nu$ are both > 0, since otherwise either (i) or (ii) is trivially satisfied. By (3.8) and (3.9), we then get for any $t \geq 0$ and $s \in [0,1]$

$$\frac{E[\alpha e^{-\alpha t}s^{\nu}]}{E\alpha} = \frac{E[\beta(dx)e^{-\alpha t}s^{\nu-1}]}{E\beta(dx)} = \frac{E\{e^{-\alpha t}s^{\nu-1}E[\beta(dx)|\alpha,\nu]\}}{E\beta(dx)}$$

$$= \frac{E\{e^{-\alpha t}s^{\nu-1}\nu E[\beta(dx)]/E\nu\}}{E\beta(dx)} = \frac{E[e^{-\alpha t}\nu s^{\nu-1}]}{E\nu}$$

so

$$E\left\{e^{-\alpha t}\,\frac{\alpha E[s^{\nu}|\alpha]}{E\alpha}\right\} = E\left\{e^{-\alpha t}\,\frac{E[\nu s^{\nu-1}|\alpha]}{E\nu}\right\}\ , \quad t \geq 0,\ 0 \leq s \leq 1,$$

and hence by the uniqueness theorem,

$$\frac{\alpha E[s^{\nu}|\alpha]}{E\alpha} = \frac{E[\nu s^{\nu-1}|\alpha]}{E\nu} \quad \text{a.s.},\ 0 \leq s \leq 1. \qquad (3.10)$$

Assuming these expectations to be calculated from some family of regular conditional distributions of ν, given α, (3.10) extends by continuity from any countable dense sub-set of s-values, so we may take the exceptional P-null set in (3.10) to be independent of s. Writing $\phi_\alpha(s) = E[s^{\nu}|\alpha]$ and

$c = E\nu/E\alpha$, we get a.s. the differential equations

$$\phi'_\alpha(s) = c\alpha\phi_\alpha(s), \quad 0 \le s \le 1,$$

and since $\phi_\alpha(1) = 1$ a.s., we obtain a.s. the unique solutions

$$\phi_\alpha(s) = e^{-c\alpha(1-s)}, \; 0 \le s \le 1 ,$$

showing that the conditional distributions of ν, given α, are a.s. Poissonian with means $c\alpha$. But this is merely another way of expressing (ii).

Conversely, assuming ξ to be such as in (i), we get by (3.1), (3.2) and Theorem 5.3 in [5], for any $t \ge 0$ and $f\colon R'_+ \to R_+$,

$$Ee^{-\eta t-\hat{\beta}f} = \frac{1}{E\beta^1 R}\, E \int_R \exp[-xt-(\beta-\delta_x)f]x\beta(dx)$$

$$= \frac{1}{E\beta^1 R} \int_R e^{-xt}\; E \exp[-(\beta_x-\delta_x)f]xE\beta(dx)$$

$$= E \exp[-(\beta_x-\delta_x)f] \int_R e^{-xt}\, \frac{E\beta^1(dx)}{E\beta^1 R} ,$$

so η and $\hat{\beta}$ are indeed independent with the asserted distributions. Similarly, for ξ as described in (ii), we get by (3.4) for any $t \ge 0$ and $f\colon R_+ \to R_+$

$$Ee^{-\eta t-\hat{\Lambda}f} = \frac{1}{E\rho MR}\, E \int_R e^{-xt-\rho Mf}\rho M(dx) = E[\rho e^{-\rho Mf}] \int_R e^{-xt}M(dx) ,$$

in conformity with our assertions. This completes the proof of Theorem 3.

It should be observed that, in the proof of the second assertion, the crucial point is to show α must be a.s. zero if ξ is not of Type IV.

In fact, assuming ξ to be of Type IV, (3.1) and (3.3) yield for any $t \geq 0$ and $f: R_+ \to R_+$

$$Ee^{-\eta t-\hat{\Lambda}f} = \frac{1}{E\Lambda R} E \int_R e^{-xt-\Lambda f} \Lambda(dx) = \int_R e^{-xt} E \exp[-\Lambda_x f] \frac{E\Lambda(dx)}{E\Lambda R} ,$$

so if η and ζ are independent, it follows by the uniqueness theorem that $E \exp[-\Lambda_x f] = E\, e^{-\Lambda f}$, $x \geq 0$ a.e. $E\Lambda$, and hence that $\Lambda_x \stackrel{d}{=} \Lambda$ a.e. Arguing as in the proof of the first assertion in Theorem 3, it is not hard to see that this implies $\Lambda = \rho M$ for some random variable $\rho \geq 0$ and some non-random $M \in \mathcal{M}(R_+)$.

REFERENCES

1. A. Benczur, *On sequences of equivalent events and the compound Poisson process*, Studia Sci. Math. Hungarica 3 (1968), 451-458. MR 39 #4905.

2. P. Billingsley, *Convergence of probability measures*, Wiley, New York, 1968. MR 38 #1718.

3. J. Hagberg, *Approximation of the summation process obtained by sampling from a finite population*, Teor. Veroyatnost. Primenen. 18 (1973), to appear.

4. P. Jagers, *On Palm probabilities*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 26 (1973), 17-32.

5. O. Kallenberg, *Characterization and convergence of random measures and point processes*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 27 (1973), to appear.

6. ______, *A canonical representation of symmetrically distributed random measures*, Mathematics and Statistics, Essays in Honour of Harald Bergström, Teknologtryck, Göteborg, 1973, pp. 41-48.

7. ______, *Canonical representations and convergence criteria for processes with interchangeable increments*, Z. Wahrscheinlichkeitstheorie und verw. Gebiete 27 (1973), to appear.

8. ______, *Path properties of processes with independent and interchangeable increments*, Tech. report, Dept. of Mathematics, Univ. of Göteborg, 1973.

9. G. Kummer and K. Matthes, *Verallgemeinerung eines Satzes von Sliwnjak II-III*, Rev. Roumaine Math. Pures et Appl. 15 (1970), 845-870, 1631-1642. MR 42 #5304, 44 #6024.

10. F. Papangelou, *The Ambrose-Kakutani theorem and the Poisson process*, Contributions to Ergodic Theory and Probability, Springer, Berlin, Heidelberg, New York, 1970, pp. 234-240.

11. S.C. Port and C.J. Stone, *Infinite particle systems*, Trans. Amer. Math. Soc. 178 (1973), 307-340.

12. B. Rosén, *Limit theorems for sampling from a finite population*, Ark. Mat. 5 (1964), 383-424. MR 31 #1700.

13. I.M. Slivnyak, *Some properties of stationary flows of homogeneous random events*, Theor. Probab. Appl. 7 (1962), 336-341, 9 (1964), 168. MR 27 #832.

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