

This research was conducted while the author, being a student in UNC at Chapel Hill, N. C. was supported by the Biostatistics Department of this University.

AN EXTENSION OF THE T-METHOD TO
UNBALANCED LINEAR MODELS OF FIXED EFFECTS

by

Yosef Hochberg

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 902

DECEMBER 1973

1. Introduction and Summary

Among the various procedures for simultaneous interval estimation, Tukey's T-method gives the shortest confidence intervals for pairwise contrasts when it is applicable. (Cf. Miller (1966) Ch. 2 and Scheffé (1959) Ch. 3). However, the T-method in its exact formulation, (excluding some conservative procedures discussed in [2]) is restricted to well balanced designs.

In this paper we discuss a generalized T-type procedure for simultaneous interval estimation of all estimable parametric functions in general fixed effects, univariate linear models. The new method, which we subsequently abbreviate as the GT method, is an extension of a generalized T-procedure for the unbalanced one way ANOVA model given in [7].

In section 2 we derive the new procedure in full rank and in less than full rank models. Some important features of the procedure are discussed in section 3. In order to gain insight into the types of models where the new procedure is expected to produce satisfactory confidence intervals for pairwise contrasts, the discussion of section 3 is followed by examples from three important designs. In section 5 we discuss a modified conservative GT procedure which constitutes a combination of a GT1 procedure as in [2] and the GT method. The proofs of some of the theorems are given in an appendix for purposes of a convenient reading.

2. The General Setup and the GT Method.

Suppose we are given a model for univariate data being linear in the fixed effects $\theta = (\theta_1, \dots, \theta_k)' \in \mathbb{R}^k$. We do not exclude from the model some other concomitant parameters, such as fixed block effects or regression coefficients on fixed covariates. For purposes of inference, the usual assumptions on the normal distribution of the error terms are made. Many of such models usually fall

under the following general setup at their inference stage. We refer the reader to [3] Ch. 11 and to [1] for two major examples, namely, the general incomplete block design and general designs with covariables, respectively.

The general univariate fixed effects linear normal model at an inference stage

Based on a class \mathcal{B} of matrices $B = ((b_{ij}))_{i,j=1,\dots,k}$, (usually \mathcal{B} is the class of generalized inverses of the matrix involved in the normal equations for θ), a class $\{\tilde{\theta}\}$ of random vectors $\tilde{\theta}$ (usually the class of solutions for θ in the normal equations) can be generated. Let $\mathcal{L}_e = \{\underline{l} \in \mathbb{R}^k : E(\underline{l}'\tilde{\theta}) = \underline{l}'\theta\}$ be invariant to any $\tilde{\theta} \in \{\tilde{\theta}\}$. These assumptions together with the following four constitute a model subsequently referred to as the general normal setup.

- i) \mathcal{B} includes symmetric p.d. (positive definite) matrices.
- ii) \mathcal{L}_e is a linear subspace of dimension e . The set $\{\underline{l}'\theta : \forall \underline{l} \in \mathcal{L}_e\}$ constitute the space of estimable parametric functions.
- iii) For all $\underline{l}_1, \underline{l}_2 \in \mathcal{L}_e$, $\text{Cov}(\underline{l}_1'\tilde{\theta}, \underline{l}_2'\tilde{\theta}) = \sigma^2 \underline{l}_1' B \underline{l}_2$ invariant to any choice of $B \in \mathcal{B}$. The quantity σ^2 is the unknown variance of the error term involved in the original model.
- iv) An unbiased estimator s_v^2 of σ^2 is available such that $\nu s_v^2 / \sigma^2$ is a central chi-square variable with ν d.f.

Let $\underline{l} = (l_1, \dots, l_k)' \in \mathbb{R}^k$, put $l_i^+ = \max\{l_i, 0\}$, $l_i^- = \max\{-l_i, 0\}$ and define

$$M(\underline{l}') = \max\{\sum l_i^+, \sum l_i^-\} \quad (2.1)$$

We denote by $R(\underline{l})$ ($\tilde{R}(\underline{l})$) the range (augmented range) of a vector \underline{l} . By $q_{k,\nu}^{(\alpha)}$ ($\tilde{q}_{k,\nu}^{(\alpha)}$) we denote the $1-\alpha$ percentile of the studentised range (augmented range) distribution with parameters k, ν . (Cf. [5] Ch. 2 for definitions).

We now describe the GT method in full rank models, i.e., when $\xi_e = \mathbb{R}^k$ which in our general normal setup implies a unique B and a unique $\tilde{\theta}$.

Theorem 2.1

Let Q: $k \times k$ be a matrix of real elements satisfying

$$Q Q' = B \quad (2.2)$$

then, the probability is $1-\alpha$ that all linear parametric functions $\underline{\xi}'\theta$ simultaneously satisfy the event

$$\{\underline{\xi}'\theta \in [\underline{\xi}'\tilde{\theta} \pm s_{\sqrt{k}, \nu} \tilde{q}_{k, \nu}^{(\alpha)} M(\underline{\xi}'Q)], \forall \underline{\xi} \in \mathbb{R}^k\} \quad (2.3)$$

Proof. First we note that B is p.d. and thus there always exists a matrix Q satisfying (2.2) and Q is non-singular. The linear combination $\underline{\xi}'(\tilde{\theta}-\theta)$ can be written as

$$\underline{\xi}'(\tilde{\theta}-\theta) = \underline{\xi}'QQ^{-1}(\tilde{\theta}-\theta) \quad (2.4)$$

The vector $Q^{-1}(\tilde{\theta}-\theta)$ is normal with zero mean and dispersion $\sigma^2 I$. Now, the event

$$\{\tilde{R}(Q^{-1}(\tilde{\theta}-\theta)) \leq s_{\sqrt{k}, \nu} \xi\} \quad (2.5)$$

is equivalent to the event

$$\{|\underline{u}'Q^{-1}(\tilde{\theta}-\theta)| \leq s_{\sqrt{k}, \nu} \xi M(\underline{u}), \forall \underline{u} \in \mathbb{R}^k\} \quad (2.6)$$

which in terms of the original $\underline{\xi}$'s can be written as

$$\{|\underline{\xi}'QQ^{-1}(\tilde{\theta}-\theta)| \leq s_{\sqrt{k}, \nu} \xi M(\underline{\xi}'Q), \forall \underline{\xi} \in \mathbb{R}^k\} \quad (2.7)$$

On equating the probability of (2.5) to $1-\alpha$ we get $\xi = \tilde{q}_{k, \nu}^{(\alpha)}$ and the Theorem now follows from the equivalence of (2.5) and (2.7) and from (2.4).

Note that Spjøtvoll and Stoline's method in [7] is a GT method in the special case when B is diagonal. These authors use for Q the matrix

$\text{Diag}(b_{11}^{1/2}, \dots, b_{kk}^{1/2})$. We also note that for any given B a matrix Q satisfying (2.2) exists but is not unique, and that the GT procedure is not invariant under all possible choices of a matrix Q. This non-invariance property exists also in Spjøtvoll and Stoline's diagonal case, and even in the original T-method which can always be subjected to an orthogonal transformation of the original vector $\tilde{\theta}$. A discussion on the multiplicity of choices of a Q matrix in (2.2) and the choice of an appropriate one is given in section 3.

A simple GT3 procedure in models of less than full rank is based on the fact that the original model can always be reparametrized to a full rank model by transforming θ to a vector $\theta_e: e \times 1$ constituting a basis for the space of estimable parametric functions. Suppose that θ_e is given by

$$\theta_e = L\theta, \quad L: e \times k \quad (2.8)$$

This brings us to a full rank model with $\tilde{\theta}_e = L\tilde{\theta}$ and $\tilde{\theta}_e \sim N(\theta_e, \sigma^2 LBL')$. The procedure embodied in Theorem 2.1 can now be used in the transformed setup to provide simultaneous interval estimation of all linear parametric functions (of θ_e) with an exact $1-\alpha$ confidence level. The fact that various different choices of a basis θ_e for the estimable functions exist does not increase the set of different possible GT procedures. A discussion of this point is given in the Appendix by Lemma A.

An alternative GT method in models of less than full rank is based on the following Theorem, the proof of which is given in the Appendix. For this alternative procedure we assume that \mathcal{L}_e is the contrasts subspace in \mathbb{R}^k . This subspace of \mathbb{R}^k is hereafter denoted by C^k .

Theorem 2.2

Let $B \in B$ be any symmetric p.d. matrix. The probability is at least $1-\alpha$ that all contrasts $c'\theta$ simultaneously satisfy

$$\{c'\theta \in [c'\tilde{\theta} \pm s_{\nu} \tilde{q}_{k,\nu}^{(\alpha)} M(c'Q_B)], \forall c \in C^k\} \quad (2.9)$$

where $Q_B Q_B' = B$ and $\tilde{\theta}$ is any element in $\{\tilde{\theta}\}$.

There are few reasons for willing to use, in some cases, a conservative procedure as given by Theorem 2.2, rather than an exact procedure obtained by first transforming to a full rank model.

1. The quantity $\tilde{q}_{k,\nu}^{(\alpha)}$ is not tabulated. It is well known that we may approximate it by $q_{k,\nu}^{(\alpha)}$ if $\alpha \leq .05$ and $k \geq 3$, (Cf. [6], p. 78). However, if one wants to be on the safe side, especially when $\alpha > .05$ is used, the conservative procedure might be more appropriate.

2. In many cases the parametric functions of interest are easier to specify in terms of the original parameters θ than in terms of the transformed vector θ_e .

3. In some problems it is easier to find an appropriate (to be discussed in the next section) Q matrix corresponding to a certain patterned $B \in B$ (any of which can be used as long as it is symmetric and p.d.) than a Q corresponding to the uniquely determined matrix LBL' .

In various important unbalanced designs, we may get an exact $1-\alpha$ confidence level for the simultaneous estimation of all contrasts based on $q_{k,\nu}^{(\alpha)}$. This is given by the following Theorem the proof of which is given in the Appendix. This Theorem assumes the setup and notation of Theorem 2.2. For convenience we write Q for Q_B .

Theorem 2.3

Let q_1, \dots, q_k constitute the k rows of Q , $q_i: 1 \times k$ ($i=1, \dots, k$) and suppose the condition

$$q_i 1_k = q_j 1_k \quad \forall (1 \leq i < j \leq k). \quad (2.10)$$

is satisfied. The probability is $1-\alpha$ that all contrasts simultaneously satisfy

$$c'\theta \in [c'\tilde{\theta} \pm s_{\nu} q_{k,\nu}^{(\alpha)} M(c'Q)], \forall c \in C^k \quad (2.11)$$

3. On the Choice of the Q Matrix

The various procedures introduced in section 2, entirely depend on the specific choices of Q matrices satisfying (2.2) for a given B matrix.

1. Let $D^{1/2} = \text{Diag}(\lambda_1^{1/2}, \dots, \lambda_k^{1/2})$ where the λ_i 's are the k positive latent roots of B and let $R = (r_1, \dots, r_k)$ where the r_i 's are any corresponding normalized latent vectors. Clearly Q can be chosen as

$$Q = RD^{1/2} \quad (3.1)$$

The multiplicity of choices for a Q matrix in this class is well known. A smaller class is the set of lower triangular matrices T satisfying

$$T T' = B \quad (3.2)$$

Clearly, Q can be chosen as any of the 2^k possible matrices T. However, many other types of matrices Q satisfying (2.2) do exist. Moreover, searches for satisfactory Q matrices in the above two classes have been non-fruitful so far, for various designs.

Can we always find a Q satisfying (2.2) which gives shorter confidence intervals for all pairwise contrasts simultaneously than any other Q? The answer to this question is that generally we cannot! A simple case when an optimal choice of Q for the class of pairwise contrasts exists, is the case when B is of the form $B=aI$. A matrix B of that structure is said to be uniform diagonal. The optimal choice in this case is $Q=a^{1/2}I$ which gives the original T-method. To see this, first note that on letting $c_{(ij)} \in C^k$ denote the vector of coefficients for the (i,j)th pairwise contrast

$$M(c_{(ij)}'Q) = M(q_i - q_j) \quad (3.3)$$

where q_i is the i th row of Q , ($i=1, \dots, k$). Next note that $(q_i - q_j)(q_i - q_j)' = 2a$ and it is easily verified that

$$\min_{\underline{q} \in \mathbb{R}^k: \underline{q}'\underline{q} = 2a} \{M(\underline{q}')\} = a^{1/2}. \quad (3.4)$$

As mentioned above, such an optimal choice for Q does not always exist. This can easily be verified in Spjøtvoll and Stoline's general diagonal case.

In crude terms we may say that in general one would try to obtain a Q matrix with a structure as close as possible to a uniform diagonal matrix. It seems that this cannot be expected in designs which are more than moderately unbalanced. However, in many important designs, Q can be chosen as to have the same structure as that of B . In such cases if B is sufficiently close to a uniform diagonal matrix, we may expect a good performance of a GT procedure based on a Q matrix chosen to be structured as B . This approach is an extension of the Spjøtvoll and Stoline's approach where they choose $Q = \text{Diag}(b_{11}^{1/2}, \dots, b_{kk}^{1/2})$ in the diagonal case when B is given by $\text{Diag}(b_{11}, \dots, b_{kk})$. We further explore this method in the next section by demonstrating it in three different designs.

4. Some Examples

Example 4.1 ANOCVA with one concomitant variable and equal sample sizes.

(See [1]).

This is an example of a full rank model where the original T-method can not be used. Let Y_{ij} ($j=1, \dots, n$; $i=1, \dots, k$) be $N=kn$ independent random variables with a common variance σ^2 and expectations given by

$$E Y_{ij} = \theta_i + bX_{ij}, \quad j=1, \dots, n; \quad i=1, \dots, k \quad (4.1)$$

Let $\bar{X}_i = \sum_{j=1}^n X_{ij}/n$, $\bar{Y}_i = \sum_{j=1}^n Y_{ij}/n$,

$$S_{xy} = \sum_{i=1}^k \sum_{j=1}^n (X_{ij} - \bar{X}_i)(Y_{ij} - \bar{Y}_i), \quad S_{xx} = \sum_{i=1}^k \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2,$$

$$\hat{b} = S_{xy}/S_{xx}, \quad SS_{\hat{b}} = S_{xy}^2/S_{xx} \quad \text{and} \quad \nu = N-k-1.$$

The least squares estimator of θ_i adjusted for the concomitant variable is $\tilde{\theta}_i$ given by

$$\tilde{\theta}_i = \bar{Y}_i - \hat{b}\bar{X}_i, \quad (i=1, \dots, k) \quad (4.2)$$

This is a full rank model. The dispersion of the $\tilde{\theta}_i$'s is given by

$$\sigma^2 \left[\frac{1}{n} I + \underline{b}\underline{b}' \right] \quad (4.3)$$

where $\underline{b} = (b_1, \dots, b_k)'$, $b_i = \bar{X}_i/S_{xx}^{1/2}$ ($i=1, \dots, k$). The unbiased estimator of σ^2 is

$$s_{\nu}^2 = (S_{yy} - SS_{\hat{b}})/\nu \quad (4.4)$$

To apply a GT procedure as given by Theorem 2.1, we look for a Q matrix with the same structure as that of B in (4.3), i.e., $Q = aI + \underline{h}\underline{h}'$, for some $a \in \mathbb{R}^1$ and $\underline{h} = (h_1, \dots, h_k)' \in \mathbb{R}^k$. To find the value of a and \underline{h} we write the equation

$$\frac{1}{n}I + \underline{b}\underline{b}' = (aI + \underline{h}\underline{h}')(aI + \underline{h}\underline{h}')' = a^2I + (2a + \underline{h}'\underline{h})\underline{h}\underline{h}' \quad (4.5)$$

We get: $a^2 = 1/n$ and $\underline{h} = k^{1/2}\underline{b}$ for some scalar k satisfying the equation

$$k(2a + k\underline{b}'\underline{b}) = 1 \quad (4.6)$$

The resulting confidence intervals for the pairwise contrasts based on the GT method are given by

$$\theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j \pm s_{\nu} \tilde{q}_{k, \nu}^{(\alpha)} ((1/n)^{1/2} + |h_i - h_j| M(\underline{h}'))] \quad (1 \leq i < j \leq k) \quad (4.7)$$

From (4.7) and the above formulas for the values of the h_i 's it is clear that the closer the values of \bar{X}_i and \bar{X}_j are, the shorter is the confidence interval for $\theta_i - \theta_j$.

Example 4.2 A group divisible PBIB design. (See [3] Ch. 12).

This is an example of a model of less than full rank where the original T-method

cannot be used. Consider a design with k treatments, b blocks, each of the treatments being tested in r blocks, each of the blocks consisting of d plots. Let n_{ij} be the number of times the i th treatment is being tested in the j th block. Put $R=r I: k \times k$ $D=dI: b \times b$ and $N=((n_{ij})): k \times b$. We assume the additive fixed effects linear model $Y_{ij} = \mu + \theta_i + \beta_j + \epsilon_{ijm}$, where the θ_i 's and β_j 's are the treatment and block effects respectively, μ is the overall mean and the ϵ_{ijm} are i.i.d. normal with a zero mean and variance σ^2 . Denote by t_i and b_j the i th treatment and the j th block totals respectively. We let $\underline{t}=(t_1, \dots, t_k)'$, $\underline{b}=(b_1, \dots, b_b)'$ and get the normal equations for the treatment effects having eliminated blocks as

$$(R-ND^{-1}N')\theta = \underline{t}-ND^{-1}\underline{b}, \quad \theta=(\theta_1, \dots, \theta_k)' \quad (4.8)$$

Suppose the design is a PBIB(2) design with parameters $b, k, r, d, \lambda_1, \lambda_2, p_{j\ell}^i$, $i, j, \ell=1, 2$. We put

$$\begin{aligned} A_{12} &= r(d-1)+\lambda_2; \quad A_{22} = (\lambda_2-\lambda_1)p_{12}^2; \quad B_{12} = \lambda_2-\lambda_1, \\ B_{22} &= r(d-1)+\lambda_2+(\lambda_2-\lambda_1)(p_{11}^1-p_{11}^2); \quad \Delta_r = A_{12}B_{22}-A_{22}B_{12} \end{aligned} \quad (4.9)$$

As shown by C. R. Rao, (cf. [3] p. 257) a solution matrix $B=((b_{ij}))_{i,j=1, \dots, k}$ for the equations (4.8) is given as

$$\left. \begin{aligned} b_{ii} &= B_{22}d\Delta_r^{-1}; \quad b_{ij} = -dB_{12}\Delta_r^{-1} \text{ if } (i,j) \text{ are first associates} \\ b_{ij} &= 0 \text{ otherwise } \quad (1 \leq i < j \leq k) \end{aligned} \right\} \quad (4.10)$$

This symmetric p.d. matrix B is one member of the class B of all solution matrices to the equations (4.8). The solution vector $\tilde{\theta}=B(\underline{t}-ND^{-1}\underline{b})$ satisfies

$$\text{cov}(c_1'\tilde{\theta}, c_2'\tilde{\theta}) = \sigma^2 c_1' B c_2, \quad \forall c_1, c_2 \in C^k \quad (4.11)$$

Clearly this setup pertains to the general normal model introduced in section 2.

The class \mathcal{B} is the set of all solution matrices to (4.8). To use GT procedures as given by Theorems 2.2 and 2.3 we may use any symmetric p.d. matrix from \mathcal{B} . However, as we subsequently exemplify, the choice of B given by (4.10) is particularly convenient for a derivation of an appropriate Q matrix. Suppose the PBIB(2) design is a group divisible design with m groups and n treatments in each group, $mn=k$. In this case B is of the form

$$B = I \otimes U, \quad I: m \times m, \quad U: n \times n \quad (4.12)$$

where $U = (a-c)I + c11'$ with a and c as given by (4.10) and \otimes denotes the direct product operation. If Q is to be of the same structure as B , we must have

$$Q = I \otimes P \quad (4.13)$$

where $P = (g-f)I + f11'$ and such that (2.2) is satisfied. This results in the equations

$$\begin{aligned} \text{I: } & (n-1)f^2 + g^2 = a \\ \text{II: } & (n-2)f^2 + 2gf = c \end{aligned} \quad (4.14)$$

which are easily solved for f and g . In this case, with that choice of Q , Theorem 2.3 can be used and the resulting confidence intervals for the pairwise contrasts are given by

$$\left. \begin{aligned} \theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j + s_{\nu} q_{k,\nu}^{(\alpha)} |g-f|], & \text{ if } (i,j) \text{ are first associates} \\ \theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j + s_{\nu} q_{k,\nu}^{(\alpha)} (|g| + (n-1)|f|)], & \text{ if } (i,j) \text{ are second associates} \end{aligned} \right\} \quad (4.15)$$

where s_{ν}^2 is the usual mean SS for error based on $\nu = bd - k - b + 1$ d.f. If $n=1$ the design is a balanced incomplete block design in which Tukey's T-method can be applied. It comes out as a special case of the GT method for $n=1$. It is obvious that for large values of m and low values of n the GT procedure can be expected to produce the best results in a comparison with any of the other possible methods. Before closing this section we will bring a numerical example to demonstrate this fact.

Example 4.3 A complete randomized block design with missing data. (See [4] p. 173).

Consider a complete randomized block design with k treatments and b blocks. Suppose that two observations are missing in one block which without loss of generality will be assumed to belong to the $(k-1)$ th and the k th treatments. The initial assumed model is

$$Y_{ij} = \theta_i + \beta_j + e_{ij}; \quad (i=1, \dots, k; j=1, \dots, b) \quad (4.16)$$

where the θ_i 's and the β_j 's are fixed unknown treatment and block effects respectively and the e_{ij} 's are i.i.d. normal variables with zero means and variance σ^2 . Denote by T_i' the total of the available observations for the i th treatment; $i=k-1, k$, and let B denote the total of the $k-2$ observations in the block where the data is not complete. Let G denote the sum of all $bk-2$ available observations. Following a well known method, due to F. Yates, we estimate the missing observations and then use the data and the estimates to analyze a 'complete' block design rather than having to handle an incomplete block design. Following that method, on denoting the estimates of the missing values by \hat{Y}_{k-1} and \hat{Y}_k we obtain the equations

$$\hat{Y}_i = (T_i' + \hat{Y}_i)/b + (B + \hat{Y}_{k-1} + \hat{Y}_k)/k - (G + \hat{Y}_{k-1} + \hat{Y}_k)/bk, \quad (i=k-1, k) \quad (4.17)$$

Solving the equations we get

$$\begin{aligned} (k-2)(b-1)\hat{Y}_{k-1} &= (k-1)T_{k-1}' + T_k' + bB - G \\ (k-2)(b-1)\hat{Y}_k &= (k-1)T_k' + T_{k-1}' + bB - G \end{aligned} \quad (4.18)$$

Denote by T_i ($i=1, \dots, k$) the total of the observations (including estimated values) for the i th treatment. The estimates of the θ_i 's are given as

$$\tilde{\theta}_i = T_i/b \quad (i=1, \dots, k) \quad (4.19)$$

The dispersion matrix of the $\tilde{\theta}_i$'s is $\sigma^2 B$ where $B = ((b_{ij}))$ is given by

$$B = \begin{pmatrix} D: (k-2) \times (k-2) & 0 \\ 0 & V: 2 \times 2 \end{pmatrix} \quad (4.20)$$

$D = (1/b)I$ and $V = (w-r)I + r11'$ where $w = \frac{1}{b}[1 + \frac{k-1}{(b-1)(b-2)}]$ and $r = 1/[b(b-1)(b-2)]$. The error mean SS here is calculated from the complete data but has two d.f. less than that in an actual complete block design. A matrix Q of the same structure satisfying $QQ' = B$ is given by

$$Q = \begin{pmatrix} (1/b)^{1/2}I: (k-2) \times (k-2) & 0 \\ 0 & (f-g)I + g11': 2 \times 2 \end{pmatrix} \quad (4.21)$$

where $g^2 = [w + (w^2 - r^2)]/2$ and $f^2 = w - g^2$. Using Theorem 2.1 we get the following intervals for the pairwise contrasts

$$\begin{aligned} \theta_i - \theta_j &\in [\tilde{\theta}_i - \tilde{\theta}_j \pm (1/b)^{1/2} s_{v-2} \tilde{q}_{k,v-2}^{(\alpha)}], \quad 1 \leq i < j \leq k-2 \\ \theta_i - \theta_j &\in [\tilde{\theta}_i - \tilde{\theta}_j \pm |f+g| s_{v-2} \tilde{q}_{k,v-2}^{(\alpha)}], \quad 1 \leq i \leq k-2 < j \leq k \\ \theta_{k-1} - \theta_k &\in [\tilde{\theta}_{k-1} - \tilde{\theta}_k \pm |f-g| s_{v-2} \tilde{q}_{k,v-2}^{(\alpha)}], \end{aligned} \quad (4.22)$$

It is almost readily obvious that for any value of b and a moderately large value of k the GT method will give shorter intervals than the S-method and the conservative procedures discussed in [2] for all pairwise contrasts.

A numerical demonstration for example 4.2.

Suppose we have the following group divisible PBIB(2) design for the six treatments 1,2,3,4,5,6 in six blocks

$$(1,2,3); (3,4,5); (2,5,6); (1,2,4); (3,4,6); (1,5,6) \quad (4.23)$$

The matrix B given by (4.12) is completely specified when specifying U . In this example we get

$$U = \begin{pmatrix} 21/48 & 3/48 \\ 3/48 & 21/48 \end{pmatrix}, \quad (4.24)$$

The matrix Q for this example is then given by

$$Q = \begin{pmatrix} .65974 & .04737 \\ .04737 & .65974 \end{pmatrix} \quad (4.25)$$

From (4.15) we obtain the following confidence intervals

$$\begin{aligned} \theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j + s_{\nu} 3.3007], & \text{ if } (i,j) \text{ are first associates} \\ \theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j + s_{\nu} 3.8113], & \text{ if } (i,j) \text{ are second associates.} \end{aligned} \quad (4.26)$$

Using Scheffé's S-method on this particular example we get

$$\begin{aligned} \theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j + s_{\nu} 3.8594] & \text{ for first associates} \\ \theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j + s_{\nu} 4.1684] & \text{ for second associates.} \end{aligned} \quad (4.27)$$

Using the GT1 method of [2] we obtain the conservative intervals

$$\theta_i - \theta_j \in [\tilde{\theta}_i - \tilde{\theta}_j + s_{\nu} 3.8113] \text{ for all } 1 \leq i < j \leq k. \quad (4.28)$$

The GT2 method discussed in [2] is based on the $1-\alpha$ percentile of the studentized maximum modulus distribution with parameters (k', ν) , where $k' = k(k-1)/2$. In this example $k'=15$ and $\nu=7$ and the appropriate constant has not yet been tabulated. However, from the discussion in section 4 of [2] it is clear that we can expect the GT1 procedure to give better results than the GT2 does in this specific example.

5. On a Modified GT Method

In various designs it is difficult to find a satisfactory Q matrix for the use of a GT procedure. This happens in many incomplete block designs other than the class of PBIB designs and in various models other than the

balanced one way ANOVA when fixed covariates are introduced. In all these cases pertaining to the general normal setup a conservative procedure is now defined which involves features of the GT method and the GT1 procedure introduced in [2].

We denote this modified GT procedure by MGT. Given a $B = (b_{ij})$ matrix let $S_i(a) = \sum_{j=1, j \neq i}^k |a - b_{ij}|$, $a \in \mathbb{R}^1$ and $B^* = \text{Diag}((b_{ii} + S_i(a) - a)^{1/2})$, $(i=1, \dots, k)$.

Theorem 5.1

The probability is at least $1 - \alpha$ that all estimable linear functions simultaneously satisfy

$$\xi' \theta \in [\xi' \tilde{\theta} \pm s_{\xi} \tilde{q}_{k, \nu}^{(\alpha)} M(\xi' B^*)], \quad \forall \xi \in \mathcal{L}_e \quad (5.1)$$

Proof. Follows from Theorem 2.1 in [2] and the above Theorems in this paper.

Note that various MGT procedures exist corresponding to different values of a . For the (i, j) th pairwise contrast we get

$$M(\tilde{c}_{(ij)}' B^*) = \max\{(b_{ii} + S_i(a) - a)^{1/2}, (b_{jj} + S_j(a) - a)^{1/2}\} \quad (5.2)$$

A reasonable MGT procedure, i.e., a reasonable choice for a is the value of a that minimizes the largest possible value of $M(\tilde{c}_{(ij)}' B^*)$. This value which we denote by a^* , clearly satisfies

$$\min_a \max_{1 < i < k} \{b_{ii} + S_i(a) - a\} = \max_{1 < i < k} \{b_{ii} + S_i(a^*) - a^*\} \quad (5.3)$$

This optimal choice a^* for a is exactly the choice for a in the GT1 procedure.

A search procedure for a^* is outlined in [2].

Appendix

Lemma A

Suppose that in an original model of less than full rank two reparametrizations are given

$$\theta_{ei} = L_1 \theta, \tilde{\theta}_{ei} = L_1 \tilde{\theta}, \tilde{\theta}_{ei} \sim N(\theta_{ei}, L_1 B L_1'), \quad i=1,2 \quad (\text{A.1})$$

The two resulting classes of GT procedures are equivalent in the sense that there exists a non-singular A such that a GT based on the second reparametrization with Q_2 , is the same as a GT based on the first with $Q_1 = A Q_2$.

Proof. Let A be the (unique) matrix satisfying

$$L_1 = A L_2 \quad (\text{A.2})$$

Since $Q_2 Q_2' = L_2 B L_2'$ it follows that

$$Q_1 Q_1' = A L_2 B L_2' A' = L_1 B L_1' \quad (\text{A.3})$$

Let $\underline{\lambda}: e \times 1$ be the coefficients of a certain linear function in terms of θ_{e2} , the same parametric function in terms of θ_{e1} has a vector of coefficients $A^{-1} \underline{\lambda}$. This implies that the argument of the function $M(\cdot)$ when using the first reparametrization with $Q_1 = A Q_2$ is $\underline{\lambda}' A^{-1} Q_1 = \underline{\lambda}' A^{-1} A Q_2 = \underline{\lambda}' Q_2$, which proves the Lemma.

Proof of Theorem 2.2

The dispersion matrix of $\tilde{\theta}$ can be expressed as

$$\sigma^2 (B + E_B) \quad (\text{A.4})$$

for some real matrix $E_B = ((e_{ij}^B))_{i,j=1,\dots,k}$. It is however clear from the assumptions underlying the general normal setup at an inference stage and the assumption $\underline{\lambda}_e = C^k$ of this Theorem that

$$e_{ij}^B = e_{ji}^B, \quad e_{ii}^B + e_{jj}^B = 2e_{ij}^B \quad \forall (1 \leq i \neq j \leq k) \quad (\text{A.5})$$

Next we show that it is always possible to construct a matrix $R: k \times k$ of the form $R = 1 \otimes \underline{h}'$, $\underline{h} = (h_1, \dots, h_k)' \in \mathbb{R}^k$ such that

$$(Q+R)(Q+R)' = B + E_B \quad (\text{A.6})$$

This is so since (A.6) together with $QQ' = B$ implies

$$E_B = QR' + RQ' + RR' = 1' \otimes Q\underline{h} + 1 \otimes \underline{h}'Q' + 11' \otimes \underline{h}'\underline{h} \quad (\text{A.7})$$

which can always be solved for \underline{h} . Following the same line of argument as in the proof of Theorem 2.1 we have that the probability is $1-\alpha$ that all $\underline{z} \in \mathbb{R}^k$ simultaneously satisfy

$$|\underline{z}'(\tilde{\theta} - E\tilde{\theta})| \leq s_{\nu} \tilde{q}_{k,\nu}^{(\alpha)} M(\underline{z}'(Q+R)) \quad (\text{A.8})$$

The theorem now follows on realizing that for all $\underline{c} \in \mathbb{C}^k$

$$M(\underline{c}'(Q+R)) = M(\underline{c}'Q), \quad E\underline{c}'\tilde{\theta} = \underline{c}'\theta \quad \text{and} \quad \underline{c} \in \mathbb{R}^k.$$

Proof of Theorem 2.3

The condition on Q implies that $\underline{c}'Q \in \mathbb{C}^k$ for any $\underline{c} \in \mathbb{C}^k$. This together with the equivalence of the events

$$\{R(\tilde{\theta} - E\tilde{\theta}) \leq s_{\nu} q_{k,\nu}^{(\alpha)}\} \quad \text{and} \quad \{|\underline{c}'(\tilde{\theta} - E\tilde{\theta})| \leq s_{\nu} q_{k,\nu}^{(\alpha)} M(\underline{c}'), \quad \forall \underline{c} \in \mathbb{C}^k\} \quad (\text{A.9})$$

and the assumption that $\mathfrak{L}_e = \mathbb{C}^k$ imply the theorem.

References

- [1] Bose, R. C. (1965). The general analysis of covariance. Lecture notes transcribed by Gary G. Koch. Available from the Biostatistics Department of UNC at Chapel Hill, N. C., U.S.A.
- [2] Hochberg, Y. (1973). Some conservative generalizations of the T-method in simultaneous inference. Contributed paper, Annual IMS meeting New York, N. Y. To be published in the J. of Mult. Anal., 1974.
- [3] John, P. W. M. (1971). Statistical design and analysis of experiments. The Macmillan Company, New York.
- [4] Kempthorne, O. (1952). The design and analysis of experiments. John Wiley Inc., New York.
- [5] Miller, R. G., Jr. (1966). Simultaneous statistical inference. McGraw-Hill, New York.
- [6] Scheffé, H. (1959). The analysis of variance. John Wiley Inc.
- [7] Spjøtvoll, E. and Stoline, M. R. (1973). An extension of the T-method of multiple comparison to include the cases with unequal sample sizes. Contributed paper, IMS meeting, Ithaca, N. Y. To be published in JASA 1974.
- [8] Tukey, J. (1953). The problem of multiple comparisons. Unpublished manuscript. Princeton University.