# OPTIMALITY WITHIN THE CLASS OF SEQUENTIAL PROBABILITY RATIO TESTS

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# ABSTRACT

This paper shows there is a very strong optimality property within the class of sequential probability ratio tests (SPRT). This property, which was discovered by B.K. Ghosh in a weaker form, is completely analogous to the classical optimality property found by Wald and Wolfowitz. The only real differences are the following. It is weaker in the sense that nothing is said about tests which are not SPRT's. But it is stronger in the sense that absolutely no assumptions are made about the data (such as being i.i.d.) and in the sense that conclusions about the numerical ordering of expected sample sizes are replaced by conclusions about the stochastic ordering of stopping variables. The property may be roughly paraphrased to state that *one can not reduce the error probabilities of an SPRT without additional sampling*. Although intuitively reasonable, this does not seem to be mathematically obvious.

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1. Introduction. B.K. Ghosh (1970) has described a type of optimality - "uniformly most efficient" - which is possessed by every sequential probability ratio test (SPRT) irrespective of assumptions on the data. It will be recalled that for data which is i.i.d. under both (simple) hypotheses and for a given SPRT, one can not reduce either error probability without resorting to a test of larger expected sample size (under both hypotheses). This expresses the well-known optimality property discovered by Wald and Wolfowitz (1948). Ghosh's result essentially says that this same property holds *without* the i.i.d. assumption *within* the class of SPRT's. Specifically, there is no other SPRT with smaller error probabilities and a smaller expected sample size.

This paper strengthens Ghosh's result and circumvents a questionable step in his proof. Let $\alpha$ and $\beta$ be the error probabilities for a given SPRT. I.e., $\alpha$ ($\beta$) is the probability that the test eventually terminates with the rejection of the null (alternative) hypothesis when it is true. Let $N$ be the sample size when the test is terminated. ($N = \infty$ if no termination occurs.) Finally, let $\alpha'$, $\beta'$ and $N'$ be the corresponding quantities for a competing SPRT. We prove the following two results:

(i) *If* $\alpha' \le \alpha$, $\beta' \le \beta$, *then* $N \le N'$ *almost surely under both hypotheses. Further, the terminal decisions are the same* *when* $N' = N < \infty$ *almost surely under both hypotheses.*[1/]

(ii) *If, in addition,* $\alpha' < \alpha$ *or* $\beta' < \beta$, *then the event* $[N < N']$ *occurs with positive probability under both hypotheses.*

That such strong results should hold is not completely surprising. They follow fairly easily for the i.i.d. case from the classical optimality property previously mentioned. However, they can be derived - even for the general case - from elementary considerations.

We shall begin by establishing a basic result which applies to any randomly stopped sequence of likelihood ratios. With this, we produce some identities which relate pairs of SPRT's. These, in turn, are used to prove (i) and (ii).

2. Randomly stopped likelihood ratios. Let $P$ and $Q$ be fixed probability measures on a measurable space $(\Omega, \mathcal{F})$ and $\mathcal{E}$ a sub - $\sigma$ - field of $\mathcal{F}$. An extended non-negative random variable $\lambda$ will be called a *likelihood ratio* for $\mathcal{E}$ if it is $\mathcal{E}$ - measurable and

(1) $$\int_E \lambda \, dP = Q(E, \lambda \neq \infty), \int_E \lambda^{-1} \, dQ = P(E, \lambda \neq 0), E \in \mathcal{E}$$

(where $\lambda^{-1} = 0$ when $\lambda = \infty$). Moreover, if $\mathcal{E}$ is generated by a random mapping $X$, $\lambda$ will be called a *likelihood ratio* for $X$ as

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1/ The terminal decisions could disagree when $N = N' < \infty$ if the continuation regions for the two SPRT's are disjoint.

well.[2/] $\lambda$ must necessarily exist, and it is unique up to a $P$ and $Q$ equivalence.

Let $\mathcal{E}_1, \mathcal{E}_2, \ldots$ be a nondecreasing sequence of sub - $\sigma$ - fields of $\mathcal{F}$, and let $N$ be a stopping variable relative to this sequence. I.e., the event $[N = n] \in \mathcal{E}_n$, $n \geq 1$. Further, let $\lambda_n$ be the likelihood ratio for $\mathcal{E}_n$ and $\mathcal{E}_N$ be the $\sigma$-field of $\mathcal{F}$ - measurable events $E$ for which $E[N = n] \in \mathcal{E}_n$, $n \geq 1$. It easily follows from (1) (by partitioning the event $[N < \infty]$) that

$$(2) \int_{E[N<\infty]} \lambda_N \, dP = Q(E, N < \infty, \lambda_N \neq \infty), \int_{E[N<\infty]} \lambda_N^{-1} \, dQ = P(E, N < \infty, \lambda_N \neq 0), E \in \mathcal{E}_N.$$

**Thus:**

(3a) $P(E) = 0 \Rightarrow Q(E) = 0$ for $E \in [N < \infty, \lambda_N \neq \infty] \, \mathcal{E}_N$.

(3b) $Q(E) = 0 \Rightarrow P(E) = 0$ for $E \in [N < \infty, \lambda_N \neq 0] \, \mathcal{E}_N$.

3. Identities. We continue using the notation of Section 2 but particularize $N$ in two different ways - one with a prime. It is convenient, but not essential, to think of $\lambda_n$ as a likelihood ratio for a sample of size $n$, taken from an infinite sequence of potential observations.

Let $S(A,B)$, $0 \leq A < B \leq \infty$, be an SPRT. Formally, it can be

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This convenient defintion is equivalent to the more conventional definition involving the ratio of densities. $\lambda = \infty$ occurs when the denominator is zero and the numerator is positive.

described in terms of a pair (N,D) where

$N$ = first $n \geq 1$ such that $\lambda_n \notin (A,B)$, $= \infty$ if no such $n$ exists;

$D = 0$ if $N < \infty$ and $\lambda_N \leq A$, $= 1$ if $N < \infty$ and $\lambda_N \geq B$.

The error probabilities are $\alpha = P(N < \infty, D = 1)$ and $\beta = Q(N < \infty, D = 0)$. Further, let $(N', D') = S(A', B')$ be a second SPRT with error probabilities $\alpha'$ and $\beta'$.

<u>Proposition 1</u>. *If* $A \leq A'$ *and* $B' \leq B$, *then*

$$\text{(4a)} \quad A(\alpha' - \alpha) + (\beta' - \beta) = \int_{[N'<N, D'=0]} (\lambda_{N'} - A)\,dP + \int_{[N'<N<\infty, D=0]} (A - \lambda_N)\,dP + AP(N'<N = \infty),$$

$$\text{(4b)} \quad (\alpha' - \alpha) + B^{-1}(\beta' - \beta) = \int_{[N'<N, D'=1]} (\lambda_{N'}^{-1} - B^{-1})\,dQ + \int_{[N'<N<\infty, D=1]} (B^{-1} - \lambda_N^{-1})\,dQ + B^{-1}\,Q(N'<N = \infty).$$

(We interpret $B^{-1}$ as zero when $B = \infty$.)

<u>Proof</u> $[N'<N, D' = 0] \in \mathcal{E}_{N'}$. By (2), the first integral of (4a) equals

$$Q(N'<N, D' = 0, \lambda_{N'} \neq \infty) - AP(N'<N, D' = 0) = Q(N'<N, D'=0) - AP(N'<N, D'=0).$$

Likewise, the second integral equals $AP(N'<N<\infty, D=0) - Q(N'<N<\infty, D=0)$. Thus it remains to check that $(\alpha' - \alpha =)$

$$P(N'<\infty, D'=1) - P(N<\infty, D=1) = P(N'<N<\infty, D=0) - P(N'<N, D'=0) + P(N'<N=\infty),$$

and $(\beta' - \beta =)$

$$Q(N'<\infty, D'=0) - Q(N<\infty, D=0) = Q(N'<N, D'=0) - Q(N'<N<\infty, D=0).$$

Both of these follow easily from geometrical considerations. In particular, one needs to observe that $N' \le N$ and that $N' = N < \infty \Rightarrow D'=D$. (4b) is proven similarly. □

Notice that each term of the right hand sides of the identities (4a) and (4b) is non-negative. Thus

$$(5) \quad A \le A',\ B' \le B \Rightarrow A(\alpha'-\alpha) + (\beta'-\beta) \ge 0,\ (\alpha'-\alpha) + B^{-1}(\beta'-\beta) \ge 0.$$

This means that it is impossible for the inequalities $\alpha' \le \alpha$, $\beta' \le \beta$ to hold with one of them strict. If $A \le 1 \le B$ (which is usually the case in practice), then $\alpha' + \beta' \ge \alpha + \beta$.

<u>Proposition 2</u>. If $A' \le A$ and $B' \le B$, then

$$(6a) \quad \beta-\beta' = \int_{[N<N']} (\lambda_N - A')\, dP + \int_{[N<N'<\infty,\ D'=0]} (A' - \lambda_{N'})\, dP + A' \{P(N<N'=\infty) + P(N<N'<\infty, D'=1)\}$$

$$+ Q(N'=N<\infty,\ D'=1,\ D=0) + Q(N'<N<\infty, D=0),$$

$$(6b) \quad \alpha'-\alpha = \int_{[N'<N]} (\lambda_{N'}^{-1} - B^{-1})\, dQ + \int_{[N'<N<\infty,\ D=1]} (B^{-1} - \lambda_N^{-1})\, dQ + B^{-1}\{Q(N'<N=\infty) + Q(N'<N<\infty, D=0)\}$$

$$+ P(N'=N<\infty,\ D'=1,\ D=0) + P(N<N'<\infty,\ D'=1).$$

<u>Proof</u> This proof is similar to the proof for Proposition 1. The geometrical implications needed for (6a) are $N<N' \Rightarrow D=0$, and that $N'<\infty,\ D'=0 \Rightarrow N \le N',\ D=0$. □

These identities yield the implication

$$A' \leq A,\ B' \leq B \ \Rightarrow\ \alpha' \geq \alpha, \beta' \leq \beta.$$

But this is obvious from geometrical considerations.

## 4. The theorem.

We are now in a position to prove the theorem described in Section 1:

**Theorem**

(i) $\alpha' \leq \alpha,\ \beta' \leq \beta \ \Rightarrow\ P(N \leq N') = Q(N \leq N') = 1,\ P(N' = N < \infty, D' \neq D)$

$$= Q(N' = N < \infty,\ D' \neq D) = 0.$$

(ii) $\alpha' \leq \alpha,\ \beta' \leq \beta$ *with at least one strict* $\Rightarrow P(N<N') > 0,\ Q(N<N') > 0.$

Proof. (i) and (ii) are immediate from the following implications:

(7) $A \leq A',\ B' \leq B,\ \alpha' \leq \alpha,\ \beta' \leq \beta \ \Rightarrow\ \alpha' = \alpha,\ \beta' = \beta,\ P(N'=N) = Q(N'=N) = 1.$

(8) $A' \leq A,\ B \leq B' \ \Rightarrow\ P(N \leq N') = Q(N \leq N') = 1.$

(9) $A' \leq A,\ B \leq B',\ \alpha' \leq \alpha,\ \beta' \leq \beta$ with one strict $\Rightarrow\ P(N<N') > 0, Q(N<N') > 0.$

(10) $A' \leq A,\ B' \leq B, \alpha' \leq \alpha \ \Rightarrow\ \alpha' = \alpha,\ P(N \leq N') = Q(N \leq N') = 1,\ P(N'=N<\infty,\ D' \neq D)$

$$= Q(N'=N<\infty,\ D' \neq D) = 0.$$

(11) $A' \leq A,\ B' \leq B,\ \alpha' \leq \alpha,\ \beta' < \beta \Rightarrow P(N' > N) > 0,\ Q(N' > N) > 0.$

(12) $A \leq A',\ B \leq B',\ \beta' \leq \beta \Rightarrow \beta' = \beta,\ P(N \leq N') = Q(N \leq N') = 1,$

$$P(N' = N < \infty, D' \neq D) = Q(N' = N < \infty, D' \neq D) = 0.$$

(13) $A \leq A',\ B \leq B',\ \alpha' < \alpha,\ \beta' \leq \beta \Rightarrow P(N' > N) > 0,\ Q(N' > N) > 0.$

Implication (8) is obvious, implications (7) and (9) follow from Proposition 1, and (10)-(13) follow from Proposition 2. Observe that the right hand sides of (4a), (4b), (6a) and (6b) consist of non-negative terms and that the first integral of each has a strictly positive integrand on its range of integration.

Proof of (7). The "given" of (7), (4a) and (4b) readily imply $\alpha' = \alpha,\ \beta' = \beta,\ P(N' < N, D' = 0) = Q(N' < N, D' = 1) = 0$. In turn, (3a) and (3b) (with $N$ replaced by $N'$) imply $Q(N' < N, D' = 0) = P(N' < N, D' = 1) = 0$. Thus $P(N \leq N') = Q(N \leq N') = 1$ and, hence, $P(N' = N) = Q(N' = N) = 1$. □

Proof of (9). (9) is equivalent to

(9') $A \leq A', B' \leq B, \alpha \leq \alpha', \beta \leq \beta'$ with one strict $\Rightarrow P(N' < N) > 0,\ Q(N' < N) > 0,$

which obviously follows from Proposition 1. □

Proof of (10) and (11). The "given" of (10) ( and hence of (11)) and (6b) imply $\alpha' = \alpha,\ Q(N' < N) = P(N' = N < \infty,\ D' \neq D) = P(N < N' < \infty, D' = 1) = 0.$ The remaining conclusions in (10) follow from (3a) and (3b).

Furthermore, (6a) simplifies to

$$\beta-\beta' = \int_{[N<N']} (\lambda_N - A')\, dP + \int_{[N<N'<\infty,\ D'=0]} (A' - \lambda_{N'})\, dP + A'\, P(N<N' = \infty),$$

from which (11) easily follows. □

Finally, (12) and (13) are proven in the same way we prove (10) and (11). This completes the proof of the theorem. □

<u>Remark</u> Notice that the value of $(\alpha,\beta)$ determines the distribution of $N$ under both hypotheses. For if $\alpha'=\alpha$ and $\beta'=\beta$, then $P(N'=N) = Q(N'=N) = 1$. This is not to say that every value $(\alpha,\beta)$ is possible. Neither does $\alpha'=\alpha$ and $\beta'=\beta$ imply $A'=A, B'=B$ in general.

## REFERENCES


[1] Ghosh, B.K. (1970). *Sequential Tests of Statistical Hypotheses*, Addison-Wesley.

[2] Wald, A. and Wolfowitz, J. (1948). "Optimum character of the sequential probability ratio test," *Ann. Math. Statist.* 19, 326-39.