NONPARAMETRIC TESTS FOR INTERCHANGEABILITY UNDER COMPETING RISKS

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Abstract

Some nonparametric tests for the hypothesis of interchangeability of the elements of a (stochastic) 2-vector under competing risks model are proposed and studied here. Both fixed sample and sequential procedures are studied. The case of progressively censored nonparametric procedures is also presented. Along with some martingale theorems on allied rank statistics, their weak convergence results are considered and incorporated in the study of the asymptotic properties of the tests. The choice of locally optimal score function is also considered.

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1. Introduction.

For a two-component system, let $F(x,y)$ be the joint distribution function (df) of the survival times X and Y of the two components. We desire to test the null hypothesis that X and Y are interchangeable, i.e.,

$$F(x,y) = F(y,x) \text{ for all } (x,y)\varepsilon E^2, \tag{1.1}$$

where E^k, $k\geq 1$, stands for the k-dimensional Euclidean space. Nonparametric tests for (1.1) are due to Sen (1967), Bell and Smith (1969), and others. In competing risks problems, instead of (X,Y), the observable random vector is (Z,Q), where

$$Z = \min(X,Y) \text{ and } Q=1, 0 \text{ or } -1 \text{ according as } Z=X,\ Z=X=Y \text{ and } Z=Y. \tag{1.2}$$

For an exposition of joint survival functions under competing risks, we may refer to Thompson et al. (1972, 1973) where other references are cited. Thus, based on a set of observable random vectors (Z_i,Q_i), $1\leq i\leq n$, our problem is to test for (1.1) against suitable alternatives. Nonparametric tests for this problem are proposed and studied here.

Three different types of tests are considered: (i) the conventional fixed sample size procedure based on all the n observations through a single statistic, (ii) the first sequential procedure based on the observations when the Z_i are observable sequentially, and (iii) the second sequential procedure suitable under progressive censoring. The first sequential procedure is suitable when the observations are not available at the same time, so that if the null hypothesis (1.1) may be rejected based on fewer than n observations, there is a reduction of the total time to perform the test. In the context of life-testing problems, when n independent systems are subject simultaneously to a continuous time-observation process and the (Z_i,Q_i) are observable only at the expiry of the lives of these systems, one may naturally be interested in monitoring the experiment with the objective of rejecting the null hypothesis with the minimum sacrifice of the lives of the units, that is, stopping the experiment at a time point where, for the first time, the accumulated evidence leads to the rejection of H_o. Unlike the other case, here the ordered random variables corresponding to $Z_1,\ldots,Z_n$ are observed sequentially, and the scheme is known as a progressively censored scheme. Our second sequential procedure applies to this situation. Thus, for both the sequential procedures, the stopping times are random variables, and the procedures may lead to reduction of time and cost of experimentation. We shall discuss these in greater detail in section 2.

The test procedures along with the preliminary notions are introduced in section 2. Some martingale theorems, invariance principles and certain basic invariance structures for the allied rank statistics are studied in section 3. Section 4 is devoted to the study of the properties of the fixed-sample and first sequential tests based on appropriate rank statistics. Their asymptotic relative efficiency (ARE) results are also considered. Parallel results for the second sequential procedure are presented in section 5. The last section is concerned with the choice of optimal scores.

2. Preliminary notions and the proposed tests. Let $\{(Z_i,Q_i),\ i\geq 1\}$ be a sequence of independent and identically distributed random vectors (iidrv), where the (Z_i,Q_i) correspond to (X_i,Y_i) as in (1.2). We assume that $F(x,y)$ possesses a density function $f(x,y)$, $\forall\,(x,y)\varepsilon E^2$, so that (i) the density function (say, $g(z)$) of Z_i exists, and (ii) $P\{X_i=Y_i\} = P\{Q_i=0\} = 0$, $\forall\, i\geq 1$. Let $G(z)$ be the df of Z_i, so that $G(z)$ is absolutely continuous in $Z\varepsilon E$. Hence, ties among $Z_1,\ldots,Z_n$ can be neglected with probablity 1.

Let $c(u)=1$ or 0 according as u is $\geq$ or <0, and let $R_{ni} = \sum_{j=1}^{n} c(Z_i-Z_j)$ be the rank of Z_i among $Z_1,\ldots,Z_n$, for $1\leq i\leq n$. Thus, $\underset{\sim}{R}_n=(R_{n1},\ldots,R_{nn})$ is some permutation of $(1,\ldots,n)$. For every $n(\geq 1)$, consider a set of real-valued rank-scores $a_n(1),\ldots,a_n(n)$, defined by

(2.1) $$a_n(i) = E\phi(U_{n_i}) \text{ or } \phi(i/(n+1)), \quad 1\leq i\leq n,$$

where $U_{n1} < \ldots < U_{nn}$ are the ordered random variables of a sample of size n from the rectangular (0,1) df [so that $EU_{ni}=i/(n+1)$, $1\leq i\leq n$], and the score-function $\phi(u)$, $0<u<1$, is assumed to be square-integrable and non-degenerate, so that

(2.2) $$0 < A^2 = \int_0^1 \phi^2(u)du < \infty.$$

Consider first the fixed-sample size test. Define the rank statistics

(2.3) $$T_n = \sum_{i=1}^n Q_i a_n(R_{ni}), \quad n\geq 1.$$

As we shall see in section 3 [cf. Lemma 3.4] that under H_o in (1.1), $\underset{\sim}{Q}_n=(Q_1,\ldots,Q_n)$ and $\underset{\sim}{R}_n$ are stochastically independent and $\underset{\sim}{Q}_n$ assumes the all possible 2^n realizations, each with the equal probability 2^{-n}. Thus, under H_o, $E(T_n)=0$ and $V(T_n) = \sum_{i=1}^n a_n^2(i) = nA_n^2$, say. On the other hand, when H_o does not hold, $n^{-1}T_n$ estimates a quantity which may be positive or negative depending on the df F(x,y). Thus, for a one-sided test, we may consider the critical region specified by

(2.4) $$n^{-\frac{1}{2}}T_n/A_n \geq C_{n,\alpha}^{(1)}, \text{ where } P\{n^{-\frac{1}{2}}T_n/A_n \geq C_{n,\alpha}^{(1)}\} = \alpha,$$

while for the two-sided test, our critical region is given by

(2.5) $$n^{-\frac{1}{2}}|T_n|/A \geq C_{n,\alpha}^{(2)}, \text{ where } P\{n^{-\frac{1}{2}}|T_n|/A_n \geq C_{n,\alpha}^{(2)}\} = \alpha.$$

In section 4, we shall see that the tests sketched above are genuinely distribution-free, so that $C_{n,\alpha}^{(i)}$, i=1,2, depend only on the level of significance α (not on F), and there exist suitable $C_\alpha^{(i)}$, i=1,2, such that

(2.6) $$\lim_{n\to\infty} C_{n,\alpha}^{(i)} = C_\alpha^{(i)}, \ i=1,2, \text{ for every } 0<\alpha<1.$$

Let us next consider the first sequential test. Here the i.i.d.r.v. (Z_i, Q_i), $i \geq 1$, are observable sequentially, so that it may be advisable to stop at an intermediate stage i.e., when (Z_i, Q_i), $1 \leq i \leq k$, are observed for some $k \leq n$, provided the statistical evidence up to that stage provocates the rejection of H_o. For every k: $1 \leq k \leq n$, define T_k by (2.3), and conventionally let $T_o = 0$. Let then

$$M_n^+ = \{\max_{0 \leq k \leq n} T_k\}/(n^{\frac{1}{2}} A_n), \quad M_n = \{\max_{0 \leq k \leq n} |T_k|\}/(n^{\frac{1}{2}} A_n), \tag{2.7}$$

so that corresponding to (2.4) and (2.5), we consider the critical regions:

$$M_n^+ \geq M_{n,\alpha}^+, \text{ where } P\{M_n^+ \geq M_{n,\alpha}^+ | H_o\} = \alpha, \tag{2.8}$$

$$M_n \geq M_{n,\alpha}, \text{ where } P\{M_n \geq M_{n,\alpha} | H_o\} = \alpha. \tag{2.9}$$

Operationally, the test procedure consists in observing sequentially the T_k, $k \geq 1$, until for the first time for some $k = N(\leq n)$, $n^{-\frac{1}{2}} A_n^{-1} T_N$ (or $n^{-\frac{1}{2}} A_n^{-1} |T_N|$) exceeds $M_{n,\alpha}^+$ (or $M_{n,\alpha}$), and rejecting H_o at that stage with the termination of the experiment. If no such $N(\leq n)$ exists, then H_o is accepted when $(Z_1, Q_1), \ldots, (Z_n, Q_n)$ are observed. We shall see in section 4 that the test procedure is distribution-free, so that $M_{n,\alpha}^+$ or $M_{n,\alpha}$ does not depend on the underlying F, and further, there exists suitable constants M_α^+ and M_α, such that

$$\lim_{n\to\infty} M_{n,\alpha}^+ = M_\alpha^+ \text{ and } \lim_{n\to\infty} M_{n,\alpha} = M_\alpha. \tag{2.10}$$

Finally, let us consider the progressively censored rank test. In this case, the experimentation starts with the continuous observation on n units and their values are recorded as they are observed sequentially. Thus, here the order statistics $Z_{n,1} \leq \ldots \leq Z_{n,n}$ (corresponding to $Z_1, \ldots, Z_n$) are observed in a sequence; by virtue of the assumed continuity of G, ties among the $Z_{n,i}$ can be neglected with probability one. We may note that

$$Z_{n,i} = Z_{S_{ni}}, \quad 1 \leq i \leq n, \tag{2.11}$$

where $\underset{\sim}{S}_n=(S_{n1},\ldots,S_{nn})$ is some permutation of $(1,\ldots,n)$. In view of the fact that $Z_i = Z_{n,R_{ni}}$, $1\leq i\leq n$, we term $\underset{\sim}{S}_n$ as the vector of anti-ranks. Also, we denote the Q_j corresponding to $Z_{S_{ni}}$ by $Q_{S_{ni}} = Q(n,S_{ni})$, for $i=1,\ldots,n$. Then, we observe that at the kth stage when $Z_{n,1},\ldots,Z_{n,k}$ have been observed, we are provided with $Q(n,S_{n1}),\ldots,Q(n,S_{nk})$, for $k=1,\ldots,n$. We denote by

(2.12) $$T_{nk} = \sum_{i=1}^{k} Q(n,S_{ni})a_n(i),\ 1\leq k\leq n.$$

Note that, by definition,

(2.13) $$T_{nn} = \sum_{i=1}^{n} R(n,S_{ni})a_n(i) = \sum_{i=1}^{n} Q_i a_n(R_{ni}) = T_n,\ n\geq 1.$$

Conventionally, we let $T_o=0$ and $T_{no}=0$, $\forall\ n\geq 0$, and define

(2.14) $$D_n^+ = \{\max_{0\leq k\leq n} T_{nk}\}/(n^{\frac{1}{2}}A_n) \text{ and } D_n = \{\max_{0\leq k\leq n} |T_{nk}|\}/(n^{\frac{1}{2}}A_n).$$

For an one-sided test, we use D_n^+ and reject H_o when

(2.15) $$D_n^+ \geq D_{n,\alpha}^+ \text{ where } P\{D_n^+ \geq D_{n,\alpha}^+ | H_o\} = \alpha,$$

and for a two-sided test, we use D_n and reject H_o when

(2.16) $$D_n \geq D_{n,\alpha} \text{ where } P\{D_n \geq D_{n,\alpha} | H_o\} = \alpha.$$

Operationally, the test procedure consists in continuing the experiment so long as $n^{-\frac{1}{2}}A_n^{-1}T_{nk}$ (or $n^{-\frac{1}{2}}A_n^{-1}|T_{nk}|$), $1\leq k\leq n$, continue to lie below $D_{n,\alpha}^+$ (or $D_{n,\alpha}$), and if $N(\leq n)$ is the smallest positive integer for which $n^{-\frac{1}{2}}A_n^{-1}T_{nN}$ is $\geq D_{n,\alpha}^+$ (or $n^{-\frac{1}{2}}A_n^{-1}|T_{nN}|$ is $\geq D_{n,\alpha}$), the experimentation is terminated along with the rejection of H_o. If no such $N(\leq n)$ exists, H_o is accepted. In section 5, we shall see that the tests based on D_n^+ and D_n are genuinely distribution-free, and there exist suitable constants D_α^+ and D_α, such that

(2.17) $$\lim_{n\to\infty} D_{n,\alpha}^+ = D_\alpha^+ \text{ and } \lim_{n\to\infty} D_{n,\alpha} = D_\alpha,\ \forall\ 0<\alpha<1.$$

For the study of the various properties of these tests, we require to study first some basic properties of $\{T_n,\ n\geq 1\}$ and $\{T_{nk},\ 1\leq k\leq n\}$. This has been accomplished in section 3.

3. Some basic results on rank statistics. The density function $g(z)$ of Z_i is given by

(3.1) $$g(z) = \int_z^\infty f(z,y)dy + \int_z^\infty f(x,z)dx, \quad -\infty<z<\infty.$$

Let $\pi(z) = P\{Z=x|Z=z\} = P\{Q=1|Z=z\}$, $-\infty<z<\infty$, so that

(3.2) $$\pi(z) = [\int_z^\infty f(z,y)dy]/g(z), \quad 0\le\pi(z)\le 1, \quad -\infty<z<\infty.$$

For every $1\le i\le n$, let

(3.3) $$\pi(i,n) = n\binom{n-1}{i-1}\int_{-\infty}^\infty \pi(z)[Gz)]^{i-1}[1-G(z)]^{n-i}dG(z), \quad \pi^*(i,n) = 2\pi(i,n)-1;$$

(3.4) $$\mu_n = n^{-1}\sum_{i=1}^n a_n(i)\pi^*(i,n) \text{ and } \mu_n^* = \sum_{k=1}^n \mu_k.$$

Note that under H_o in (1.1), $\pi(z)=\frac{1}{2}$ for all $-\infty<z<\infty$, so that

(3.5) $$\pi(i,n) = \tfrac{1}{2}, \quad \pi^*(i,n) = 0, \quad 1\le i\le n \text{ and } \mu_n=\mu_n^*=0, \ \forall\ n\ge 1.$$

For every $n\ge 1$, let $\mathcal{B}_n$ be the σ-field generated by $(\underset{\sim}{Q}_n, \underset{\sim}{R}_n)$ where $\underset{\sim}{Q}_n$ and $\underset{\sim}{R}_n$ are defined in section 2. Note that $\mathcal{B}_n$ is $\uparrow$ in $n(\ge 1)$. Then, we have the following:

Theorem 3.1. For $a_n(i)=E\phi(U_{ni})$, $1\le i\le n$ and ϕ integrable inside $[0,1]$, $\{T_n-\mu_n^*, \mathcal{B}_n;\ n\ge 1\}$ is a martingale.

Proof. By (2.3), (3.3) and (3.4), $E(T_1-\mu_1)=0$, while for $n\ge 2$,

(3.6) $$E(T_n-\mu_n^*|\mathcal{B}_{n-1}) = [\sum_{i=1}^{n-1} E\{Q_i a_n(R_{ni})|\mathcal{B}_{n-1}\}-\mu_{n-1}^*] + [E\{Q_n a_n(R_{nn})|\mathcal{B}_{n-1}\}-\mu_n].$$

Now for $1\le i\le n-1$, given $\mathcal{B}_{n-1}$, R_{ni} can be either R_{n-1i} or $(R_{n-1i}+1)$ with respective conditional probabilities $1-n^{-1}R_{n-1i}$ and $n^{-1}R_{n-1i}$, and Q_i is fixed, so that

(3.7) $$E\{Q_i a_n(R_{ni})|\mathcal{B}_{n-1}\} = Q_i\{(1-n^{-1}R_{n-1i})a_n(R_{n-1i})+n^{-1}R_{n-1i}a_n(R_{n-1i}+1)\} = Q_i a_{n-1}(R_{n-1i}), \quad 1\le i\le n-1,$$

where the last step follows from the well-known and easily verifiable identity:

$$n^{-1}[(n-i)E\phi(U_{n_i}) + iE\phi(U_{n_i+1})] = E\phi(U_{n-1i}),\ 1\le i\le n-1. \tag{3.8}$$

Thus, from (3.6) and (3.7), we have

$$E\{T_n-\mu_n^*|\mathcal{B}_{n-1}\} = T_{n-1}-\mu_{n-1}^* + E\{Q_n a_n(R_{nn})|\mathcal{B}_{n-1}\}-\mu_n. \tag{3.9}$$

Now, given $\mathcal{B}_{n-1}$, the possible values of $Q_n a_n(R_{nn})$ are $\pm a_n(j)$, $j=1,\ldots,n$, and $Q_n a_n(R_{nn}) = a_n(j)$ with probability $\frac{1}{n}\pi(j,n)$, and $-a_n(j)$ with probability $\frac{1}{n}(1-\pi(j,n))$, for $j=1,\ldots,n$, so that

$$E\{Q_n a_n(R_{nn})|\mathcal{B}_{n-1}\} = \sum_{j=1}^n a_n(j)\frac{1}{n}\{2\pi(j,n)-1\} = \mu_n, \text{ by (3.4)}. \tag{3.10}$$

Hence, the theorem follows from (3.9) and (3.10). Q.E.D.

Since $Q_i^2=1$ with probability 1, for every $i\ge 1$, by (2.3) we have

$$\begin{aligned} E(T_n^2) &= E[\textstyle\sum_{i=1}^n Q_i a_n(R_{ni})]^2 \\ &= \textstyle\sum_{i=1}^n a_n^2(i) + \sum_{1\le i\ne j\le n} E\{Q_i Q_j a_n(R_{ni}) a_n(R_{nj})\}, \end{aligned} \tag{3.11}$$

where by (2.1) and (2.2), as $n\to\infty$,

$$\frac{1}{n}\sum_{i=1}^n a_n^2(i) = A_n^2 \to A^2 = \int_0^1 \phi^2(u)du. \tag{3.12}$$

We let $\pi^*(z) = \{2\pi(z)-1\}$, $-\infty<z<\infty$, and for $1\le k<q\le n(\ge 1)$, define

$$\pi^*(k,q;n) = \frac{n!}{(k-1)!(q-k-1)!(n-q)!} \iint_{-\infty<u<v<\infty} \pi^*(u)\pi^*(v)[G(u)]^{k-1}[G(v)-G(u)]^{q-k-1} [1-G(v)]^{n-q}dG(u)dG(v). \tag{3.13}$$

Note that under H_o, $\pi^*(k,q;n) = 0$, $\forall\ 1\le k<q\le n$. Since the $Z_1,\ldots,Z_n$ are iidrv, by some standard arguments, it follows that for $1\le i\ne j\le n$,

$$(3.14)\quad E\{Q_iQ_j a_n(R_{ni})a_n(R_{nj})\} = \sum_{s\neq t=1}^{n} a_n(s)a_n(t)\left\{\sum_{u=0}^{1}\sum_{v=0}^{1}(-1)^{u+v} P[R_{ni}=s,\ R_{nj}=t,\ Q_i=(-1)^u,\ Q_j=(-1)^v]\right\}$$

$$= \binom{n}{2}^{-1}\sum_{1\le k<q\le n} a_n(k)a_n(q)\pi^*(k,q;n).$$

Thus, from (3.11) and (3.14) and by Theorem 3.1 ($\Longrightarrow ET_n=\mu_n^*$), we have

$$(3.15)\qquad V(T_n) = nA_n^2 + 2\sum_{1\le k<q\le n} a_n(k)a_n(q)\pi^*(k,q;n)-(\mu_n^*)^2.$$

It readily follows from (3.5) and the fact that under H_o, $\pi^*(k,q;n)=0$, that

$$(3.16)\qquad V(T_n|H_o) = nA_n^2 \text{ and } n^{-1}V(T_n|H_o) \to A^2 \text{ as } n\to\infty.$$

To simplify (3.15) for large n when H_o is not necessarily true, we assume that the following conditions are satisfied:

(I) $\phi(u) = \phi_1(u)-\phi_2(u)$ where $\phi_j(u)$ is non-decreasing and absolutely continuous inside [0,1], and

$$(3.17)\qquad \int_0^1 |\phi_j(u)|\{u(1-u)\}^{-\frac{1}{2}}du < \infty \text{ for } j=1,2,$$

and (II) $\pi(z)$ is absolutely continuous in z for all $0<G(z)<1$.

Let us then define

$$(3.18)\qquad \sigma^2 = \sigma^2(F) = \int_0^1 \phi^2(u)du - \left(\int_{-\infty}^{\infty}\pi^*(z)\phi(G(z))dG(z)\right)^2 +$$

$$2\Big[\iint_{-\infty<u<v<\infty} \pi^*(u)\pi^*(v)[G(u)\{1-G(v)\}\phi'(G(u))\phi'(G(v))+\phi(G(u))\{1-G(v)\}\phi'(G(v))$$

$$- G(u)\phi'(G(u))\phi(G(v))]dG(u)dG(v)\Big].$$

Note that $|\pi^*(z)| \le 1$, $\forall\ -\infty<z<\infty$, so that some standard computations yield that $\sigma^2(F)<\infty$ for every F. Then, we have the following.

<u>Theorem 3.2</u>. <u>Under</u> (2.1), (3.17) <u>and conditions</u> I <u>and</u> II,

(3.19) $$n^{-1}V(T_n) \to \sigma^2 \text{ as } n\to\infty.$$

Proof. By virtue of (3.17), we obtain, on proceeding as in Hoeffding (1973), that

(3.20) $$|\sum_{i=1}^{n} Q_i E\phi(U_{nR_{ni}}) - \sum_{i=1}^{n} Q_i \phi((n+1)^{-1}R_{ni})|$$
$$\leq \sum_{i=1}^{n} |E\phi(U_{ni}) - \phi(i/(n+1))| = o(n^{\frac{1}{2}}), \text{ as } n\to\infty.$$

Consequently, if we prove the theorem for $a_n(i) = E\phi(U_{ni})$, $1\leq i\leq n$, the result applies as well to the other case of $a_n(i) = \phi(i/(n+1))$, $1\leq i\leq n$. We let $T_o=0$, and for $n\geq 1$,

(3.21) $$L_n = T_n - T_{n-1} - \mu_n,\ q_n^2 = E(L_n^2|\mathcal{B}_{n-1}),\ n\geq 1.$$

Then, by Theorem 3.1, we have $n^{-1}V(T_n) = n^{-1}\sum_{i=1}^{n} E[q_n^2]$, so to prove (3.19), it suffices to show that as $n\to\infty$,

(3.22) $$E(q_n^2) \to \sigma^2.$$

By (2.3), (3.21) and a few steps we obtain that

$$q_n^2 = E[a_n^2(R_{nn})|\mathcal{B}_{n-1}] - \mu_n^2 + \sum_{i=1}^{n-1} E\{[a_n(R_{ni}) - a_{n-1}(R_{n-1i})]^2|\mathcal{B}_{n-1}\}$$

(3.23) $$+ \sum_{1\leq i\neq j\leq n-1} Q_i Q_j E\{[a_n(R_{ni}) - a_{n-1}(R_{n-1i})][a_n(R_{nj}) - a_{n-1}(R_{n-1j})]|\mathcal{B}_{n-1}\} +$$
$$2\sum_{i=1}^{n-1} Q_i E\{Q_n a_n(R_{nn})[a_n(R_{ni}) - a_{n-1}(R_{n-1i})]|\mathcal{B}_{n-1}\}.$$

Now, as in the proof of Theorem 3.1, we have

(3.24) $$E[a_n^2(R_{nn})|\mathcal{B}_{n-1}] = A_n^2 \to A^2 = \int_0^1 \phi^2(u)du \text{ as } n\to\infty,$$

(3.25) $$E\{[a_n(R_{ni}) - a_{n-1}(R_{n-1i})]^2|\mathcal{B}_{n-1}\} =$$
$$[n^{-2}R_{n-1i}(n - R_{n-1i})][a_n(R_{n-1i}+1) - a_n(R_{n-1i})]^2, \quad 1\leq i\leq n-1,$$

(3.26) $$E\{[a_n(R_{ni}) - a_{n-1}(R_{n-1i})][a_n(R_{nj}) - a_{n-1}(R_{n-1j})]|\mathcal{B}_{n-1}, i\neq j\}$$
$$= n^{-2}\omega_1(n-\omega_2)[a_n(\omega_1+1) - a_n(\omega_1)][a_n(\omega_2+1) - a_n(\omega_2)],$$

where $\omega_1 = \min(R_{n-1i}, R_{n-1j})$ and $\omega_2 = \max(R_{n-1i}, R_{n-1j})$. Finally, for $i \le n-1$,

$$(3.27) \quad E\{Q_n a_n(R_{nn})[a_n(R_{ni}) - a_{n-1}(R_{n-1i})] \mid \mathcal{B}_n\}$$
$$= \sum_{j=1}^{n} a_n(j) \tfrac{1}{n} \pi^*(j,n)\{[a_n(R_{n-1i} + c(R_{n-1i} - j)) - a_{n-1}(R_{n-1i})]\}$$
$$= [a_n(R_{n-1i}+1) - a_n(R_{n-1i})]\{n^{-2}(n - R_{n-1i}) \sum_{j=1}^{R_{n-1i}} \pi^*(j,n) a_n(j)$$
$$- n^{-2} R_{n-1i} \sum_{j=R_{n-1i}+1}^{n} \pi^*(j,n) a_n(j)\}.$$

Also, note that for $1 \le i \ne j \le n-1$,

$$(3.28) \quad E(Q_i \mid \underset{\sim}{R}_{n-1}) = 2\pi(R_{n-1i}, n-1) - 1 = \pi^*(R_{n-1i}, n-1),$$

$$(3.29) \quad E(Q_i Q_j \mid \underset{\sim}{R}_{n-1}) = \pi^*(R_{n-1i}, R_{n-1j}; n-1), \text{ if } R_{n-1i} < R_{n-1j}$$
$$= \pi^*(R_{n-1j}, R_{n-1i}; n-1), \text{ if } R_{n-1i} > R_{n-1j}.$$

Thus, writing $E(q_n^2) = E\{E(q_n^2 \mid \underset{\sim}{R}_{n-1})\}$, and using (3.23) through (3.29) that

$$(3.30) \quad E(q_n^2) = A_n^2 - \mu_n^2 + \sum_{i=1}^{n-1} n^{-2} i(n-i)[a_n(i+1) - a_n(i)]^2 +$$
$$2 \sum_{1 \le i < j \le n-1} \pi^*(i,j,n-1) n^{-2} i(n-j)[a_n(i+1) - a_n(i)][a_n(j+1) - a_n(j)] +$$
$$2 \sum_{i=1}^{n-1} \pi^*(i,n-1)[a_n(i+1) - a_n(i)]\{n^{-2}(n-i) \sum_{j=1}^{i} \pi^*(j,n) a_n(j) -$$
$$n^{-2} i \sum_{j=i+1}^{n} \pi^*(j,n) a_n(j)\}.$$

Note that $\pi^*(z) = 2\pi(z) - 1$ is a bounded and absolutely continuous function of z, so that by the well-known bounds for expected order statistics, we have

$$(3.31) \quad \pi^*(i,n) = \pi^*(G^{-1}(i/(n+1))) + o(n^{-\frac{1}{2}}), \quad 1 \le i \le n,$$

$$(3.32) \quad \pi^*(i,j;n) = \pi^*(G^{-1}(i/(n+1)))\pi^*(G^{-1}(j/(n+1))) + o(n^{-\frac{1}{2}}), \quad 1 \le i < j \le n.$$

Thus, by (3.4), (3.31) and Hoeffding (1953, 1973), we obtain that

$$(3.33) \quad \mu_n \to \mu(F) = \int_{-\infty}^{\infty} \{2\pi(z) - 1\} \phi(G(z))\, dG(z) \text{ as } n \to \infty.$$

Also, by the recent results of Hoeffding (1973), the third term on the rhs (right hand side) of (3.30) converges to 0 as $n\to\infty$. By (3.32) and some standard steps, the fourth term on the rhs of (3.30) converges to (as $n\to\infty$)

$$(3.34)\qquad 2\iint_{-\infty<x<y<\infty} \pi^*(x)\pi^*(y)G(x)[1-G(y)]\phi'(G(x))\phi'(G(y))dG(x)dG(y),$$

and similarly, the last term converges (as $n\to\infty$) to

$$(3.35)\qquad 2\iint_{-\infty<x<y<\infty} \pi^*(x)\pi^*(y)\{\phi(G(x))[1-G(y)]\phi'(G(y))-G(x)\phi'(G(x))\phi(G(y))\}dG(x)dG(y).$$

The proof of (3.22) follows from (3.18), (3.12), (3.28), (3.33), (3.34) and (3.35). Q.E.D.

Now, by virtue of Theorems 3.1 and 3.2, for every $0\le s\le t\le 1$,

$$(3.36)\qquad n^{-1}V(T_{[nt]}-T_{[ns]}) = n^{-1}V(T_{[nt]})-n^{-1}V(T_{[ns]})\to\sigma^2(t-s),\text{ as } n\to\infty.$$

In the sequel, it will be assumed that σ^2 is strictly positive, so that

$$(3.37)\qquad 0<\sigma^2<\infty.$$

Let $I=[0,1]$, $W_n(0)=0$, $n\ge 1$, and define

$$(3.38)\qquad W_n(k/n) = n^{-\frac{1}{2}}[T_k-\mu_k^*]/\sigma,\ k=1,\dots,n.$$

Consider then a stochastic process $W_n=\{W_n(t),t\in I\}$, where for $\frac{k}{n}\le t\le\frac{k+1}{n}$,

$$(3.39)\qquad W_n(t) = W_n(\tfrac{k}{n})+(nt-k)[W_n(\tfrac{k+1}{n})-W_n(\tfrac{k}{n})],\ k=0,\dots,n-1.$$

Thus, for every $n(\ge 1)$, W_n belongs to the space $C[0,1]$ with which we associate the uniform topology specified by the metric

$$(3.40)\qquad \rho(x,y) = \sup_{t\in I}|x(t)-y(t)|,\ x,y\in C[0,1].$$

Finally, let $W=\{W(t),t\in I\}$ be a standard Brownian motion on I, so that $EW(t)=0$ and $E[W(s)W(t)]=\min(s,t)$ for every $s,t\in I$.

Theorem 3.3. Under (3.37) and the conditions of Theorem 3.2, (3.41) $$W_n \overset{\mathcal{D}}{\to} W, \text{ \underline{in the uniform topology on} } C[0,1].$$

Proof. As in Hájek (1968) and Hoeffding (1973), for every $\eta>0$, there exists [under (3.17)] a decomposition

(3.42) $$\phi(u) = \phi_{(1)}(u) + \phi_{(2)}(u) - \phi_{(3)}(u), \; 0<u<1,$$

where $\phi_{(1)}$ is a polynomial, $\phi_{(2)}$ and $\phi_{(3)}$ are non-decreasing, and

(3.43) $$\left\{ \sum_{j=2}^{3} \int_0^1 |\phi_{(j)}(u)|\{u(1-u)\}^{-\frac{1}{2}}du \right\} < \eta[\int_0^1 |\phi(u)|\{u(1-u)\}^{-\frac{1}{2}}du.$$

Now, in (3.18), on replacing $\phi(u)$ by $\phi_{(j)}(u)$ everywhere and denoting the corresponding quantity by σ_j^2, $j=1,2,3$, it can be shown that (3.43) implies that

(3.44) $$\sigma_2^2/\sigma^2 < \tfrac{1}{2}\eta', \; \sigma_3^2/\sigma^2 < \tfrac{1}{2}\eta' \text{ and } |\sigma_1^2/\sigma^2-1| < \eta',$$

where $\eta'(>0)$ depends on η, and $\eta'\to 0$ as $\eta\to 0$. Also, by virtue of (3.20), $\max_{1\le k\le n} |n^{-\frac{1}{2}}\{\sum_{i=1}^{k} Q_i a_k(R_{ki}) - \sum_{i=1}^{k} Q_i \phi((k+1)^{-1}R_{ki})\}| = o(1)$, for $a_k(i)$ defined by (2.1). Hence, here also, it suffices to work with $a_n(i)=E\phi(U_{ni})$, $1\le i\le n(\ge 1)$. Suppose now in (2.3) and (3.4), we replace the score function $a_n(i)$ by $a_{n,j}(i) = E\phi_{(j)}(U_{ni})$, $1\le i\le n$, $j=1,2,3$, and denote the corresponding quantities by $T_{n,j}$, $\mu_{n,j}$ and $\mu^*_{n,j}$, respectively, for $j=1,2,3$. Similarly, in (3.38)-(3.39), we replace T_k, μ^*_k and σ by $T_{k,j}, \mu^*_{k,j}$ and σ_j, respectively, and define the resulting process by $W_{nj} = \{W_{nj}(t), t\in I\}$, for $j=1,2,3$. Then, by (3.42), we have

(3.45) $$W_n = (\sigma_1/\sigma)W_{n1} + (\sigma_2/\sigma)W_{n2} - (\sigma_3/\sigma)W_{n3}.$$

Note that Theorem 3.1 applies to each of $\{T_{n,j}-\mu^*_{n,j}, \mathcal{B}_n; n\ge 1\}$, $j=1,2,3$, and by definition, $\sup_{t\in I}|W_{nj}(t)| = \max_{0\le k\le n} n^{-\frac{1}{2}}|T_{k,j}-\mu^*_{k,j}|/\sigma_j$, so that by the Kolmogorov-inequality for martingales, we have

(3.46) $$P\{\sup_{t\in I}|W_{nj}(t)| \ge K\} = P\{\max_{0\le k\le n}|T_{k,j}-\mu^*_{k,j}| \ge K\sqrt{n}\sigma_j\}$$
$$\le (nK^2)^{-1}E[T_{n,j}-\mu_{n,j}]^2/\sigma_j^2 \to K^{-2}, \text{ as } n\to\infty, \; j=1,2,3.$$

by Theorem 3.2. By virtue of (3.44), (3.45) and (3.46), for every $\varepsilon>0$ and $\varepsilon'>0$, there exists an $\eta>0$, such that under (3.17) and (3.42),

$$(3.47) \qquad P\{\sup_{t\in I}|W_n(t)-(\sigma_2/\sigma)W_{n2}(t)+(\sigma_3/\sigma)W_{n3}(t)|>\varepsilon'\}<\varepsilon.$$

Consequently, by (3.44) and (3.47), it suffices to prove that as $n\to\infty$,

$$(3.48) \qquad W_{n1} \overset{\mathcal{D}}{\to} W, \text{ in the uniform topology on } C[0,1],$$

and for this purpose, we use a functional central limit theorem for martingales [cf. Theorem 3 of Brown (1971)] according to which it suffices to show that as $n\to\infty$, for every $\varepsilon>0$,

$$(3.49) \qquad n^{-1}\sum_{i=1}^{n} E\{L_{i,1}^2 I(|L_{i,1}| \geq \varepsilon\sigma_1\sqrt{n})\} \to 0,$$

$$(3.50) \qquad (\sum_{i=1}^{n} q_{i,1}^2)/(n\sigma_1^2) \overset{P}{\to} 1,$$

where $L_{n,1}$ and $q_{n,1}$ are defined by (3.21) for $\phi=\phi_{(1)}$ and $I(A)$ stands for the indicator function of a set A. Let $\phi_{(1)}^{(r)}(u) = (d^r/du^r)\phi_{(1)}(u)$, $r=0,1,2$. Since $\phi_{(1)}$ is a polynomial and is absolutely continuous, we have

$$(3.51) \qquad \sup_{0<t<1}|\phi_{(1)}^{(r)}(t)| = K_r(<\infty), \text{ for } r=0,1,2.$$

Then, $|L_{1,1}| \leq |T_{1,1}| + |a_{1,1}(1)| = 2|\int_0^1 \phi_{(1)}(u)du| < \infty$, and for $n\geq 2$,

$$(3.52) \qquad |L_{n,1}| \leq \sum_{i=1}^{n-1}|a_{n,1}(R_{ni}) - a_{n-1,1}(R_{n-1i})| + |a_{n,1}(R_{nn})| + |\mu_{n,1}|.$$

Note that R_{ni} is either R_{n-1i} or $R_{n-1i}+1$, so that on using (3.8) and (3.51),

$$(3.53) \qquad |a_{n,1}(R_{ni})-a_{n-1,1}(R_{n-1i})| \leq |a_{n,1}(R_{n-1i}+1)-a_{n,1}(R_{n-1i})|$$
$$\leq \max_{1\leq k\leq n-1}|a_{n,1}(k+1)-a_{n,1}(k)| = O(n^{-1}),$$

as under (3.51), $n[a_{n,1}(i+1)-a_{n,1}(i)] = \tilde{a}_{n,1}(i)$ is bounded, and

$$(3.54) \qquad |\tilde{a}_{n,1}(i)-\phi_{(1)}^{(1)}(i/(n+1)| \to 0, \text{ as } n\to\infty, \forall\ 1\le i\le n.$$

Similarly, $|\tilde{a}_{n,1}(R_{nn})| \le \max_{1\le k\le n} |a_{n,1}(k)| = O(1)$ and $|\mu_{n,1}| = O(1)$. Consequently, by (3.52), (3.53) and the above, we have that for every $\varepsilon>0$, there exist an integer n_ε, such that

$$(3.55) \qquad |L_{n,1}| < \varepsilon\sigma_1\sqrt{n} \text{ for every } n \ge n_\varepsilon.$$

On the other hand,

$$(3.56) \qquad n^{-1}\sum_{i=1}^{n_\varepsilon} E\{L_{i,1}^2 I(|L_{i,1}| > \varepsilon\sigma_1\sqrt{n}\} \le n^{-1}\sum_{i=1}^{n_\varepsilon} E(L_{i,1}^2)$$

$$= n^{-1}V[T_{n_\varepsilon,1}] \sim \sigma_1^2(n_\varepsilon/n) \to 0 \text{ as } n\to\infty.$$

Hence, (3.49) follows from (3.55) and (3.56).

To prove (3.50), we use (3.23) through (3.27) for $\phi=\phi_{(1)}$ i.e., $a_{n,1}(i)$, $1\le i\le n$. Writing then $\omega_{ij} = \min(R_{n-1i},R_{n-1j})$ and $\omega_{ij}^* = \max(R_{n-1i},R_{n-1j})$, we have

$$(3.57) \qquad q_{n,1}^2 = n^{-1}\sum_{i=1}^{n} a_{n,1}^2(i)-\mu_{n,1}^2+\sum_{i=1}^{n-1} n^{-2}i(n-i)[a_{n,1}(i+1)-a_{n,1}(i)]^2 +$$

$$2\sum_{1\le i<j\le n-1} Q_iQ_j n^{-2}\omega_{ij}(n-\omega_{ij}^*)[a_{n,1}(R_{n-1i}+1)-a_{n,1}(R_{n-1i})][a_{n,1}(R_{n-1j}+1)-a_{n,1}(R_{n-1j})]$$

$$+ 2\sum_{i=1}^{n-1}Q_i[a_{n,1}(R_{n-1i}+1)-a_{n,1}(R_{n-1i})]\{n^{-2}(n-R_{n-1i})\sum_{j=1}^{R_{n-1i}} \pi^*(j,n)a_{n,1}(j)$$

$$-n^{-2}R_{n-1i}\sum_{j=R_{n-1i}+1}^{n} \pi^*(j,n)a_{n,1}(j)\}.$$

The first term on the rhs of (3.57) converges to $\int_0^1 \phi_{(1)}^2(u)du$ as $n\to\infty$, and the second term to $\mu_1^2(F) = (\int_{-\infty}^{\infty} \pi^*(z)\phi_{(1)}(G(z))dG(z))^2$. By (3.53), the third term goes to 0 as $n\to\infty$, while the fourth term can be written as

$$(3.58) \qquad 2n^{-2}\sum_{1\le i<j\le n-1} Q(n-1,S_{n-1i})Q(n-1,S_{n-1j})n^{-2}i(n-j)\tilde{a}_{n,1}(i)\tilde{a}_{n,1}(j),$$

where the S_{ni} and $Q(n,S_{ni})$ are defined by (2.11) and shortly after that. Note that as in (3.31),

$$(3.59) \qquad E[Q(n-1,S_{n-1i})Q(n-1,S_{n-1j})] = \pi^*(i,j;n-1), \text{ for } 1\le i<j\le n-1,$$

so that by (3.54), (3.32) and (3.59), the expected value of (3.58) converges (as $n\to\infty$) to

$$(3.60)\qquad 2\iint\limits_{-\infty<x<y<\infty} \pi^*(x)\pi^*(y)G(x)[1-G(y)]d\phi_{(1)}(G(x))d\phi_{(1)}(G(y)).$$

On the other hand, $Q(n-1,S_{n-1i})$, $1\le i\le n-1$, are interchangeable and bounded (by 1) random variables, so that on evaluating the 4th moment of (3.58), using the Markov-inequality and the Borel-Cantelli Lemma, it follows that (3.58) converges almost surely to (3.60). In a similar manner, it follows that the last term on the rhs of (3.57) converges almost surely (as $n\to\infty$) to

$$(3.61)\qquad 2\iint\limits_{-\infty<x<y<\infty} \pi^*(x)\pi^*(y)[\phi_{(1)}(G(x))[1-G(y)]\phi^{(1)}_{(1)}(G(y)) - G(x)\phi^{(1)}_{(1)}(G(x))\phi_{(1)}(G(y))]dG(x)dG(y).$$

Thus, $q^2_{n,1} \to \sigma^2_1$ almost surely as $n\to\infty$, and this implies (3.50). Q.E.D.

<u>Remark</u>. On using (3.42)-(3.43) and the recent results of Hoeffding (1973), it can be shown that under (3.17), (3.37) can be improved to:

$$(3.62)\qquad |n^{\frac{1}{2}}[\mu_n-\mu(F)]| \to 0 \text{ as } n\to\infty,$$

so that $\{\max_{1\le k\le n} |k\mu(F)-\mu^*_k|/\sqrt{n}\sigma\} \to 0$ as $n\to\infty$. Consequently, in (3.38), it is possible to replace μ^*_k by $k\mu(F)$ for $1\le k\le n$.

Let us now consider the situation when H_o holds. Let $\underset{\sim}{j}_n=((-1)^{j_1},\ldots,(-1)^{j_n})$ where j_i is either 0 or 1, $1\le i\le n$, and let $\underset{\sim}{J}_n=\{\underset{\sim}{j}_n\colon j_i=0,1,\ 1\le i\le n\}$. Also, let $\mathcal{S}_n$ be the set of all possible n! realizations of $\underset{\sim}{S}_n$, defined after (2.11). Finally, let $\underset{\sim}{Q}(\underset{\sim}{S}_n) = (Q(n,S_{n1}),\ldots,Q(n,S_{nn}))$. Then, we have the following.

<u>Lemma 3.4</u>. <u>Under</u> H_o <u>in</u> (1.1) $\underset{\sim}{Q}_n=(Q_1,\ldots,Q_n)$ <u>and</u> $\underset{\sim}{R}_n=(R_{n1},\ldots,R_{nn})$ <u>are stochastically independent, and for every</u> $\underset{\sim}{S}_n\varepsilon\mathcal{S}_n$,

$$(3.63)\qquad P\{\underset{\sim}{Q}(\underset{\sim}{S}_n) = \underset{\sim}{j}_n\} = 2^{-n}, \text{ for every } \underset{\sim}{j}_n\varepsilon\underset{\sim}{J}_n.$$

<u>Proof</u>. Now $Z_1,\ldots,Z_n$ are iidrv, so that $\underset{\sim}{R}_n$ can have all possible n! permutations of $(1,\ldots,n)$ with the common probability 1/n!. On the other hand, if $\underset{\sim}{r}_n$

is any permutation of $(1,\ldots,n)$, then

$$\text{(3.64)} \quad P\{\underset{\sim}{Q}_n=\underset{\sim}{j}_n,\ \underset{\sim}{R}_n=\underset{\sim}{r}_n|H_o\}$$

$$= \underset{(\underset{\sim}{s}_n)}{\int\cdots\int}\{\prod_{i=1}^{n}\{g(z_i)[\pi(z_i)]^{1-j_i}[1-\pi(z_i)]^{j_i}\}dz_i,$$

where the n fold integration extends over the domain $\{-\infty<Z_{s_{n1}}<\ldots<Z_{s_{nn}}<\infty\}$ and $\underset{\sim}{s}_n$ is the anti-rank vector corresponding to $\underset{\sim}{r}_n$. Since, by (3.1) and (3.2), under H_o, $\pi(z)=\frac{1}{2}$ for all Z, (3.64) reduces to

$$\text{(3.65)} \quad 2^{-n}\underset{(\underset{\sim}{s}_n)}{\int\cdots\int} g(z_1)\ldots g(Z_n)dz_1\ldots dz_n = 2^{-n}(n!)^{-1} = 2^{-n}P\{\underset{\sim}{R}_n=\underset{\sim}{r}_n\}.$$

Hence, $P\{\underset{\sim}{Q}_n=\underset{\sim}{j}_n|\underset{\sim}{R}_n=\underset{\sim}{r}_n,H_o\} = 2^{-n}$, $\forall\ \underset{\sim}{r}_n$, and this implies the independence of $\underset{\sim}{R}_n$ and $\underset{\sim}{Q}_n$. Hence

$$\text{(3.66)} \quad P\{\underset{\sim}{Q}_n = \underset{\sim}{j}_n|H_o\} = 2^{-n},\ \forall\ \underset{\sim}{j}_n\varepsilon \underset{\sim}{J}_n.$$

By virtue of the fact that $\underset{\sim}{S}_n$ is the anti-rank corresponding to some $\underset{\sim}{R}_n$, we have $P\{\underset{\sim}{Q}(\underset{\sim}{S}_n)=\underset{\sim}{j}_n|H_o\} = P\{\underset{\sim}{Q}_n=\underset{\sim}{j}_n|\underset{\sim}{R}_n,H_o\}$, and hence, (3.63) follows from (3.66) and the independence of $\underset{\sim}{Q}_n$ and $\underset{\sim}{R}_n$. Q.E.D.

Let $\underset{\sim}{S}_n^{(k)}=(S_{n1},\ldots,S_{nk})$, $\underset{\sim}{Q}(S_n^{(k)}) = (Q(n,S_{n1}),\ldots,Q(n,S_{nk}))$, and let $\mathcal{B}^*_{nk}$ be the σ-field generated by $(\underset{\sim}{S}_n^{(k)},\underset{\sim}{Q}(S_n^{(k)}))$, when H_o holds, for $k=1,\ldots,n$.

<u>Lemma 3.5</u>. <u>For every</u> $n\geq 1$, $\{T_{nk},\mathcal{B}^*_{nk},\ 1\leq k\leq n\}$ <u>is a martingale</u>.

<u>Proof</u>. By (2.12), for every $k>q$,

$$\text{(3.67)} \quad E(T_{nk}|\mathcal{B}^*_{nq}) = E\{\textstyle\sum_{i=1}^{k} a_n(i)Q(n,S_{ni})|\mathcal{B}^*_{nq}\}$$

$$= T_{nq} + \textstyle\sum_{i=q+1}^{k} a_n(i)E[Q(n,S_{ni})|\mathcal{B}^*_{nq}].$$

By Lemma 3.4, for every $i>q$, $E[Q(n,S_{n_i})|\mathcal{B}^*_{nq}] = 0$, so that by (3.67), $E(T_{nk}|\mathcal{B}^*_{nq}) = T_{nq}$ for every $k>q$. Q.E.D.

We let $T_{no}=0$, and for $1\leq k\leq n$,

$$\text{(3.68)} \quad W^*_n(k/n) = n^{-\frac{1}{2}}A_n^{-1}T_{nk},$$

and by linear interpolation between $[k/n,(k+1)/n]$, for $k=0,1,\ldots,n-1$, we complete the definition of $W_n^*=\{W_n^*(t),t\in I\}$.

Theorem 3.6. Under (1.1), (2.2) and the condition that

$$\max_{1\le k\le n}\{|a_n(k)|/\sqrt{n}\}\to 0 \text{ as } n\to\infty, \tag{3.69}$$

$W_n^* \xrightarrow{\mathcal{D}} W$, in the uniform topology on $C[0,1]$.

[Note that (3.17) implies (3.69) but the converse is not true.]

Proof. Let $\xi_{nk} = T_{nk}-T_{nk-1}$, $1\le k\le n$. Then, by (2.12) and Lemmas 3.4 and 3.5, we have

$$E\xi_{nk} = 0,\ E[\xi_{nk}|\mathcal{B}^*_{nk-1}] = 0 \text{ and } E[\xi^2_{nk}|\mathcal{B}^*_{nk-1}] = a_n^2(k), \tag{3.70}$$

for $1\le k\le n$, so that $E\,T_{nk}=0$, $1\le k\le n$, and

$$V_n^2 = \sum_{k=1}^n E(\xi^2_{nk}|\mathcal{B}^*_{nk-1}) = \sum_{k=1}^n a_n^2(k) = n\,A_n^2, \tag{3.71}$$

where $A_n^2 \to A^2 = \int_0^1 \phi^2(u)du$, as $n\to\infty$. Also, by (3.69), $|\xi_{nk}| = |a_n(k)| \le \max_{1\le i\le n}|a_n(i)| = o(V_n)$, for all k: $1\le k\le n$. Hence, for every $\varepsilon>0$,

$$n^{-1}\sum_{i=1}^n E\{\xi^2_{ni}I(|\xi_{ni}|\ge \varepsilon V_n)\}\to 0 \text{ as } n\to\infty. \tag{3.72}$$

The convergence of the finite dimensional distributions of W_n^* to those of W follows directly from (3.71), (3.72) and Theorem 2.1 of Dvoretsky (1972). By virtue of Lemma 3.5 and the Doob upcrossing inequality for semi-martingales, the proof of the tightness of W_n^* follows along the lines of Section 6 of Brown (1971), and hence, the details are omitted.

4. Properties of the tests based on T_n, M_n^+ and M_n.

By virtue of Lemma 3.4, under H_o in (1.1), $\underset{\sim}{Q}_n$ and $\underset{\sim}{R}_n$ are stochastically independent, and $P\{\underset{\sim}{Q}_n=\underset{\sim}{j}_n|H_o\} = 2^{-n}$ for every $\underset{\sim}{j}_n\in\underset{\sim}{J}_n$. Thus, if we let

$$\tilde{T}_n = \sum_{i=1}^n a_n(i)U_i,\ n\ge 1, \tag{4.1}$$

where U_i, $1\le i\le n$, are iidrv, and $P\{U_i=\pm 1\} = \frac{1}{2}$, $i\ge 1$, we conclude that T_n has the same distribution (under H_o) as of $\tilde{T}_n$. On the other hand, $\tilde{T}_n$ involves a linear combination of iidrv, and hence, its distribution can be traced without much problem. In fact, if one keeps in mind the classical one-sample problem, then the corresponding rank order test statistic [viz. Hájek and Sidák (1967, p. 108)] has the same distribution (under the null hypothesis of symmetry) as of $\tilde{T}_n$. Consequently, the available tables for this situation [viz., Owen (1962)] for various common scores and small sample sizes provide the necessary tables for our case too. Since, the distribution of $\tilde{T}_n$ depends only on $a_n(1),\ldots,a_n(n)$, we conclude that under H_o in (1.1), T_n is genuinely distribution-free. On the other hand, by Theorem 3.3, it follows that for every real x,

$$\lim_{n\to\infty} P\{n^{-\frac{1}{2}}T_n/A_n \le x | H_o\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-\tfrac{1}{2}t^2\}dt = \Phi(x), \tag{4.2}$$

where $\Phi(x)$ is the standard normal df. Thus, if $\Phi(\tau_\alpha) = 1-\alpha$, $0<\alpha<1$, we obtain from (2.4), (2.5) and (4.2) that

$$\lim_{n\to\infty} C_{n,\alpha}^{(1)} = \tau_\alpha \text{ and } \lim_{n\to\infty} C_{n,\alpha}^{(2)} = \tau_{\alpha/2},\ 0<\alpha<1. \tag{4.3}$$

With a view to studying the ARE of the proposed test for various score functions, we first consider the Bahadur-efficiency of the tests. For this, we first consider the following.

<u>Lemma 4.1</u>. <u>Under</u> (3.17), $n^{-1}T_n \to \mu(F)$ a.s., <u>as</u> $n\to\infty$, <u>where</u> $\mu(F)$ <u>is defined by</u> (3.33).

<u>Proof</u>. Using (3.42), we rewrite $T_n = T_{n,1} + T_{n,2} - T_{n,3}$, where by the Schwarz inequality and (3.43), for j=2,3,

$$|n^{-1}T_{n,j}| \le (n^{-1}\textstyle\sum_{i=1}^n Q_i^2)^{\frac{1}{2}}(n^{-1}\sum_{i=1}^n a_{n,j}^2(i))^{\frac{1}{2}} = [\frac{1}{n}\sum_{i=1}^n a_{n,j}^2(i)]^{\frac{1}{2}} < \tfrac{1}{2}\eta', \tag{4.4}$$

for $n \ge n_o(\eta)$, where $\eta'>0$ and $\eta'\to0$ as $\eta\to0$. Thus, by choosing η (and hence, η') sufficiently small, it suffices to show that $n^{-1}T_{n,1} \to \mu(F)$ a.s., as $n\to\infty$. By the same decomposition (i.e., (3.42)), we can show that

(4.5) $$|\mu(F)-\mu_1(F)| < \eta' \text{ where } \mu_1(F) = \int_{-\infty}^{\infty} \pi^*(z)\phi_{(1)}(G(z))dG(z).$$

Hence, it suffices to show that $n^{-1}T_{n,1} \to \mu_1(F)$ a.s., as $n\to\infty$. To do this, we consider the empirical df

(4.6) $$G_n(z) = n^{-1}\sum_{i=1}^{n} c(z-Z_i),\ n\geq 1,\ z\varepsilon E.$$

Then, by (3.20) and (2.3), we have

(4.7) $$\begin{aligned} n^{-1}T_{n,1} &= n^{-1}\sum_{i=1}^{n} Q_i\phi_{(1)}((n+1)^{-1}R_{ni}) + o(n^{-\frac{1}{2}}) \\ &= n^{-1}\sum_{i=1}^{n} Q_i\phi_{(1)}(\tfrac{n}{n+1}G_n(Z_i)) + o(n^{-\frac{1}{2}}) \\ &= n^{-1}\sum_{i=1}^{n} Q_i\phi_{(1)}(G(Z_i)) + n^{-1}\sum_{i=1}^{n} Q_i[\phi_{(1)}(\tfrac{n}{n+1}G_n(Z_i)) - \phi_{(1)}(G(Z_i))] + o(n^{-\frac{1}{2}}). \end{aligned}$$

Now, $\chi_i = Q_i\phi_{(1)}(G(Z_i))$, $i\geq 1$, are iidrv with mean $\mu_1(F)$, and hence, by the Kintchine strong law of large numbers,

(4.8) $$n^{-1}\sum_{i=1}^{n} Q_i\phi_{(1)}(G(Z_i)) \to \mu_1(F) \text{ a.s., as } n\to\infty.$$

Also, by the Glivenko-Cantelli theorem, $\sup_{z\varepsilon E}|G_n(z)-G(z)|\to 0$ a.s., as $n\to\infty$, so that on noticing that $|Q_i| \leq 1$, $\forall\ i\geq 1$ and $\phi_{(1)}$ is a polynomial, we immediately conclude that the second term on the rhs of (4.7), being bounded by $\max_{1\leq i\leq n}|\phi_{(1)}(\frac{n}{n+1}G_n(Z_i)) - \phi_{(1)}(G(Z_i))|$, converges a.s. to 0 as $n\to\infty$. So, the proof is complete.

By (4.2), Lemma 4.1 and the definition of Bahadur (1960) efficiency [cf. Puri and Sen (1971, p. 122)], we conclude that the BARE (Bahadur ARE) of $\{T_n\}$ based on the score function ϕ with respect to $\{T_n^*\}$ based on the score function ϕ^* is given by

(4.9) $$e_1(\phi,\phi^*) = [\mu(F,\phi)A(\phi^*)/\mu(F,\phi^*)A(\phi)]^2,$$

where $\mu(F,\phi) = \int_{-\infty}^{\infty} \pi^*(z)\phi(G(z))dG(z)$, $A^2(\phi) = \int_0^1 \phi^2(u)du$ and similar expression for $\mu(F,\phi^*)$ and $A^2(\phi^*)$ hold for $\phi=\phi^*$. Notice that one may rewrite

(4.10) $$\mu^2(F,\phi)/A^2(\phi) = \{\int_{-\infty}^{\infty}[\pi^*(z)]^2 dG(z)\}\{\frac{[\int_{-\infty}^{\infty}\pi^*(z)\phi(G(z))dG(z)]^2}{[\int_0^1 \phi^2(u)du][\int_{-\infty}^{\infty}[\pi^*(z)]^2 dG(z)]}\}$$

$$= (\int_0^1 \psi^2(u)du)\{[\int_0^1 \psi(u)\phi(u)du]^2/[\int_0^1 \psi^2(u)du][\int_0^1 \phi^2(u)du]\}$$

$$= \rho^2(\psi,\phi)\cdot(\int_0^1 \psi^2(u)du),$$

where $\psi(u) = \pi^*(G^{-1}(u)) = 2\pi(G^{-1}(u))-1)$, $0<u<1$. Thus, (4.9) reduces to

(4.11) $$e_1(\phi,\phi^*) = \rho^2(\psi,\phi)/\rho^2(\psi,\phi^*).$$

Thus, from the BARE point of view, the optimal choice of $\phi(u)$ is $\psi(u)$, $0<u<1$, and as a result,

(4.12) $e(\phi,\psi) = \rho^2(\psi,\phi)$ is always bounded by 1.

We could have also considered the Pitman ARE, where we conceive of a sequence $\{H_n\}$ of alternative hypotheses, such that under H_n. $F(x,y) = F_{(n)}(x,y)$, is such that $Z_1,\ldots,Z_n$ are iidrv with a df $G_{(n)}(z)$ (dependent on n) and $\pi(z) = \pi_{(n)}(z)$ also may depend on z, in such a way that

(4.13) $\lim_{n\to\infty} G_{(n)}(z) = G(z)$ exists, and $\pi_{(n)}(z) = \frac{1}{2}+n^{-\frac{1}{2}}\gamma(z)$, $z\varepsilon E$,

and $\int_{-\infty}^{\infty}|\gamma(z)||\phi(G(z))|dG(z)<\infty$. Then, if we let

(4.14) $$\psi^*(u) = \gamma(G^{-1}(u)),\ 0<u<1,\ A(\gamma) = \int_0^1 [\psi^*(u)]^2 du;$$

(4.15) $$\rho(\psi^*,\phi) = (\int_0^1 \psi^*(u)\phi(u)du)/[A(\phi)A(\gamma)],$$

it follows by some routine steps that the Pitman ARE of $\{T_n\}$ with respect to $\{T_n^*\}$ is

(4.16) $$e_2(\phi,\phi^*) = \rho^2(\psi^*,\phi)/\rho^2(\psi^*,\phi^*).$$

In this case, the asymptotically optimal score function is $\phi=\psi^*$.

Let us now consider the tests based on M_n^+ and M_n. Note that here also the null hypothesis distribution of M_n^+ or M_n is generated by the $2^n n!$ equally likely realizations of $(\underset{\sim}{Q}_n, \underset{\sim}{R}_n)$. [It may be remarked that given $\underset{\sim}{Q}_n$ and $\underset{\sim}{R}_n$, the vector $(T_1,\ldots,T_n)$ assumes a particular value dependent only on the score function and $(\underset{\sim}{Q}_n, \underset{\sim}{R}_n)$.] Thus, here, one can enumerate the distribution of M_n^+ or M_n by direct evaluation of all the $2^n n!$ equally likely realizations of $(\underset{\sim}{Q}_n, \underset{\sim}{R}_n)$; by this constitution, the statistics M_n and M_n^+ are distribution-free under H_o. The process of evaluating the exact null distribution of M_n^+ or M_n becomes prohibitively laborious as n increases. However, for large n, by virtue of Theorem 3.3 and well-known results on the boundary crossing probabilities for a standard Brownian motion, we obtain that for every $x \geq 0$,

$$\lim_{n\to\infty} P\{M_n^+ \leq x | H_o\} = 2\Phi(x)-1, \tag{4.17}$$

$$\lim_{n\to\infty} P\{M_n \leq x | H_o\} = \sum_{k=-\infty}^{\infty} (-1)^k [\Phi((2k+1)x)-\Phi((2k-1)x)]. \tag{4.18}$$

Note that if W_α be the upper $100\alpha\%$ point of the df in (4.18), then by (2.8), (2.9), (4.17) and (4.18),

$$\lim_{n\to\infty} M_{n,\alpha}^+ = \tau_{\alpha/2} \text{ and } \lim_{n\to\infty} M_{n,\alpha} = W_\alpha: \quad 0<\alpha<1. \tag{4.19}$$

If we denote the rhs of (4.17) and (4.18) by $H^+(x)$ and $H(x)$, respectively, we note that by (4.17), for large x,

$$-\log[1-H^+(x)] = \tfrac{1}{2}x^2\{1+o(1)\}. \tag{4.20}$$

Also, noting that $1-H^+(x) \leq 1-H(x) \leq 2[1-H^+(x)]$, we have for large x,

$$-\log[1-H(x)] = \tfrac{1}{2}x^2\{1+o(1)\}. \tag{4.21}$$

Further, by Lemma 4.1 and (3.12), it follows that as $n\to\infty$

(4.22) $$n^{-\frac{1}{2}}M_n^+ \to \mu(F)/A \text{ a.s., and } n^{-\frac{1}{2}}M_n \to |\mu(F)|/A \text{ a.s.}$$

Hence, the efficacy [in the sense of Bahadur (1960)] of either M_n^+ or M_n is

(4.23) $$\mu^2(F)/A^2 = (\int_0^1 \psi(u)\phi(u)du)^2/(\int_0^1 \phi^2(u)du),$$

where $\psi(u)$ is defined after (4.10). As such the BARE of M_n^+ (or M_n) with respect to T_n in (2.4 [or (2.5)] is equal to 1, when the same score function $\phi(u)$ is employed in both the cases. On the other hand, in (2.4)-(2.5), our sample size is prefixed and equal to n, while in (2.8)-(2.9), it is a random variable N_n, and N_n can be smaller than n with a positive probability. In fact, by Lemma 4.1 and (4.19), it follows that for every $\varepsilon>0$,

(4.24) $$\begin{aligned} P\{N_n > \varepsilon n|\mu(F)\neq 0\} &\le P\{n^{-\frac{1}{2}}T_{[n\varepsilon]}/A_n > M_{n,\alpha}^+|\mu(F)\neq 0\} \\ &= P\{n^{-1}T_{[n\varepsilon]}/A_n > n^{-\frac{1}{2}}M_{n,\alpha}^+|\mu(F)\neq 0\}\to 0, \text{ as } n\to\infty, \end{aligned}$$

and a similar result follows for M_n. Consequently, when H_o is not true, one may expect a considerable amount of reduction of the ASN of the 1st sequential procedure, without any loss of the BARE.

5. Properties of the tests based on D_n^+ and D_n. Note that by Lemma 3.4, under H_o, $\underset{\sim}{Q}(\underset{\sim}{S}_n)$ assumes all possible 2^n realizations $\underset{\sim}{j}_n \varepsilon \underset{\sim}{J}_n$, each with the equal probability 2^{-n}. By a look at (2.12) and (2.14), we observe that the set of realizations of $(T_{n1},\dots,T_{nn})$, and hence, of D_n^+ or D_n, generated by the set of 2^n equally likely realizations of $\underset{\sim}{Q}(\underset{\sim}{S}_n)$, can be traced, and the exact null distribution can be computed. By virtue of this constitution, the tests based on D_n^+ and D_n are distribution-free.

It follows from Theorem 3.6 that as $n\to\infty$,

(5.1) $$P\{D_n^+ \le x\} \to P\{\sup_{t\varepsilon I} W(t) \le x\}, \ \forall\ 0\le x<\infty;$$

(5.2) $$P\{D_n \le x\} \to P\{\sup_{t\in I} |W(t)| \le x\},\ \forall\ 0\le x<\infty.$$

As a result, D_n^+ and M_n^+ (or D_n and M_n) both have the same limiting null distribution given by (4.17) (or (4.18)). Consequently, as in (4.19),

(5.3) $$\lim_{n\to\infty} D^+_{n,\alpha} = \tau_{\alpha/2} \text{ and } \lim_{n\to\infty} D_{n,\alpha} = W_\alpha:\quad 0<\alpha<1,$$

and (4.20)-(4.21) also apply to these statistics.

Let us now denote by

(5.4) $$\tau(x) = \int_{-\infty}^{x} \pi^*(z)\phi(G(z))dG(z),\ -\infty<x<\infty.$$

<u>Lemma 5.1</u>. <u>Under</u> (3.17) <u>and the conditions of Theorem 3.2</u>,

(5.5) $$n^{-\frac{1}{2}}D_n^+ \to \sup_x \tau(x)/A \text{ a.s., as } n\to\infty,$$

(5.6) $$n^{-\frac{1}{2}}D_n \to \sup_x |\tau(x)|/A \text{ a.s., as } n\to\infty.$$

<u>Proof</u>. As in the proof of Lemma 4.1, we write, on using (3.42), $T_{nk} = T_{nk,1} + T_{nk,2} - T_{nk,3}$. Then, for j=2 or 3,

(5.7) $$\max_{1\le k\le n} |n^{-1}T_{nk,j}| \le (n^{-1}\textstyle\sum_{i=1}^{k} Q_i^2)^{\frac{1}{2}}(n^{-1}\sum_{i=1}^{k} a^2_{n,j}(i))^{\frac{1}{2}}$$
$$\le (k/n)\{\tfrac{1}{n}\textstyle\sum_{i=1}^{n} a^2_{n,j}(i))^{\frac{1}{2}} \le [\tfrac{1}{n}\sum_{i=1}^{n} a^2_{n,j}(i)]^{\frac{1}{2}} < \tfrac{1}{2}\eta',$$

where $\eta'(>0)$ depends on $\eta(>0)$ in (3.43), and $\eta'\to 0$ as $\eta\to 0$. Consequently, it suffices to replace T_{nk} by $T_{nk,1}$ and ϕ by $\phi_{(1)}$ in D_n^+, D_n and $\tau(x)$, respectively, where by (3.42), $\phi_{(1)}$ is a polynomial, and hence, (3.51) holds.

For some arbitrary $\varepsilon>0$, choose a set of $m_\varepsilon(=m)$ points

(5.8) $$z_0,\ldots,z_{m+1},\ z_0=-\infty,\ z_{m+1}=\infty,\ G(z_j)=j\varepsilon,\ j=1,\ldots,m,$$

where $\varepsilon m_\varepsilon \ge 1-\varepsilon$. We also, denote by

(5.9) $$k_{nj} = [nj\varepsilon]+1, \text{ for } j=1,\ldots,m,\ k_{nm+1}=n.$$

Note that $\pi^*(z)$ is absolutely continuous and bounded, $\phi_{(1)}$ is a polynomial and

G is absolutely continuous with $G(z_j)-G(z_{j-1}) \leq \varepsilon$, $1\leq j\leq m+1$. Hence, for every $\delta>0$, there exists an $\varepsilon>0$, such that

(5.10) $$|\tau(x)-\tau(y)|<\tfrac{1}{2}\delta \text{ for every } x,y\varepsilon[z_{j-1},z_j],\ 1\leq j\leq m+1.$$

On the other hand, (3.17) insuring (3.69), and $|\pi^*(z)|\leq 1$, imply that for every $k_{nj-1}\leq k<q\leq k_{nj}$,

(5.11) $$|n^{-1}(T_{nq,1}-T_{nk,1})|\leq n^{-1}\textstyle\sum_{i=k+1}^{q}|a_{n,1}(i)|$$
$$\leq \{n^{-1}\textstyle\sum_{i=k+1}^{q} a^2_{n,1}(i)\}^{\frac{1}{2}} \leq \{n^{-1}\sum_{i=k_{nj-1}}^{k_{nj}} a^2_{n,1}(i)\}^{\frac{1}{2}}$$
$$\to (\int_{(j-1)\varepsilon}^{j\varepsilon} \phi^2_{(1)}(u)du)^{\frac{1}{2}} < \tfrac{1}{2}\delta, \text{ for every } 1\leq j\leq m+1.$$

Consequently, it suffices to show that

(5.12) $$\max_{1\leq j\leq m+1} |\{\max_{1\leq i\leq k_{nj}} [n^{-1}T_{ni,1}]-\tau(Z_j)|\to 0, \text{ a.s., as } n\to\infty.$$

The proof of (5.12) follows along the lines of Lemma 4.1, and hence, the details are omitted.

Let us now denote by

(5.13) $$\tau^o_+ = \sup_x \tau(x) \text{ and } \tau^o = \sup_x |\tau(x)|.$$

Then, by (5.1), (5.2), (4.17), (4.18), (4.20), (4.21), Lemma 5.1 and (5.13), it follows that the efficacy of D_n^+ (or D_n) in the sense of Bahadur (1960) is given by

(5.14) $$(\tau^o_+)^2/A^2 \ [\text{or } (\tau^o)^2/A^2].$$

Here also, we note that if we let for $-\infty<x<\infty$,

(5.15) $$\psi_x(u) = \pi^*(G^{-1}(u)) \text{ if } G^{-1}(u) \leq x,$$
$$= 0, \quad \text{otherwise},$$

and if we let

(5.16) $$A_x^2(\psi) = \int_0^1 \psi_x^2(u)du, \quad -\infty<x<\infty$$

then we have

(5.17) $$(\tau_+^o)^2/A^2 = \sup_x \, [A_x^2(\psi)\rho^2(\psi_x,\phi)],$$

where $\rho^2(\psi_x,\phi) \leq 1$. Note that $A_x^2(\psi)$ is non-decreasing, so that if we let $\phi(u) = \psi_\infty(u) = \pi^*(G^{-1}(u))$, $0<u<1$, the rhs of (5.17) is maximized; for any other $\phi(u)$ (not proportional to $\psi_\infty(u)$), the rhs of (5.17) is bounded from above by $A_\infty^2(\psi) = \int_0^1 \psi_\infty^2(u)du$, so that

(5.18) $$\sup_{\{\phi\}} \{\tau_+^o/A^2\} = A_\infty^2(\psi) = \int_{-\infty}^{\infty} [\pi^*(z)]^2 dG(z).$$

Hence, here also, maximizing the BARE leads us to the asymptotically optimal score function $\psi_\infty(u) = \pi^*(G^{-1}(u))$, $0<u<1$. A similar result holds for D_n. In the next section, we shall study the optimal score function, in little more details, for some important cases.

6. Asymptotically optimal score function. In some important special cases, $\pi^*(z)$ can be written in more explicit forms, and the optimal score functions can be obtained in simpler forms too.

6.1. Stochastically independent components. Here X and Y are stochastically independent, so that for all $(x,y)\varepsilon E^2$

(6.1) $$F(x,y) = F(x,\infty)F(\infty,y) = F_1(x)F_2(y), \text{ say.}$$

Let f_1 and f_2 be the density functions for F_1 and F_2 respectively. Then by (3.1) and (3.2),

(6.2) $$g(z) = f_1(z)[1-F_2(z)]+f_2(z)[1-F_1(z)], \quad \pi(z) = f_1(z)[1-F_2(z)]/g(z);$$

(6.3) $$\pi^*(z) = [f_1(z)[1-F_2(z)]-f_2(z)[1-F_1(z)]]/[f_1(z)[1-f_2(z)]+f_2(z)[1-F_1(z)]]$$
$$= [r_1(z)-r_2(z)]/[r_1(z)+r_2(z)],$$

where the hazard rates $r_1(z)$ and $r_2(z)$ are defined by

(6.4) $$r_i(z) = f_i(z)/[1-F_i(z)],\ z\varepsilon E,\ \text{for } i=1,2.$$

Now, under H_o in (1.1), $F_1=F_2$, so that $r_1(z) = r_2(z)$ for all z. We consider two special cases where F_1 and F_2 may differ in locations or scales. First consider the model

(6.5) $$F_2(x) = F_1(x-\theta),\ -\infty<x<\infty,\ \theta \text{ real.}$$

Then $r_2(z) = r_1(z-\theta)$, so that by (6.3),

(6.6) $$\pi^*(z) = [r_1(z)-r_1(z-\theta)]/[r_1(z)+r_1(z-\theta)],\ z\varepsilon E.$$

For small θ, (6.6) yields (whenever $r_1(z)$ is differentiable)

(6.7) $$r^*(z) \simeq (\theta/2)[\frac{d}{dz} \log r_1(z)],\ z\varepsilon E.$$

Thus, for local translation alternatives, the asymptotically optimal score function is

(6.8) $$\psi_\infty(u) = [(d/dz) \log r_1(z)]_{z=G^{-1}(u)},\ 0<u<1.$$

We may recall that the classical two-sample location problem [viz., Hájek and Sidak (1967, p. 66)], the locally most powerful rank test corresponds to the score function

(6.9) $$\tilde{\psi}(u) = -f_1'(F_1^{-1}(u))/f_1(F_1^{-1}(u)),\ 0<u<1.$$

In general, (6.8) and (6.9) are different from each other. To show this, let us consider the general exponential type of df's for which F_1, f_1 and f_1' exist and the following hold:

(6.10) $$\frac{d}{dx}\left\{\frac{1-F_1(x)}{f_1(x)}\right\} \to 0 \text{ as } x\to\infty \text{ and } \frac{d}{dx}\left\{\frac{F_1(x)}{f_1(x)}\right\} \to 0 \text{ as } x\to-\infty.$$

Note that (6.10) implies that $-[1-F_1(x)]f_1'(x)/f_1^2(x)\to 1$ as $x\to\infty$ and $F_1(x)f_1'(x)/f_1^2(x)\to 1$ as $x\to\infty$, so that as $x\to\infty$,

(6.11) $$\frac{d}{dx}\log r_1(x) = f_1'(x)/f_1(x) + f_1(x)/[1-F_1(x)]$$
$$= [f_1'(x)/f_1(x)]\{1+f_1^2(x)/[1-F_1(x)]f_1'(x)\}$$
$$= [f_1'(x)/f_1(x)]\{o(1)\},$$

and as $x\to-\infty$,

(6.12) $$\frac{d}{dx}\log r_1(x) = [f_1'(x)/f_1(x)]\left\{1+\frac{f_1(x)}{1-F_1(x)}\cdot\frac{f_1(x)}{f_1'(x)}\right\}$$
$$= [f_1'(x)/f_1(x)]\left\{1+\frac{f_1(x)}{1-F_1(x)}\cdot\frac{F_1(x)}{f_1(x)}[1+o(1)]\right\}$$
$$= [f_1'(x)/f_1(x)]\{1+F_1(x)[1-F_1(x)]^{-1}[1+o(1)]\}$$
$$= [f_1'(x)/f_1(x)]\{1+o(1)\}.$$

Thus, $\psi_\infty(u)$ behaves alike $\tilde{\psi}(u)$ as $u\to 0$, but differently when $u\to 1$. In particular for normal df, $f_1'(x)/f_1(x) = -x$, so that it appears that $\psi_\infty(u)$ attaches more weight when u is small and less as $u\to 1$. From one point of view this is quite important too. If the null hypothesis is not true, with greater weight for small u, the T_{nk} will be crossing the barrier $D^+_{n,\alpha}$ (or $\pm D_{n,\alpha}$) faster than the other case where $\psi_\infty(u)$ would have attached more weight to the upper tail. Thus, we would expect an early termination in such a case, and hence, the ASN for the progressively censored test will be smaller when H_o does not hold.

Consider now the scale model where

(6.13) $$F_2(x) = F_1(x/\theta),\ \theta>0 \text{ and } H_o:\ \theta=1.$$

In this case, $r_2(z) = \theta^{-1}r_1(z/\theta)$, $z\varepsilon E$, so that

(6.14) $$\pi^*(z) = [r_1(z)-\theta^{-1}r_1(z/\theta)]/[r_1(z)+\theta^{-1}r_1(z/\theta)],$$

and hence for $\theta=1+\delta$, δ small, (6.14) tends to

(6.15) $$(-\delta/2)\{1+z(d/dz)\log r_1(z)\},\ z\varepsilon E.$$

Consequently, for local scalar alternatives, the asymptotically optimal score function is

(6.16) $$\psi_\infty(u) = 1+[z(d/dz)\log r_1(z)]_{z=G^{-1}(u)},\quad 0<u<1.$$

By arguments similar to (6.10)-(6.12), it follows that (6.16) is generally different from the optimal score function for the classical two-sample scale problem.

6.2. Interchangeable components model. Here we assume that (1.1) holds under H_o and under alternative, X and Y-θ are interchangeable for some real θ. Thus, under alternative,

(6.17) $$F(x,y) = F_o(x,y-\theta),\quad (x,y)\varepsilon E^2,$$

where $F_o(x,y) \equiv F_o(y,x)$ for all (x,y). Let us denote the joint survival function by

(6.18) $$\bar{F}(x,y) = 1 - F(x,\infty) - F(\infty,y) + F(x,y),\quad (x,y)\varepsilon E^2.$$

Then, note that under (6.17) and small θ,

(6.19) $$\pi^*(z) = \theta[f_o(z,z) - \int_z^\infty [(\partial/\partial u)f_o(x,u)]_{u=z}dx]/[2\int_z^\infty f_o(x,z)dx] + o(\theta),$$

where f_o is the density function corresponding to F_o. (6.19) reduces to (6.7) when $f_o(z,z) = f_o^2(z), \forall\ z\varepsilon E$. For specific f_o, such as the bivariate normal density, (6.19) may be evaluated and the corresponding $\psi(u)$ can be determined. In general, these are quite complicated.

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