AN ALMOST SURE INVARIANCE PRINCIPLE FOR THE EXTREMA OF CERTAIN SAMPLE FUNCTIONS

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ABSTRACT

For a general class of statistics expressible as extrema of certain sample functions, an almost sure invariance principle, particularly useful in the context of the law of iterated logarithm and the probabilities of moderate deviations, is established, and its applications are stressed.

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Key Words and Phrases: Almost sure invariance principle, bundle strength of filaments, extrema of sample functions, law of iterated logarithm, multivariate Kolmogorov-Smirnov statistics, probability of moderate deviations.

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1. Introduction.

Sen et al. (1973) established the asymptotic normality of a general class of extrema of sample functions expressible as

$$Z_n = \sup\{L(x, F_n(x)) : x \in A\}, \text{ for some } A \subset E^p, \tag{1.1}$$

where F_n is the empirical distribution function (df) of a random sample of size n from a continuous df $F(x)$, $x \in E^p$, the $p(\geq 1)$-dimensional Euclidean space, and $L(x,y)$ satisfies certain regularity conditions. An important member of this class relates to the bundle strength of filaments where p=1, the basic random variables are non-negative, and $nZ_n = \max_{1 \leq i \leq n} [(n-i+1)X_{n,i}]$, where $X_{n,1} \leq \cdots \leq X_{n,n}$ are the ordered random variables. Here, F is defined on $A=[0,\infty)$ and $L(x,y) = x(1-y)$ for $x \in A$ and $0 \leq y \leq 1$; see Bhattacharyya et al. (1970) for a detailed exposition. In the context of sequential estimation and testing of the mean (per unit) bundle strength [i.e., $\theta = \sup\{x[1-F(x)] : 0 \leq x < \infty\}$], Sen (1973a,c) has considered certain almost sure (a.s.) representation for $\{Z_n\}$, and has shown that the usual a.s. invariance principle [viz., the embedding of Wiener processes] for sums of independent random variables or martingales [cf. Strassen (1967)] also holds for $\{Z_n\}$, though Z_n is not a martingale nor involves a sum of independent random variables. In fact, a little deeper examination reveals that the same proof goes through for general $L(x,y)$ satisfying certain mild regularity conditions; we shall comment on it in section 5.

The object of the present investigation is to show that parallel to Theorem 1.4 of Strassen (1967) an a.s. invariance principle (particularly useful in the context of the law of iterated logarithm and probabilities of moderate deviations) holds for $\{Z_n\}$ under essentially the same conditions as pertaining to its asymptotic normality. The main theorem along with the preliminary notions is presented in section 2. Section 3 deals with certain basic lemmas which are needed for the proof of the main theorem, and the latter is sketched in section 4. In this context, some results on (multivariate) Kolmogorov-Smirnov statistics considered earlier by Kiefer (1961), Sethuraman (1964) and Sen (1973b) are made use of. The last section deals with certain concluding remarks and a few applications of the main theorem.

2. The main theorem. Let $\{X_i, i\geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) rv's defined on a probability space (Ω, A, P) where X_i has a continuous df $F(x)$, $x\varepsilon E^p$, for some $p\geq 1$, and our desired $A\subset E^p$. We assume that for $x\varepsilon A$, $L(x,F(x))$ assumes a unique maximum θ at a unique point x_o, so that

$$\theta = \sup\{L(x,F(x)): x\varepsilon A\} = L(x_o,F(x_o)) \text{ where } x_o\varepsilon A, \tag{2.1}$$

and for every $\varepsilon>0$, there exists an $\eta>0$, such that

$$L(x,F(x)) < \theta-\varepsilon \text{ for every } x: |x-x_o| > \eta. \tag{2.2}$$

Further, we assume that for some $\delta(>0)$, sufficiently small, there exist four positive constants C_i, $k_i(\geq 1)$, $i=1,2$, such that

$$\theta-C_1|x-x_o|^{k_1} \leq L(x,F(x)) \leq \theta-C_2|x-x_o|^{k_2}, \quad (k_1\leq k_2), \tag{2.3}$$

for all x: $|x-x_o|\leq\delta$, where in (2.2)-(2.3), $|u|$ stands for the Euclidean norm of the vector $u(\varepsilon E^p)$. Also, we assume that F admits of a continuous (joint) density function $f(x)$ for all x: $|x-x_o|<\delta$, and

$$0 < p_o = F(x_o) < 1, \quad 0 < f(x_o) < \infty. \tag{2.4}$$

Let then $A^*=\{(x,y): x\varepsilon A\subset E^p, 0\leq y\leq 1\}$, and for every $\delta\varepsilon[0,1]$, let

$$A^*_\delta = \{(x,y): x\varepsilon A, \max(0,F(x)-\delta) \leq y \leq \min(1,F(x)+\delta)\}. \tag{2.5}$$

We assume that $L(x,y)$ is defined for every $(x,y)\varepsilon A^*$, and for some $\delta>0$, $(x,y)\varepsilon A^*_\delta$, $L(x,y)$ possesses a continuous partial (with respect to y) derivative $L_{01}(x,y)$ which satisfies the following conditions:

$$(2.6)\qquad |L_{01}(x,y)| \leq g(x),\ \forall\ (x,y)\varepsilon A^*_\delta,$$

$$(2.7)\qquad \int_A g^2(x)dF(x) \leq \lambda^2 < \infty,\ \forall\ A\subset E^p,$$

$$(2.8)\qquad L_{01}(x_o,F(x_o)) = \xi_o \text{ where } 0 < |\xi_o| < \infty.$$

These conditions are slightly less restrictive than the ones in Sen et al. (1973) where some additional conditions were needed to suit the non-stationarity of the basic process. Let us then define

$$(2.9)\qquad \sigma^2 = \xi_o^2 p_o(1-p_o) \text{ so that } 0 < \sigma < \infty.$$

From the results of Sen et al. (1973) it follows that under (2.1)-(2.8),

$$(2.10)\qquad \mathcal{L}(n^{\frac{1}{2}}[Z_n-\theta]) \to N(0,\sigma^2),\text{ as } n\to\infty.$$

Let $\phi=\{\phi(t): 0\leq t<\infty\}$ be a positive function with a continuous derivative $\phi'(t)$ such that (i) as $t\to\infty$ with $s/t\to 1$, $\phi'(s)/\phi'(t)\to 1$, (ii) for some $\frac{1}{2}<h<3/5$,

$$(2.11)\qquad t^{-h}\phi(t) \text{ is } \downarrow \text{ but } \psi(t) = t^{-\frac{1}{2}}\phi(t) \text{ is } \uparrow \text{ in } t,$$

and (iii) the Kolmogorov-Petrovski-Erdös criterion holds, i.e.,

$$(2.12)\qquad \int_n^\infty t^{-1}\psi(t)\exp\{-\tfrac{1}{2}k^2\psi^2(t)\}dt < \infty,\ \forall\ k\geq 1 \text{ and } n\geq 1.$$

Finally, we let

$$(2.13)\qquad \nu_k(t) = \tfrac{1}{2}k^2\psi^2(t) - \log\psi(t) - \log\log t,\quad k\geq 1,\ t\geq 1,$$

and assume that for every $\eta>0$, there exists an $\varepsilon>0$, such that

(2.14) $|\nu_k(n)/\nu_1(n)-1| < \eta$ for every $1 \leq k \leq 1+\varepsilon$, uniformly in n.

Let then

(2.15) $$P_n(\psi) = P\{m^{\frac{1}{2}}|Z_m-\theta| > \sigma\psi(m) \text{ for some } m \geq n\},$$

and on replacing $|Z_m-\theta|$ by $(Z_m-\theta)$ and $(\theta-Z_m)$, we define the corresponding probabilities by $P_n^+(\psi)$ and $P_n^-(\psi)$, respectively. Then, along the lines of Theorem 1.4 of Strassen (1967), our main theorem may be presented as follows.

Theorem 1. *Under* (2.1)-(2.8) *and* (2.11)-(2.14),

(2.16) $$\lim_{n\to\infty}\{[\log P_n(\psi)]/\nu_1(n)\} = -1,$$

and, if, in addition, $\lim_{n\to\infty}(\log\log n)/\psi^2(n) = 0$, *then*

(2.17) $$\lim_{n\to\infty}\{[\log P_n(\psi)]/\psi^2(n)\} = -\tfrac{1}{2}$$

The same results hold for $\{P_n^+(\psi)\}$ *and* $\{P_n^-(\psi)\}$.

The proof of the theorem is sketched in section 4; certain other results required in this context are presented in section 3.

3. Some useful results. First, we consider the following lemma which follows as a corollary to Theorem 1 of Sethuraman (1964).

Lemma 3.1. *For every* $B \subset E^p$ *and* $\varepsilon>0$, *there exists a* $\rho=\rho(B,\varepsilon)$, *such that* $0 < \rho(B,\varepsilon) \leq \rho(E^p,\varepsilon) < 1$, *and*

(3.1) $$\lim_{n\to\infty} [n^{-1}\log\{\sup_{x\varepsilon B}|F_n(x)-F(x)|>\varepsilon\}] = \log \rho.$$

Let us now define

(3.2) $$Z_n^* = n^{\frac{1}{2}}[L(x_o,F_n(x_o))-\theta]/\sigma,$$

where σ is defined by (2.9) and x_o by (2.1). Then, we have the following.

Lemma 3.2. Under (2.4)-(2.8) and (2.11)-(2.14),

(3.3) $$\lim_{n\to\infty}[\{\nu_1(n)\}^{-1} \log P\{Z_m^* > \psi(m) \text{ for some } m\geq n\}] = -1,$$

and, in (3.3), Z_m^* may also be replaced by $-Z_m^*$ or $|Z_m^*|$.

Proof. By (2.1), (2.4)-(2.9) and (3.2), we have

(3.4) $$Z_n^* = \gamma_n\{n^{\frac{1}{2}}[F_n(x_o)-F(x_o)]/\sqrt{p_o(1-p_o)}\} = \gamma_n U_n, \text{ say,}$$

where $\gamma_n = L_{01}(x_o, uF_n(x_o)+(1-u)F(x_o))/L_{01}(x_o, F(x_o))$, $0<u<1$, and U_n is the standardized form of a binomial random variable. Thus, for every $\varepsilon>0$, we have

(3.5) $$\begin{aligned} &P\{Z_m^* > \psi(m) \text{ for some } m\geq n\} \\ &\leq P\{Z_m^* \geq (1+\varepsilon)\psi(m), \text{ for some } m\geq n,\ \gamma_m \leq 1+\tfrac{1}{2}\varepsilon,\ \forall\ m\geq n\} \\ &\quad + P\{\gamma_m > 1+\tfrac{1}{2}\varepsilon, \text{ for some } m\geq n\} \\ &\leq P\{U_m \geq (1+\varepsilon')\psi(m) \text{ for some } m\geq n\} + P\{\gamma_m > 1+\tfrac{1}{2}\varepsilon \text{ for some } m\geq n\}, \end{aligned}$$

where $1 < (1+\varepsilon)/(1+\frac{1}{2}\varepsilon) = 1+\varepsilon' < 1+\varepsilon/2$. By Theorem 1.4 of Strassen (1967), we obtain that as $n\to\infty$,

(3.6) $$\begin{aligned} &P\{U_m \geq (1+\varepsilon')\psi(m) \text{ for some } m\geq n\} \\ &\sim (1+\varepsilon')(2\pi)^{-\frac{1}{2}} \int_n^\infty \phi'(t)t^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}(1+\varepsilon')\psi^2(t)\}dt = I_n(1+\varepsilon'), \text{ say.} \end{aligned}$$

Also, by Lemma 3.1 of Sen (1973b), we have

(3.7) $$\lim_{n\to\infty}\{[\log I_n(1+\varepsilon')]/\nu_{1+\varepsilon'}(n)\} = -1.$$

Further, by virtue of the assumed continuity of $L_{01}(x,y)$ (in A_δ^*), for every $\eta>0$ (sufficiently small), there exists a $\delta>0$, such that $|L_{01}(x_o,y)-L_{01}(x_o,F(x_o))|<\eta$ whenever $|y-F(x_o)|<\delta$. Since $L_{01}(x_o,F(x_o)) = \xi_o \neq 0$, η can always be so chosen that $\eta/|\xi_o| < \frac{1}{2}\varepsilon$. Hence, $\gamma_m < 1+\frac{1}{2}\varepsilon$ whenever $|F_m(x_o)-F(x_o)|<\delta$. Thus,

$$(3.8)\qquad P\{\gamma_m > 1+\tfrac{1}{2}\varepsilon \text{ for some } m\geq n\}$$

$$\leq P\{|F_m(x_o)-F(x_o)| > \delta \text{ for some } m\geq n\}$$

$$\leq C_\delta[\rho(\delta)]^n,\ 0<\rho(\delta)<1,\ 0<C_\delta<\infty,$$

where $C_\delta \leq C[1-\rho(\delta)]^{-1}$ depends on δ, and the last step follows from Theorem 1 of Hoeffding (1963). Note that by (2.11) and (2.13), $\nu_k(n)$ is bounded by $n^{1/5}$ for large n, while $[\rho(\delta)]^n = \exp\{n \log \rho(\delta)\}$ decreases exponentially with n. Consequently, for n sufficiently large,

$$(3.9)\qquad I_n(1+\varepsilon') + C_\delta[\rho(\delta)]^n = I_n(1+\varepsilon')\{1+o(1)\},$$

and hence, by (3.5), (3.6), (3.7), (3.8) and (3.9), we obtain that

$$(3.10)\qquad \lim_n \sup[\{\nu_{1+\varepsilon'}(n)\}^{-1} \log P\{Z^*_m \geq (1+\varepsilon)\psi(m) \text{ for some } m\geq n\}] \leq -1.$$

Now, we may write $\{\nu_{1+\varepsilon}(n)\}^{-1} \log P\{Z^*_m \geq (1+\varepsilon)\psi(m) \text{ for some } m\geq n\} = [\nu_{1+\varepsilon'}(n)/\nu_{1+\varepsilon}(n)]\cdot$ $[\{\nu_{1+\varepsilon'}(n)\}^{-1} \log P\{Z^*_m \geq (1+\varepsilon)\psi(m) \text{ for some } m\geq n\}]$. Also, $\varepsilon'<\varepsilon/2$. Hence, by letting ε to be arbitrarily small, we obtain from (2.14) and (3.10) that for every $\eta>0$,

$$(3.11)\qquad \lim_n \sup[\{\nu_1(n)\}^{-1} \log P\{Z^*_m > \psi(m) \text{ for some } m\geq n\}] \leq -1+\eta.$$

On the other hand, by (3.4),

$$(3.12)\qquad P\{Z^*_m > \psi(m) \text{ for some } m\geq n\}$$

$$\geq P\{Z^*_m > \psi(m) \text{ for some } m\geq n,\ \gamma_m \geq 1-\tfrac{1}{2}\varepsilon,\ \forall\ m\geq n\}$$

$$\leq P\{U_m \geq (1+\varepsilon'')\psi(m) \text{ for some } m\geq n\} - P\{\gamma_m < 1-\tfrac{1}{2}\varepsilon \text{ for some } m\geq n\},$$

where $1 < (1-\frac{1}{2}\varepsilon)^{-1} = 1+\varepsilon'' < 1+\frac{1}{2}\varepsilon(1+\varepsilon)$ for every $0<\varepsilon<1$. Thus, proceeding as in

(3.6) through (3.9), it follows that for sufficiently large n, the right hand side of (3.12) can be written as

$$I_n(1+\varepsilon'')\{1 + o(1)\}. \tag{3.13}$$

Hence, by (3.7), (3.12) and (3.13), we obtain that

$$\liminf_n [\{\nu_{1+\varepsilon''}(n)\}^{-1} \log P\{Z^*_m > \psi(m) \text{ for some } m \geq n\}] \geq -1. \tag{3.14}$$

Also, $\varepsilon'' < \frac{1}{2}\varepsilon(1+\varepsilon)$ can be made small by letting ε to be small. So, on using (2.14), we obtain from (3.14) that for every $\eta > 0$,

$$\liminf_n [\{\nu_1(n)\}^{-1} \log P\{Z^*_m > \psi(m) \text{ for some } m \geq n\}] \geq -1-\eta. \tag{3.15}$$

Since $\eta(>0)$ is arbitrary, the proof of (3.3) follows from (3.10) and (3.15). The other two cases follows on parallel lines. Q.E.D.

Note that in (2.3), $k_2 \geq k_1 \geq 1$. We write $d_n = \max(\log n, \psi^2(n))$, and let

$$B_n = B_n(x_o,\alpha) = \{x: |F(x)-F(x_o)| \leq cn^{-\alpha}d_n\}, \ \alpha = (2k_2)^{-1} (< \tfrac{1}{2}) \text{ and } c>0, \tag{3.16}$$

$$G^*_n = \sup\{n^{\frac{1}{2}}|L(x,F_n(x))-L(x,F(x))-L(x_o,F_n(x_o))+L(x_o,F(x_o))|: x\varepsilon B_n\}. \tag{3.17}$$

<u>Lemma 3.3</u> <u>Under</u> (2.3)-(2.11), <u>for every</u> $\varepsilon>0$, <u>as</u> $n\to\infty$,

$$P\{G^*_m > \varepsilon\psi(m) \text{ for some } m \geq n\} = o(\exp\{-\psi^2(n)\}). \tag{3.18}$$

<u>Proof</u>. On writing $\gamma_n(x) = L_{01}(x,uF_n(x)+(1-u)F(x))$, $0<u<1$, $x\varepsilon B_n$, we have

$$\begin{aligned} G^*_n &= \sup\{n^{\frac{1}{2}}|\gamma_n(x)\{F_n(x)-F(x)-F_n(x_o)+F(x_o)\}+[\gamma_n(x)-\gamma_n(x_o)]\{F_n(x_o)-F(x_o)\}|: x\varepsilon B_n\} \\ &\leq \sup\{n^{\frac{1}{2}}|F_n(x)-F(x)-F_n(x_o)+F(x_o)|: x\varepsilon B_n\}\sup\{|\gamma_n(x)|: x\varepsilon B_n\} \\ &\quad + \sqrt{n}|L(x_o,F_n(x_o))-L(x_o,F(x_o))|[\sup\{|\gamma_n(x)/\gamma_n(x_o)-1|: x\varepsilon B_n\}]. \end{aligned} \tag{3.19}$$

By virtue of the assumed continuity of $L_{01}(x,y)$ [for $(x,y)\varepsilon A^*_\delta$], (2.6)-(2.8), Lemma 3.1 and Lemma 3.2, on denoting by

$$G_n = \sup\{n^{\frac{1}{2}}|F_n(x)-F_n(x_o)-F(x)+F(x_o)|: x\varepsilon B_n\}, \tag{3.20}$$

it suffices to show that for every $\varepsilon>0$, as $n\to\infty$,

$$P\{G_m > \varepsilon\psi(m) \text{ for some } m\geq n\} = o(\exp\{-\psi^2(n)\}). \tag{3.21}$$

We consider the two cases: (i) ψ: $\psi^2(n) < C \log n$ for some $C>1$, and (ii) ψ: $C \log n \leq \psi^2(n) \leq o(n^{1/5})$, separately. In case (i), basically following the proof of Lemma 1 of Bahadur (1966) with modifications as in Theorem 3.2 of Sen (1973a) and extending the proof to the $p(\geq 1)$-variate case in a straightforward manner, it follows that for every $s(>0)$ there exist a constant c_s and an integer n_s, such that

$$P\{G_n > c_s n^{-\alpha/2}(\log n)\} < n^{-s-1}, \text{ for every } n\geq n_s. \tag{3.22}$$

We note that for every $\alpha>0$, $n^{-\alpha/2} \log n \to 0$ as $n\to\infty$, while by (2.11), for every $\varepsilon>0$, $\varepsilon\psi(n)$ is $\uparrow$ in n. Hence, there exists an integer n_ε, such that $\varepsilon\psi(n) \geq c_s n^{-\alpha/2} \log n$ for $n\geq n_\varepsilon$. As a result, by (3.22), for $n \geq \max(n_s,n_\varepsilon)$,

$$\begin{aligned} &P\{G_m > \varepsilon\psi(m) \text{ for some } m\geq n\} \\ &\leq \textstyle\sum_{m=n}^{\infty} P\{G_m > \varepsilon\psi(m)\} \leq \sum_{m=n}^{\infty} P\{G_m > c_s m^{-\alpha/2} \log m\} \\ &\leq \textstyle\sum_{m=n}^{\infty} n^{-s-1} = O(n^{-s}) = o(n^{-C}) = o(\exp\{- C \log n\}) \\ &= o(\exp\{-\psi^2(n)\}), \text{ by choosing } s>C. \end{aligned} \tag{3.23}$$

Thus, (3.21) holds for case (i). For case (ii), we consider first the case of $p=1$. We define a set of points $\{x_{n,j}\}$ by $F(x_o) = p_o$ and

$$F(x_{n,j}) = p_o + n^{-\frac{1}{2}}\psi(n)j\varepsilon/2, \quad j=0,\pm 1,\ldots,\pm N_n, \tag{3.24}$$

where N_n is the least positive integer (k) for which $F(x_{n,k}) > p_o + n^{-\alpha}d_n$, so that $N_n = O(n^{\frac{1}{2}-\alpha}d_n)$. Using the monotonicity of the df F_n and F_o we obtain by a few simple steps that for $x\varepsilon[x_{n,j},x_{n,j+1}]$,

(3.25) $$\sqrt{n}|F_n(x) - F_n(x_o) - F(x) + F(x_o)|$$
$$\leq \max_{\ell=j,j+1} \{\sqrt{n}|F_n(x_{n,\ell})-F_n(x_o)-F(x_{n,\ell})+F(x_o)|\} + \varepsilon\psi(n)/2,$$

for $j=-N_n,\ldots,N_n-1$. Consequently,

(3.26) $$P\{G_m > \varepsilon\psi(m) \text{ for some } m\geq n\}$$
$$\leq \sum_{m=n}^{\infty} \sum_{j=-N_m}^{N_m} P\{\sqrt{m}|F_m(x_{m,j})-F_m(x_o)-F(x_{m,j})+F(x_o)| > \tfrac{1}{2}\varepsilon\psi(m)\}.$$

Since, for $x<y$, $m[F_m(y)-F_m(x)]$ has the binomial distribution with parameters $(m, F(y)-F(x))$, by Theorem 1 of Hoeffding (1963) [viz., his (2.2) and (2.4)], we obtain on noting that $|F(x_{n,j}) - F(x_o)| \leq m^{-\alpha}\ \forall\ |j| \leq N_m$, that

(3.27) $$P\{\sqrt{m}|F_m(x_{n,j})-F_m(x_o)-F(x_{m,j})+F(x_o)| > \tfrac{1}{2}\varepsilon\psi(m)\}$$
$$\leq \exp\{-\tfrac{1}{4}\varepsilon^2\psi^2(m)[\alpha \log m - C^*]\},\ \forall\ |j| \leq N_m,$$

whenever $m^{-\alpha} < \frac{1}{2}$, and C^* is a positive constant. Thus, by (3.24), the right hand side of (3.26) is bounded by

(3.28) $$\textstyle\sum_{m=n}^{\infty} \exp\{-\tfrac{1}{4}\varepsilon^2\psi^2(m)[\alpha \log m-C^*] + \log(2N_m)\}.$$

As $\log(2N_m) = O(\log m)$ while $\psi^2(m) \geq C \log m$, $C>1$, for every $\varepsilon>0$, $\alpha>0$, there exists an $n_o = n_o(\varepsilon,\alpha)$, such that for $n \geq n_o$,

(3.29) $$\tfrac{1}{4}\varepsilon^2\psi^2(m)[\alpha \log m-C^*]-\log(2N_m) \geq (1+\delta)\psi^2(m) \text{ where } C\delta = 1+\eta,\ \eta>0.$$

Hence, by (3.28) and (3.29), the right hand side of (3.23) is bounded (for $n\geq n_o$) by

$$(3.30)\qquad \sum_{m=n}^{\infty} \exp\{-(1+\delta)\psi^2(m)\} = \sum_{m=n}^{\infty} \exp\{-\psi^2(m)\}\exp\{-\delta\psi^2(m)\}$$

$$\leq \exp\{-\psi^2(n)\}\sum_{m=n}^{\infty} \exp\{-\delta\psi^2(m)\}$$

$$\leq \exp\{-\psi^2(n)\}\sum_{m=n}^{\infty} m^{-c\delta} \quad (\text{as } \psi^2(n) \geq c \log n)$$

$$= \exp\{-\psi^2(n)\}O(n^{-\eta}) \quad (\text{as } c\delta = 1+\eta,\ \eta>0)$$

$$= o(\exp\{-\psi^2(n)\}).$$

Thus, (3.21) holds for $p=1$. For $p>1$, we need to replace the $(2N_n+1)$ points in (3.24) by $(2N_n+1)^p$ blocks and $\varepsilon/2$ by $\varepsilon/2p$, so that the increment of the df F over the two corners of a block is $\leq \varepsilon\psi(n)/2$. The rest of the proof follows on parallel lines. Hence, the proof of the lemma is complete.

Let us now define

$$(3.31)\qquad Z_n^{(1)} = \sup\{L(x,F_n(x)):\ x\varepsilon B_n\}.$$

Note that, $P\{\sqrt{m}(Z_m^{(1)}-\theta)>\sigma\psi(m)$ for some $m\geq n\} \leq P\{\sup[L(x,F_n(x))-L(x_o,F(x_o)):\ x\varepsilon B_m] > m^{-\frac{1}{2}}\sigma\psi(m)$ for some $m\geq n\} = P\{Z_m^* + G_m^* > \sigma\psi(m)$ for some $m\geq n\}$, where G_m^* is defined by (3.17). Hence, by Lemma 3.2 and Lemma 3.3, we arrive at the following.

Lemma 3.4. Under (2.3)-(2.9) and (2.11)-(2.14),

$$(3.32)\qquad \lim_{n\to\infty}[\{\nu_1(n)\}^{-1}\log P\{\sqrt{m}\,|Z_m^{(1)}-\theta| > \sigma\psi(m) \text{ for some } m\geq n\}] = -1,$$

and in (3.32), $|Z_m^{(1)}-\theta|$ may be replaced by $(Z_m^{(1)}-\theta)$ or by $(\theta-Z_m^{(1)})$.

Finally, let us denote by

$$(3.33)\qquad B_n(\delta) = \{x:\ |x-x_o|\leq\delta \text{ but } x\notin B_n(x_o,\alpha)\},$$

$$(3.34)\qquad G_n^* = \sup\{\sqrt{n}\,|L(x,F_n(x))-L(x,F(x))|:\ x\varepsilon B_n(\delta)\},$$

where δ satisfy (2.3). Also, let $\{a_n\}$ be any sequence of positive numbers.

Then, by virtue of (2.5)-(2.8) and Theorem 1-m of Kiefer (1961) we obtain the following:

Lemma 3.5. Under (2.5)-(2.8), for every $n(\geq 1)$,

(3.35) $$P\{G_n^* \geq a_n\} \leq c_1 \exp\{-c_2 a_n^2\},$$

where c_1 and c_2 are both finite positive numbers, and for large n, c_2 may be replaced by $(2-\varepsilon)/\xi_o^2$, for some $\varepsilon>0$.

4. Proof of the main theorem. We prove (2.16)-(2.17) only for $\{P_n^+(\psi)\}$; the proof for $\{P_n^-(\psi)\}$ follows on parallel lines, while by noting that $P_n^+(\psi)\leq P_n(\psi)\leq P_n^+(\psi)+P_n^-(\psi)$, for all n, the proof for $\{P_n(\psi)\}$ follows immediately.

Note that, by definition

(4.1) $$\begin{aligned} P_n^+(\psi) &= P\{m^{\frac{1}{2}}(Z_m-\theta) > \sigma\psi(m) \text{ for some } m\geq n\} \\ &= 1-P\{m^{\frac{1}{2}}(Z_m-\theta) \leq \sigma\psi(m),\ \forall\ m\geq n\} \\ &\geq 1-P\{Z_m^* \leq \psi(m),\ \forall\ m\geq n\} = P\{Z_m^* > \psi(m) \text{ for some } m\geq n\}, \end{aligned}$$

as $Z_m \geq L(x_o, F_n(x_o)) = \theta+\sigma n^{-\frac{1}{2}}Z_m^*$, with probability one. Hence, by (4.1) and Lemma 3.2, we obtain that

(4.2) $$\liminf_n\{[\log P_n^+(\psi)]/\nu_1(n)\} \geq -1.$$

Let us define B_n and $B_n(\delta)$ as in (3.16) and (3.33), and let

(4.3) $$\tilde{A}_\delta = \{x:\ x\varepsilon A \text{ but } |x-x_o|>\delta\}.$$

Note that by virtue of (2.6) and (2.7), for every $(x,y)\varepsilon A_\delta^*$,

(4.4) $$|L(x,y)-L(x,F(x))| \leq g(x)|y-F(x)|,$$

so that whenever $\sup_x |F_n(x)-F(x)|<\delta$, by (4.4),

$$|L(x,F_n(x))-L(x,F(x))| \le g(x)|F_n(x)-F(x)|. \tag{4.5}$$

As such, by a straightforward generalization of Theorem 2.1 of Sen (1971a) [under our (2.7)], it follows that for every $\varepsilon>0$, there exist positive constants $C(<\infty)$, $\rho^*(\varepsilon)$: $0<\rho^*(\varepsilon)<1$ and an integer $n_o(\varepsilon)$, such that for $n \ge n_o(\varepsilon)$,

$$P\{\sup_{x\varepsilon\tilde{A}_\delta} |L(x,F_n(x))-L(x,F(x))| > \tfrac{1}{2}\varepsilon\} \le C[\rho^*(\varepsilon)]^n. \tag{4.6}$$

As a result,

$$\begin{aligned} (4.7)\qquad & P\{\sup_{x\varepsilon\tilde{A}_\delta} L(x,F_m(x)) > \theta-\tfrac{1}{2}\varepsilon \text{ for some } m\ge n\} \\ &\le \sum_{m=n}^{\infty} P\{\sup_{x\varepsilon\tilde{A}_\delta} L(x,F_m(x) > \theta-\tfrac{1}{2}\varepsilon\} \\ &= \sum_{m=n}^{\infty} P\{\sup_{x\varepsilon\tilde{A}_\delta} [L(x,F(x))+\{L(x,F_m(x))-L(x,F(x))\}] > \theta-\tfrac{1}{2}\varepsilon\} \\ &\le \sum_{m=n}^{\infty} P\{\sup_{x\varepsilon\tilde{A}_\delta} \{L(x,F_m(x))-L(x,F(x))\} > \tfrac{1}{2}\varepsilon\} \text{ (by (2.2))} \\ &\le C^*[\rho^*(\varepsilon)]^n, \text{ where } C^* = C\{1-\rho^*(\varepsilon)\}^{-1} \quad (<\infty). \end{aligned}$$

Let us then define $Z_n^{(1)}$ as in (3.28), and let

$$Z_n^{(2)} = \sup\{L(x,F_n(x)): x\varepsilon B_n(\delta)\},\ Z_n^{(3)} = \sup\{L(x,F(x)): x\varepsilon\tilde{A}_\delta\}, \tag{4.8}$$

so that $Z_n = \max\{Z_n^{(1)},Z_n^{(2)},Z_n^{(3)}\}$. Then, by (2.3), (3.16) and (3.33),

$$\begin{aligned} (4.9)\qquad Z_n^{(2)} &= \sup\{L(x,F_n(x)): x\varepsilon B_n(\delta)\} \\ &\le \theta-c_2 n^{-\frac{1}{2}} d_n^{k_2} + \sup\{|L(x,F_n(x))-L(x,F(x))|: x\varepsilon B_n(\delta)\}, \end{aligned}$$

so that by (4.9) and (3.34)-(3.35), we have

$$(4.10) \qquad P\{Z_m^{(2)} \geq \theta - \tfrac{1}{2}c_2 m^{-\frac{1}{2}} d_m^{k_2} \text{ for some } m \geq n\}$$

$$\leq P\{\sup\{\sqrt{m}|L(x,F_m(x)) - L(x,F(x))| : x \in B_m(\delta)\} \geq \tfrac{1}{2}c_2 d_m^{k_2} \text{ for some } m \geq n\}$$

$$\leq \sum_{m=n}^{\infty} [c_1 \exp\{-\tfrac{1}{4}c_2^2 d_m^{2k_2}\}] = o(\exp\{-\psi^2(n)\}), \text{ as } n\to\infty,$$

as $d_n^{2k_2} = [\max\{\log n, \psi^2(n)\}]^{2k_2} \geq (\log n)^{k_2}[\psi^2(n)]^{k_2}$ and $k_2 \geq 1$. Finally, by (3.2) and (3.31),

$$(4.11) \qquad P\{Z_m^{(1)} < \theta - \tfrac{1}{2}c_2 m^{-\frac{1}{2}} d_m^{k_2}, \text{ for some } m \geq n\}$$

$$\leq P\{Z_m^* < -\tfrac{1}{2}c_2 d_m^{k_2}, \text{ for some } m \geq n\}$$

$$= o(\exp\{-\psi^2(n)\}), \text{ as } n\to\infty,$$

where the last step follows from Lemma 3.2 by noting that $d_n^{k_2} \geq [\psi^2(n)]^{k_2}$ is large compared to $\psi(n)$, by (2.11). Hence, from (4.7), (4.10) and (4.11), we have

$$(4.12) \qquad P\{Z_m \neq Z_m^{(1)} \text{ for some } m \geq n\}$$

$$\leq C^*[\rho^*(\varepsilon)]^n + 2[o(\exp\{-\psi^2(n)\})] = o(\exp\{-\psi^2(n)\}),$$

as $\psi^2(n)/n \log \rho^*(\delta) \to 0$ as $n\to\infty$. Then, we have

$$(4.13) \qquad P_n^+(\psi) \leq P\{\sqrt{m}(Z_m^{(1)} - \theta) > \sigma\psi(m), \text{ for some } m \geq n\} +$$

$$P\{Z_m \neq Z_m^{(1)}, \text{ for some } m \geq n\},$$

so that by Lemma 3.4, (4.12) and (4.13), we have

$$(4.14) \qquad \lim_n \sup[\{\nu_1(n)\}^{-1} \log P_n^+(\psi)] \leq -1,$$

and hence, (2.16) follows from (4.2) and (4.14). Also, by (2.13), $\lim_{n\to\infty}(\log\log n)/\psi^2(n) = 0$

implies that $\lim_{n\to\infty}\{\nu_1(n)/\psi^2(n)\} = \frac{1}{2}$, so that (2.17) follows from (2.16). Q.E.D.

5. A few applications and concluding remarks. (i) The law of iterated logarithm. We let $\psi^2(t) = 2(1+\varepsilon)\log\log t$ for $t\geq 3$; $\psi^2(t) = 1$, $0\leq t<3$, where $\varepsilon>0$. Then, for large n, $\nu_1(n) = \varepsilon \log\log n - O(\log\log\log n)\to\infty$ as $n\to\infty$, so that by (2.16),

$$\text{(5.1)} \qquad \lim_{n\to\infty} P\{m^{\frac{1}{2}}(Z_m-\theta) > \sigma\sqrt{2(1+\varepsilon)\log\log m} \text{ for some } m\geq n\} = 0, \ \forall\ \varepsilon>0.$$

On the other hand, $Z_n \geq L(x_o, F_n(x_o))$, $\forall\ n\geq 1$, with probability 1, so that for every $\varepsilon>0$,

$$\text{(5.2)} \qquad \begin{aligned} &P\{m^{\frac{1}{2}}(Z_m-\theta) > \sigma\sqrt{2(1-\varepsilon)\log\log m} \text{ for some } m\geq n\} \\ &\geq P\{Z_m^* > \sqrt{2(1-\varepsilon)\log\log m} \text{ for some } m\geq n\} \\ &\to 1, \text{ as } n\to\infty, \end{aligned}$$

where the last step follows from (i) (2.6)-(2.8) and Lemma 3.1 implying that for every $u\varepsilon(0,1)$, $L_{01}(x_o, uF_n(x_o) + (1-u)F(x_o)) \to \xi_o(\neq 0)$ a.s., as $n\to\infty$, and (ii) $\{\sqrt{m}|F_m(x_o)-F(x_o)|\}$ is attracted by the usual law of iterated logarithm. Hence, by (5.1) and (5.2), we have on letting $\varepsilon(>0)$ to be arbitrarily small,

$$\text{(5.3)} \qquad P\{\limsup_n n^{\frac{1}{2}}(Z_n-\theta)/[2\sigma^2\log\log n]^{\frac{1}{2}} = 1\} = 1.$$

Similarly, it can be shown that

$$\text{(5.4)} \qquad P\{\liminf_n n^{\frac{1}{2}}(Z_n-\theta)/[2\sigma^2\log\log n]^{\frac{1}{2}} = -1\} = 1.$$

Thus, $\{Z_n\}$ obeys the law of iterated logarithm.

(ii) Probability of moderate deviations. Here, we let $\psi^2(n) = c^2\log n$, for some $0<c<\infty$. Then, by (2.17), we obtain that

$$\text{(5.5)}\qquad \lim_{n\to\infty}[(\log n)^{-1}\log P\{m^{\frac{1}{2}}(Z_m-\theta)/\sigma > c\log m \text{ for some } m\geq n\}] = -\tfrac{1}{2}c^2,$$

which is stronger version of the usual result

$$\text{(5.6)}\qquad \lim_{n\to\infty}[(\log n)^{-1}\log P\{n^{\frac{1}{2}}(Z_n-\theta)/\sigma > c\log n\}] = -\tfrac{1}{2}c^2,$$

and the latter is classically known as the probability of moderate deviation.

(iii) A. s. convergence to Wiener processes. Let us now impose another condition on $L(x,y)$, namely, that the first (partial) derivative $L_{01}(x,y)$ satisfies a local Lipschitz condition for all (x,y): $|x-x_o|<\delta_1$, $|y-F(x_o)|<\delta_2$, where δ_1, δ_2 are sufficiently small. Specifically, we assume that

$$\text{(5.7)}\qquad |L_{01}(x,y)-L_{01}(x_o,p_o)|\leq K_1|x-x_o|^{d_1}+K_2|y-p_o|^{d_2},\ \forall\ |x-x_o|<\delta_1, |y-p_o|<\delta_2,$$

where d_1 and d_2 are positive numbers. Then, by virtue of Lemma 3.1, Lemma 3.2, (3.16), (3.17) and (5.7), we conclude that there exists a $d>0$, such that for n sufficiently large, the second term on the right hand side of (3.19) is

$$\text{(5.8)}\qquad O(n^{-d}\log n) \text{ almost surely,}$$

and by Lemma 1 of Bahadur (1966) along with Theorem 3.2 of Sen (1973a), the first term is

$$\text{(5.9)}\qquad O(n^{-(4k_2)^{-1}}(\log n)) \text{ almost surely, as } n\to\infty.$$

So that, under (5.7), for some $d(\leq (4k_2)^{-1})$, by (3.17) and (3.19),

$$\text{(5.10)}\qquad G_n^* = O(n^{-d}(\log n)) = o(1) \text{ almost surely, as } n\to\infty.$$

As a result, from (4.12), (3.2), (3.17) and (5.10),

$$\text{(5.11)}\qquad \sqrt{n}(Z_n-\theta)/\sigma = Z_n^* + O(n^{-d}\log n) \text{ a.s., as } n\to\infty.$$

Let now $W=\{W(t): 0\leq t<\infty\}$ be a standard Wiener process on $[0,\infty)$. Then, using well-known results on $\{\sqrt{n}[F_n(x_o)-F(x_o)], n\geq 1\}$ along with the a.s. convergence of $L_{01}(x_o, uF_n(x_o) + (1-u)F(x_o))$ to $\xi_0(\neq 0)$, for $0<u<1$, we claim that as $n\to\infty$,

$$\sqrt{n}\ Z_n^* = W(n) + O(n^a \log n) \text{ a.s., for some } 0<a<\tfrac{1}{2}. \tag{5.12}$$

From (5.11) and (5.12), we claim that as $n\to\infty$

$$\sqrt{n}\ (Z_n-\theta)/\sigma = n^{-\frac{1}{2}}W(n) + O(n^{-d} \log n) \text{ a.s.,} \tag{5.13}$$

for some $d>0$. Thus, the main results in Sen (1973a,c), deduced for the particular case of bundle strength of filaments, also hold for general $L(x,F(x))$, $x\varepsilon A$, provided (5.7) holds.

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