*The paper was written while second author was on leave from Leningrad State University, Leningrad.

AN EXAMPLE OF SINGULAR STATISTICAL EXPERIMENTS ADMITTING LOCAL EXPONENTIAL APPROXIMATION

by

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SUMMARY. In this paper the authors study the asymptotic minimax properties of statistical estimates constructed from independent observations with a density having jumps to zero.

1. Introduction. Statement of problem. Conditions.

Consider a sequence of indentically distributed independent random observations

$$X_1, X_2, \ldots$$

with common distribution P_θ depending on unknown parameter θ. We shall suppose that X_j are real random variables and that the distributions P_θ are continuous with respect to Lebesgue measure. We denote by $f(x;\theta)$ the density of P_θ relative to Lebesgue measure.

In the "regular case" of smooth dependent of $f(x;\theta)$ on θ the joint density $\prod_1^n f(X_j;\theta)$, after proper normalization, may be approximated when

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$n \to \infty$ by normal family (see [1], Appendix). This idea of local asymptotic normality was developed in important papers [2], [3] of L. LeCam. It was brilliantly explored in J. Hajek's paper [1].

In this paper we wish to investigate a singular case of a discontinuous density f, when however the joint density $\prod_1^n f(X_j;\theta)$ admits the approximation by a nonnormal but also by a simple exponential family. Such a more general concept of local asymptotic exponentiality was also introduced by LeCam. See, in particular, the Example 3 of part 6 of [4] which is very close to the theme of this paper.

We now give precise statements of the restrictions which will be imposed on the density function $f(x;\theta)$.

I. The function $f(x;\theta)$ is defined and measurable on the closed rectangle $F' \times \Theta^c$ where the parametric set Θ is an open interval on the real line. For any points $\theta, \theta' \in \Theta$, $\theta \neq \theta'$,

$$\int_{-\infty}^{\infty} |f(x;\theta) - f(x;\theta')| \, dx > 0 .$$

II. The function $f(x;\theta)$ is absolutely continuous in θ for fixed x in each of regions $x < x_1(\theta)$, $x_1(\theta) < x < x_2(\theta)$, ... $x > x_2(\theta)$, where $x_1(\theta)_1$ ... $x_2(\theta)$ are monotone differentialbe pairwise nonintersecting curves defined for $\theta \in \Theta^c$ and $0 < |x_j'(\theta)| < \infty$, $\text{sign } x_j'(\theta) = \text{sign } x_k'(\theta)$.

III. The following limits exist uniformly in θ for θ in any compact subset of Θ :

$$\lim_{x \downarrow x_k(\theta)} f(x;\theta) = p_k(\theta) , \quad \lim_{x \uparrow x_k(\theta)} f(x;\theta) = q_k(\theta), \quad k = 1,\ldots r ,$$

where the functions $p_k(\theta)$, $q_k(\theta)$ are continuous on Θ and

$$\sum |p_k(\theta) - q_k(\theta)| > 0 .$$

IV. Let $f'(x;\theta)$ denote the derivative of the absolutely continuous component of $f(x;\theta)$. The integrals $\int_{-\infty}^{\infty} f'(x;\theta)dx$, $\int_{-\infty}^{\infty} |f'(x;\theta)|dx$ are continuous function of θ.

The asymptotic behavior of the estimates of the parameter θ under the conditions stated was investigated in our paper [5]. Here we add one more condition

V. Either all $q_k(\theta) \equiv 0$ or all $p_k(\theta) \equiv 0$.

The exponential distribution $\exp\{-(x-\theta)\}$, $x > \theta$, gives an example of density which satisfyies all conditions I - V. The uniform distribution on the interval $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$ satisfyies the conditions I - IV but not V.

We prove in part 2 that under the conditions I - V the normalized likelihood ratio

$$Z_n(u) = \prod_1^n f(X_j;\theta + u/n)/\prod_1^n f(X_j;\theta)$$

admits the good approximation by an exponential family (see Th. 2.1).

In part 3 we investigate general statistical experiments which admit the approximation by one sided exponential families. Our results are analogous to those which Hájek discovered for locally asymptotic normal families (see [1], [8]).

In part 4 we return to the case of independent identically distributed observations and establish some asymptotic minimax properties of Bayes estimates among all sequential estimates $[\{t_n\}, \sigma]$ with $E_\theta \sigma \leq n$.

Here and below $P_\theta\{\cdot\}$ and $E_\theta\{\cdot\}$ denote probability and expectation generated by all sequence of observations when θ is the true value of the parameter.

We use also below the following notation. If A is an event, then $\chi(A)$ always denotes the indicator of A. For example, if ξ is a random variable,

$$\chi\{\xi > x\} = \begin{cases} 1, & \xi > x \\ 0, & \xi \le x . \end{cases}$$

2. Asymptotic exponentiality of normalized likelihood ratio.

Define random function $Z_n(u)$ by

$$Z_n(u) = \prod_1^n \frac{f(X_j;\theta+u/n)}{f(X_j;\theta)} ,$$

where θ is a "true" value of parameter. We investigate here the asymptotic behavior of $Z_n(u)$ when $n \to \infty$. We will consider below only the case sign $x'_k(\theta) > 0$, $q_k(\theta) \equiv 0$. Three other cases reduce to this one by mappings $(x,\theta) \to (x,-\theta)$, $(x,\theta) \to (-x,-\theta)$.

Define random moments τ_n as

$$\tau_n = \inf\{u\colon x_k(\theta+u/n) > X_j > x_k(\theta) \text{ for some } k = 1,\dots r,\ j = 1,\dots n\} .$$

Let further $p(\theta) = p = \sum x'_k(\theta)\, p_k(\theta)$ and let the random function

$$\tilde{Z}_n(u) = \begin{cases} e^{pu}, & u < \tau_n , \\ 0 & u > \tau_n . \end{cases}$$

Theorem 2.1. Assume that conditions I to V are satisfied (and sign $x'_k > 0$, $q_k = 0$). Then the following relations hold

$$\lim_{n\to\infty} P\{\tau_n > u\} = \begin{cases} e^{-pu}, & u \ge 0 \\ 1 , & u \le 0 \end{cases} \tag{2.1}$$

(2.2) $$Z_n(u) - \tilde{Z}_n(u) \xrightarrow[n\to\infty]{} 0 \quad \text{in probability ;}$$

and moreover

(2.3) $$\sup_{|u|\le H} E_\theta |Z_n(u) - \tilde{Z}_n(u)| \underset{n\to\infty}{\longrightarrow} 0 \,, \quad 0 < H < \infty$$

Proof. The (2.1) follows from the Poisson approximation of binomial distribution. The proof of (2.3) is broken down into a few lemmas.

Lemma 2.1. If $\{\xi_n\}$ is a sequence of non-negative random variables converging in distribution to a random variable ξ then

(i) $\liminf_n E|\xi_n| \ge E|\xi|$

(ii) $E|\xi_n| \to E|\xi| < \infty \Longleftrightarrow |\xi_n|$ are uniformly integrable

For proof see [6], p. 183. Define now the random processes $\eta_j(u)$ and random variables ℓ_j, T_j by

$$\eta_j(u) = \frac{1}{u}\left[\frac{f(X_j;\ \theta+u)}{f(X_j;\theta)} - 1\right], \quad \ell_j = \frac{f'(X_j;\ \theta)}{f(X_j;\theta)} \,,$$

$$T_j = \inf\{v:\ x_k(\theta+v) > X_j > x_k(\theta) \text{ for some } k = 1,\dots r\} \,.$$

Evidently, $\tau_n = n \min(T_1,\dots T_n)$.

Lemma 2.2. Under conditions I - V

(2.4) $$\lim_{u\to 0} E_\theta |\chi(T_j > u)\, \eta_j(u) - \ell_j| = 0$$

Proof. Evidently, $\lim_{u\to 0} \chi(T_j > u)\, \eta_j(u) = \ell_j$ with P_θ-probability 1. Further,

$$E_\theta \chi(T_j > u)\, \eta_j(u) \le E_\theta\{\chi(T_j > u)\, \frac{1}{u}\int_\theta^{\theta+u} |f_z'(X_j;z)|\, dz \le$$

$$\le \frac{1}{u}\int_\theta^{\theta+u} dz \int_{-\infty}^{\infty} |f'(x;z)|dx + \int_{-\infty}^{\infty} |f'(x;\theta)|dx = E|\ell_j| \,.$$

Because of Lemma 2.1 (i), it follows that

$$\lim_{u} E_\theta\{\chi(T_j > u)|\eta_j(u)|\} = \int |f'(x;\theta)|dx = E_\theta|\ell_j| .$$

According to (ii) of Lemma 2.1, the $\chi(T_j > u)|\eta_j(u)|$ are uniformly integrable and therefore

$$E_\theta|\chi(T_j > u)\eta_j(u) - \ell_j| \to 0 .$$

Lemma 2.3. If the conditions I - V are satisfied, then for every $\delta > 0$

$$P_\theta\{T_j > u,\ |\eta_j(u)| > \delta/|u|\} = o(|u|) ,\quad u \to 0 ,$$

$$P_{\theta+u}\{T_j > u,\ |\eta_j(u)| > \delta/|u|, f(X_j;\theta) \neq 0\} = o(|u|) ,\quad u \to 0 .$$

Proof. Since (2.4), it follows that

$$P_\theta\{T_j > u,\ |\eta_j(u)| > \delta/|u|\} \le P_\theta\{T_j > u,\ |\eta_j(u) - \ell_j| > \delta/2|u|\} +$$
$$+ P_\theta\{|\ell_j| > \delta/2|u|\} \le \frac{2|u|}{\delta} E_\theta\{\chi(T_j > u)\ |\eta_j(u) - \ell_j|\} +$$
$$+ \frac{2|u|}{\delta} E_\theta\{\chi(|\ell_j| > \delta/2|u|)\ |\ell_j|\} = o(|u|) .$$

As for the second assertion, if $|f(x;\theta+u)/f(x;\theta) - 1| > \delta$ then

$$f(x;\theta+u) < \frac{1}{\delta_1} |f(x;\theta+u) - f(x;\theta)|,\ \delta_1 > \min(\delta/2,\ 1/2) .$$

Hence

$$P_{\theta+u}\{T_j > u,\ |\eta_j(u)| > \delta/|u|,\ f(X_j;\theta) \neq 0\} \le$$
$$\le \frac{|u|}{\delta_1} E_\theta\{\chi(T_j > u,\ |\eta_j(u)| > \delta/|u|)\ |\eta_j(u)|\} \le$$
$$\le \frac{|u|}{\delta_1} E_\theta\{\chi(T_j > u)\ |\eta_j(u) - \ell_j|\} +$$
$$+ \frac{|u|}{\delta_1} E_\theta\{\chi(|\eta_j(u)| > \delta/|u|)\ |\ell_j|\} = o(|u|) .$$

Lemma 2.4. Let the conditions I - V are fulfilled, then

$$E_\theta \ell_j = \int_{-\infty}^{\infty} f'(x;\theta)dx = \sum_1^r x_k'(\theta)\, p_k(\theta) = p .$$

Proof. The direct calculation.

The next two lemmas give the necessary estimates of the deviation of $Z_n(u)$ from $\tilde{Z}_n(u)$.

Lemma 2.5. Under conditions I - V for every $0 < H < \infty$

$$\sup_{|u|\le H} E_\theta\{|Z_n(u) - e^{pu}|\ \chi(\tau_n > u)\} \to 0,\ n \to 0 . \tag{2.6}$$

Proof. Fix a small positive number $\delta < \frac{1}{2}$. If all differences

$$\left|\frac{f(X_j;\theta+u/n)}{f(X_j;\theta)} - 1\right| < \infty,\ j = 1,2,\ldots n ,$$

then

$$|\ell n\ Z_n(u)\ pu| \le \frac{|u|}{n} |\sum_1^n(\ell_j p)| + \frac{|u|}{n}\sum_1^n |\eta_j(u/n) - \ell_j| +$$
$$+ 2\delta \frac{|u|}{n}\sum_1^n |\eta_j(u/n)| .$$

Hence, uniformly in $|u| \le H$.

$$E_\theta\{|Z_n(u) - e^{pu}|\ \chi(\tau_n > u,\ \bigcap_1^n\{|\frac{f(X_j;\theta+u/n)}{f(X_j;\theta)} - 1| < \delta\}\} \le$$
$$\le |u|e^{pu}\ E_\theta\{\frac{1}{n}\ |\sum_1^n(\ell_j - E_\theta\ell_j)|\} + \tag{2.7}$$
$$+ |u|\ E_\theta\{|\eta_1(u/n) - \ell_1|\ \chi(T_1 > u/n)\} + 2\delta|u|\ E_\theta|\eta_1(u/n)| \le$$
$$\le B\delta + O(1) .$$

We used here the law of large numbers to estimate the first righthand term, and lemma 2.2 to estimate the second and third ones.

Further, by (2.5)

$$E_\theta\{|Z_n(u) - e^{pu}|\ \chi(\tau_n > u,\ \bigcup_1^n\{|\frac{f(X_j;\theta+u/n)}{f(X_j;\theta)} - 1| > \delta\}\} \le$$

$$\le n\, P_{\theta+u/n}\{|\eta_1(u/n)| > \delta n/|u|\ ,\ T_1 > u/n\ ,\ f(X_1;\theta) \neq 0\} +$$

$$+ e^{-pu}\, n\, P_\theta\{|\eta_1(u/n)| > \delta n/|u|\ ,\ T_1 > u/n\} = o(1)\ ,\ n \to \infty \tag{2.8}$$

uniformly in $|u| \le H$. The inequalities (2.7), (2.8) give us the desired result (2.6).

Lemma 2.6. Under the conditions I - V

$$\sup_{|u|\le H} E_\theta\{|Z_n(u) - \tilde{Z}_n(u)|\ \chi(\tau_n < u)\} = o(1),\ n \to \infty\ .$$

Proof. We have

$$E_\theta\{|Z_n(u) - \tilde{Z}_n(u)|\ \chi(\tau_n < u)\} = E_\theta\{Z_n(u)\ \chi(\tau_n < u)\} \le P_{\theta+u/n}\{\tau_n < u\}\ .$$

By definition of τ_n the last probability is

$$P_{\theta+u/n}\{\bigcup_{j=1}^n \bigcup_{k=1}^r \{x_k(\theta + u/n) > X_j > x_k(\theta)\}\} \le$$

$$\le n \sum_{k=1}^r \int_{x_k(\theta)}^{x_k(\theta+u/n)} f(x;\theta+u/n)dx$$

Then since $\lim f(x;\theta+u/n) = 0\ ,\ \ x \uparrow x_k(\theta+u/n)$ uniformly in u (conditions III and IV)

$$\int_{x_k(\theta)}^{x_k(\theta+u/n)} f(x;\theta+u/n)dx \le o(1)\ |x_k(\theta + u/n) - x_k(\theta)| = o(1/n)\ .$$

This last inequality proves the lemma. The assertions (2.2) and (2.3) are evident consequences of lemmas 2.5, 2.6 and theorem 1 is proved.

3. Locally asymptotically one sided exponential families of distributions. Limiting distributions of regular estimates; local asymptotic minimax property of Bayesian estimates.

Let us consider (following Hájek [1]) a sequence of statistical experiments $\{X^n, A^n, P^n_\theta\}$, $n \geq 1$, where θ runs through an open set $\Theta \subseteq R^1$. Let $\alpha(n) \uparrow \infty$ be a sequence of normalizing factors. Take a point $\theta \in \Theta$ and assume it to be the true value of parameter. Denote $Z_n(u)$ the Radon-Nikodym derivative of the absolutely continuos part of $P^n_{\theta+u/\alpha(n)}$ with respect to P^n_θ. For the sake of brevity we shall write below P_θ instead P^n_θ and $P_{\theta,u}$ instead $P^n_{\theta+u/\alpha(n)}$. Note that the definition $Z_n(u)$ is just the same as in Section 2.

Assumption 3.1. Assume that either

$$(3.1) \qquad Z_n(u) = \begin{cases} e^{pu} + o_p(1), & u < \tau_n \\ o_p(1), & u > \tau_n \end{cases}$$

or

$$(3.2) \qquad Z_n(u) = \begin{cases} e^{-pu} + o_p(1), & u > -\varepsilon_n \\ o_p(1), & u < -\varepsilon_n \end{cases}$$

where p is a positive number, the positive random variables τ_n are A^n measurable and satisfy $\lim_n P_\theta\{\tau_n > u\} = e^{-pu}$, and $o_p(1)$ denotes random variables which go to zero in probability when $n \to \infty$.

Theorem 2.1 of Section 2 gives us an example of statistical experiments satisfying Assumption 3.1. We shall consider below (without special warning) the case (3.1) of Assumption 3.1 only. Denote

$$\tilde{Z}_n(u) = \begin{cases} e^{pu}, & u < \tau_n \\ 0, & u > \tau_n . \end{cases}$$

Define now the measure $\tilde{P}_{\theta,u}$ on A^n by the formula

$$\tilde{P}_{\theta,u}(A) = \int_A \tilde{Z}_n(u)\, dP_\theta \,, \quad A \in A^n \,.$$

Theorem 3.1. Under Assumption 3.1 for every $u > 0$

$$(3.3) \qquad \int_{X^n} |dP_{\theta,u} - d\tilde{P}_{\theta,u}| \to 0 \,, \quad n \to \infty \,.$$

Proof. For every fixed u both sequences $\{Z_n(u)\}$, $\{\tilde{Z}_n(u)\}$ tends in distribution to the random variable

$$Z(u) = \begin{cases} e^{pu} \,, & u < \tau \,, \\ 0 \,, & u > \tau \,, \end{cases}$$

where $P\{\tau > y\} = e^{-py}$. Futher, for $u \geq 0$

$$\lim E_\theta\, Z_n(u) = \lim E_\theta\, \tilde{Z}_n(u) = 1 = E_\theta\, Z(u) \,,$$

and by Lemma 2.1 the random variables $Z_n(u)$, $\tilde{Z}_n(u)$ are uniformly integrable with respect to P_θ, so that

$$(3.4) \qquad E_\theta |Z_n(u) - \tilde{Z}_n(u)| \to 0 \,, \quad n \to \infty \,.$$

Next, let $P^{(r)}_{\theta,u}$, $P^{(s)}_{\theta,u}$ denote the regular and the singular parts of $P_{\theta,u}$ with respect to P_θ. Then for $u > 0$

$$P^{(s)}_{\theta,u}\{X^n\} \leq 1 - P^{(r)}_{\theta,u}\{\tau_n > u\} = 1 - E_\theta\{Z_n(u)\cdot\chi(\tau_n > u\} \to 0 \,, \quad n \to \infty \,.$$

Therefore,

$$\int_{X^n} |dP_{\theta,u} - d\tilde{P}_{\theta,u}| \leq E_\theta |Z_n(u) - \tilde{Z}_n(u)| + P^{(s)}_{\theta,u}\{X^n\} \to 0 \,, \quad n \to \infty \,.$$

Theorem 3.2. Let us assume that (3.1) holds. Consider a sequence $\{t_n\}$ of estimates of parameter θ satisfying

(3.5) $P_{\theta,u}\{\alpha(n)(t_n - \theta - u/\alpha(n)) < y\} \to F(y)$ for every $u \in R'$

in continuity points of some distribution function $F(y)$, $y \in R'$.

Then we have

$$F = H_p * G ,$$

where H_p is the exponential distribution on $(0,\infty)$ with parameter p , i.e. $1 - H_p(y) = e^{-py}$, $y > 0$, and G is a certain distribution function in R^1.

Proof. Let $f(s)$ denotes the characteristic function of F and rewrite (3.5) in the form

$$E_{\theta,u}\{\exp\{is\zeta_n - isu\}\} \to f(s) ,$$

where $\zeta_n = \alpha(n)(t_n - \theta)$. We have from this and (3.3)

$$f(s) = e^{pu + ius} \int_{X^n} e^{is\zeta_n} \chi(\tau_n > u)\, dP_\theta + o(1) , \quad n \to \infty .$$

Multiplying both part of the last equality on $e^{-\lambda u}$, $\lambda = -\mu + iv$, $\mu > 0$, and integrating relative to u, we find

$$\frac{f(s)}{p+\lambda-is} = \frac{1}{\lambda}\int_{X^n} e^{is\zeta_n} (1 - e^{-\lambda t_n}) dP_\theta + o(1) =$$

$$= \frac{f(s)}{\lambda} - \frac{1}{\lambda}\int_{X^n} e^{is\zeta_n - \lambda t_n}\, dP_\theta + o(1) .$$

Now we can choose $v = s$, $\mu = 0$ and then we have

$$f(s) = \frac{p}{is-p} \lim \int_{X^n} e^{is(\zeta_n - t_n)}\, dP_\theta = \frac{p}{is-p}\int_{-\infty}^{\infty} e^{isy}\, dG(y) = \frac{p}{is-p}\cdot g(s) ,$$

where $g(s)$ is the characteristic function of G. Because $p/is-p$ is the characteristic function of H_p the theorem is proved.

Suppose that the loss resulting from replacement of the true vaule of the parameter θ by its estimate t_n is assumed to be $w(\alpha(n)(t_n-\theta))$, when $w(\cdot)$ is a given function. The mean loss, the reisk function, is then

(3.6) $$E_\theta^{(n)} \, w(\alpha(n)(t_n - \theta)) \, .$$

We shall assume below that the function w in (3.6) satisfies the following condition:

(3.7) $$w(y) = w(|y|) \; ; \; w(y) \uparrow \, , \; y \geq 0 \; ; \; w(y) \geq 0 \; , \; w \not\equiv \text{const.}$$

(3.8) $$\int_0^\infty w(y) \, e^{-\varepsilon y} \, dy < \infty \, , \quad \varepsilon > 0 \, .$$

Theorem 3.3. Under assumption 3.1, any sequence of estimates $\{t_n\}$ for θ satisfies (for the w of (3.7), (3.8))

(3.9) $$\lim_{\delta \to 0} \liminf_{n \to \infty} \sup_{|\theta - u| < \delta} E^{(n)}\{w(\alpha(n)(t_n - u))\} \geq \min_u P \int_{-\infty}^{0} w(y-u) \, e^{py} \, dy \, .$$

Proof. Denote trough $w_a(y)$ a truncated version of w:

$$w_a(y) = \min\{w(y), a\} \, .$$

We have for any $b < \delta \cdot \alpha(n)$

$$\sup_{|u-\theta|<\delta} E_u^{(n)}\{w\{\alpha(n)(t_n - u)\}\} \leq \frac{\alpha(n)}{b} \int_\theta^{\theta + b/\alpha(n)} E_u^{(n)}\{w(\alpha(n)(t_n - u))\} \, du =$$

$$= \frac{1}{b} \int_0^b E_{\theta,u} \, \{w(\zeta_n - u)\} du \, ,$$

where we denote $\alpha(n)(t_n - \theta) = \zeta_n$. Further, by Theorem 3.1 for $n \to \infty$

$$\frac{1}{b} \int_0^b E_{\theta,u} \, \{w(\zeta_n - u)\} du \geq \frac{1}{b} \int_0^b E_{\theta,u} \, \{w_a(\zeta_n - u)\} du =$$

$$= \frac{1}{b} E_\theta \int_0^b w_a(\zeta_n - u) \, \tilde{Z}_n(u) du + o(1) =$$

$$= \frac{1}{b} E_\theta \int_0^{\min(b, \tau_n)} w_a(\zeta_n - u) e^{pu} du + o(1) \geq$$

$$\geq \frac{1}{b} E_\theta\{\chi(\tau_n<b) \int_{-\infty}^{\tau_n} w_a(\zeta_n-u)e^{pu}du\} -$$

$$- a/bp + 0(1) \geq \min_y \int_{-\infty}^{0} w_a(y - u)e^{pu}du \cdot$$

$$\cdot \frac{1}{b} E_\theta\{\chi(\tau_n<b)e^{p\tau_n}\} - a/bp + o(1) =$$

$$= \min_y p \int_{-\infty}^{0} w_a(y-u)e^{pu}du - a/bp + o(1) .$$

Hence, putting here $b = a^2$ we have

$$\liminf_{n\to\infty} \sup_{|\theta-u|<\delta} E_u^{(n)}\{w(\alpha(n)(t_n-u))\} \geq$$

$$\geq \min_y p \int_{-\infty}^{0} w_a(y-u)e^{pu}du - 1/ap .$$

Finally, under conditions (3.7), (3.8)

$$\lim_{a\to\infty} \min_y p \int_{-\infty}^{0} w_a(y-u)d^{pu}du = \min_y \int_{-\infty}^{0} w(y-a)e^{pu}du$$

and Theorem 3.3 is proved.

Return to the case of independent identically distributed observations satisfying conditions I - V. We have noted already that under conditions I - V the assumption 3.1 is always fulfilled with $\alpha(n) = n$. Define now the sequence of estimates $\hat{t}_n(w) = \hat{t}_n$ in such a way that

$$\int w(\hat{t}_n-u) \prod_1^n f(X_j;u)du = \min_y \int w(y-u) \prod_1^n f(X_j;u)du$$

It is easy to deduce from [5] that under a wide conditions upon f and w (but a little more restrictive that I - V and (3.7), (3.8)) the estimates $\hat{t}_n$ have the following properties:

(i) the limit distribution of $n(\hat{t}_n-\theta)$ coincides with the distribution of $\tau + y_w$ when y_w is the point of minimum of $p\int_{-\infty}^{0} w(y-u)e^{pu}du$ and $P\{\tau>y\} = e^{-py}$, $y > 0$.

(ii)
$$\lim_n E_\theta w(n(t_n-\theta)) = p \int_{-\infty}^{0} w(y_w-u)e^{pu}du .$$

So from the point of view of Theorem 3.2, 3.3 the estimates $\hat{t}_n(w)$ are asymptotically "good" estimates with respect to the loss function w.

4. The case of sequential estimation.

We continue to consider the sequence $\{X_j\}$ of independent identically distributed observations satisfying the conditions I - V. Assume we are given: 1) a stopping time σ; 2) a sequence of statistics $\{t_n\}$, $t_n = t_n(X_1,\dots X_n)$. As an estimate of parameter θ we use the random variable $t_\sigma(X_1,\dots X_\sigma)$. We call a pair $d = [\{t_n\}, \sigma]$ a sequential estimation plan. We want to prove here that from the point of view of Theorem 3.3 sequential schemes are not better than fixed sample schemes.

Theorem 4.1. Denote $\mathcal{D}_n$ the collection of all sequential plans $d = [\{t_n\}, \sigma]$ with $E_\theta\sigma \le n$, $\theta \in \Theta$. Under the conditions I - V

$$q_a = \lim_{\delta\to 0} \liminf_{n\to\infty} \inf_{\mathcal{D}_n} \sup_{|\theta-u|\le\delta} n^a E_u|t_\sigma - u|^a \ge \ge \inf_y p\cdot\int_{-\infty}^{0} |y-u|^a e^{pu}du , \quad a > 0 . \tag{4.1}$$

Proof. We will at first show that we need consider only such plans $d \in \mathcal{D}_n$ for which

$$P\{\sigma \ge \varepsilon n\} = 1 , \quad \varepsilon > 0 . \tag{4.2}$$

Lemma 4.1. For any $\alpha > 0$ there exist a positive number $\varepsilon = \varepsilon(\alpha) > 0$ and a sequence of plans $d_n = [\{t_k^{(n)}\}, \sigma(n)]$ such that $P_u\{\sigma_n \ge \varepsilon n\} = 1$, $u \in \theta$ and

$$\lim_{\delta} \lim_{n} \sup_{|\theta-u|\le\delta} n^a E_u|t_{\delta(n)}^{(n)} - u|^a \le q_a + \alpha .$$

The proof of Lemma 4.1 coincides exactly with the proof of Lemma 2.5 of [7] and we omit it.

Lemma 4.2. The following relation holds

$$(4.2)\qquad E_\theta\{\tilde{Z}_n(u)|X_1,\ldots X_k\} = \tilde{Z}_k(u\cdot\frac{k}{n})(1+\rho(k,n,u,)),\ k\le n\ ,$$

where

$$\sup_{0<u<H}\ \max_{1\le k\le n} |\rho(k,n,u)| \to 0,\ n\to\infty\ .$$

Proof. By definition of $\tilde{Z}_n(u)$,

$$E_\theta\{Z_n(u)|X_1,\ldots X_k\} = e^{pu}\ E_\theta\{\chi(\tau_n>u)|X_1,\ldots X_k\} =$$

$$= e^{pu}\prod_1^k \chi(T_j>u/n)\prod_{k_H}^n E_\theta\chi(T_j>u/n) =$$

$$= e^{pu}\ \chi(\tau_k>uk/n)\ (1 - P\{\bigcup_1^r\{x_k(\theta+u/n) > X_j > x_k(\theta)\}\})^{n-k} =$$

$$= \tilde{Z}_k(uk/n)(1+\rho(k,n,u))\ .$$

The lemma is proved. Let now $d\in\mathcal{D}_n$. We may (and will) suppose by lemma 4.1 that d satisfies (4.2). Let us fix also two positive numbers $N>\varepsilon$ and $b>N$. We specify the choice of the numbers ε, N, b later. Using Lemma 4.2, we obtain after a slight modification of the first half of (3.10).

$$n^a \sup_{|\theta-u|\le\delta} E_u|t_\sigma-u|^a \ge \frac{1}{b}\int_0^b E_{\theta+u/Nn}|n(t_\sigma-\theta) - u/N|^a du \ge$$

$$\ge \frac{1}{b}E_\theta\sum_{\varepsilon n}^{N_n}\chi(\sigma=k)\int_0^b |n(t_\sigma-\theta) - u/N|^a\ \tilde{Z}_{nN}(u)du +$$

$$(4.3)\qquad + 0(1) = \frac{1}{b}E_\theta\sum_{\varepsilon n}^{Nn}\chi(\sigma=k)\int_0^b |n(t_\sigma-\theta) - u/N|^a\ \cdot$$

$$\cdot\ \tilde{Z}_k(u\frac{k}{nN})du + 0(1) \ge \min_x p\int_{-\infty}^0 |x-u|e^{pu}du\ \frac{N}{pb}\sum_{k=n\varepsilon}^{Nn} E_\theta(\frac{n}{k})^{a+1}\chi(\sigma=k)\ \cdot$$

$$\chi(\tau_k < \frac{kb}{Nn})\ e^{p\tau_k} + o(1) + 0(1/b)\ .$$

Hence, to prove the theorem it is sufficient to show that for every $\alpha > 0$ it is possible to choose N, b, ε in such a way that

$$(4.4) \qquad \sum_{k=n\varepsilon}^{Nn} \left(\frac{n}{k}\right)^a \psi_k \geq 1 - \alpha, \; \psi_k = \frac{N}{pb} \cdot E_\theta \frac{n}{k} \chi(\sigma=k) \; \chi(\tau_k < \frac{kb}{Nn}) \; e^{p\tau_k} .$$

To estimate the left side of (4.4) we use the following results.

Lemma 4.3. For any $\alpha > 0$ there exists such a choice of numbers $0 < \varepsilon < N < b$ in (4.3) that

$$(4.5) \qquad \sum \psi_k \geq 1 - \alpha \quad \sum k\psi_k \leq n(1-\alpha)$$

Lemma 4.5. Let us suppose we are given positive numbers $\psi_k \geq 0$ which satisfy conditions (4.5) and let $g(u)$ be a convex decreasing function of $u > 0$. Then

$$(4.6) \qquad \sum g(k/n) \; \psi_k \geq (1-\alpha) \; g\left(\frac{1+\alpha}{1-\alpha}\right)$$

and hence $(g(u) = u^{-a})$

$$(4.7) \qquad \sum (n/k)^a \; \psi_k \geq (1-\alpha) \left(\frac{1+\alpha}{1-\alpha}\right)^a$$

The inequality (4.4) and hence the assertion of the theorem is a simple consequence of the relations (4.5) and (4.6). The last of them follows with ease from Jensen's inequality. Namely,

$$\sum g(k/n) \; \psi_k \geq (1-\alpha) \sum g(k/n) \cdot \frac{\psi_k}{\sum \psi_j} \geq$$

$$\geq (1-\alpha) \; g \; \left(\frac{1}{n} \sum \frac{k\psi_k}{\sum \psi_j}\right) \geq (1-\alpha) \; g \, . \left(\frac{1+\alpha}{1-\alpha}\right)$$

and (4.6) is proved.

To prove the first inequality (4.5) we define the number $\beta = b - \sqrt{b}$ and note that because of lemma 4.1 for all sufficiently large n

$$\sup_{0\le u\le\beta} P_{\theta+u/Nn}\{\tau_\sigma > b\sigma/Nn\} = \sup_{0\le u\le\beta} P_{\theta+u/Nn}\{\bigcap_1^\sigma\{T_j > b/Nn\}\} \le$$

$$\le \sup_{0\le u\le\beta} P_{\theta+u/Nn}\{\bigcap_1^{[\varepsilon n]}\{T_j > b/Nn\}\} =$$

$$= \sup_{0\le u\le\beta}(P_{\theta+u/Nn}\{T_1 > b/Nn\})^{[\varepsilon n]} \le \exp\{-p\varepsilon\sqrt{b}/2N\} .$$

Using this last inequaltiy and Tchebichev's inequaltiy, we obtain

$$\inf_{0\le u\le\beta} P_{\theta+u/Nn}\{\sigma \le Nn,\ \tau_\sigma \le b\sigma/Nn\} \ge 1 - \exp\{-p\varepsilon\sqrt{b}/2N\} - 1/N .$$

Applying once more the arguments we used to derive (4.3), we find

$$\inf_{0\le u\le\beta} P_{\theta+u/Nn}\{\sigma \le Nn,\ \tau_\sigma \le b\sigma/Nn\} \le \frac{1}{\beta}\int_0^\beta P_{\theta+u/Nn}\{\sigma \le Nn,\ \tau_\sigma \le b\sigma/Nn\}du =$$

$$= \frac{1}{\beta} E_\theta\{\sum_{\varepsilon n}^{Nn} \chi(\sigma = k)\ \chi(\tau_k \le bk/Nn) \int_0^\beta \tilde{Z}_{Nn}(u)du\} +$$

$$+ o(1) \le \frac{1}{\beta} E_\theta\{\sum_{\varepsilon n}^{Nn} \chi(\sigma = k)\ \chi(\tau_k \le bk/Nn) \cdot$$

$$\cdot \int_0^\beta \tilde{Z}_k(uk/Nn)du\} + 0(1) \le \frac{N}{p\beta}\sum_{\varepsilon n}^{Nn} \frac{n}{k} E_\theta\{\chi(\sigma = k) \cdot$$

$$\cdot\ \chi(\tau_k \le bk/Nn)e^{p\tau_k}\} + 0(1) .$$

Hence,

$$\sum \psi_k \ge (1-\exp\{-p\varepsilon\sqrt{b}/2N\} - 1/N)(1-1/\sqrt{b}) + o(1) . \tag{4.8}$$

The estimate of $\sum k\psi_k$ may be obtained in the same way. At first, since $E_{\theta+u/Nn}\ \sigma \le n$ we have

$$\sup_u E_{\theta+u/Nn}\{\sigma\cdot\chi(\tau_k \le bk/Nn)\} \le n .$$

Therefore

$$n \ge \frac{1}{b}\int_0^b E_{\theta+u/Nn}\{\sigma\ \chi(\tau_\sigma \le b\sigma/Nn)\}du \ge$$

$$\ge \frac{Nn}{bp}\sum_{\varepsilon n}^{Nn} E_\theta\{\chi(\tau_k \le bk/Nn)\cdot\chi(\sigma = k)e^{p\tau_k}\ u\}$$

$$- \frac{Nn}{bp}\sum_{\varepsilon p}^{Nn} E_\theta\{\chi(\tau_k \le b_k/Nn)\cdot\chi(\sigma = k)\} + o(1) \ge \sum k\psi_k - nN/bp + o(1)$$

and

$$\sum k\psi_k \le n(1 + N/bp) + o(1) .$$

The inequalities (4.8) and (4.9) prove (4.5) because we can simultaneously make $\exp\{-\varepsilon\sqrt{b}/2N\}$, N/b, $1/N$ as small as we want.

Remark. The theorem 4.1 is an analogue of the theorem 3.3 for the case $w(x) = |x|^a$, $a > 0$. In fact, we proved a little more. Indeed, for a function w satisfying (3.7), (3.8) define

$$g_w(\lambda) = g(\lambda) = \min_x \int_{-\infty}^{0} w(\lambda^{-1}(x-v))e^{pv}dv$$

Then, like (4.3),

$$\sup_{|\theta-u|\le\delta} E_u\, w(n^a(t_\sigma-u)) \ge \sum \psi_k\, g(k/n) .$$

If we suppose that $g(\lambda)$ is convex and continuous at $\lambda = 1$ then by lemma 4.5 we find for every $\alpha > 0$

$$\sum \psi_k\, g(k/n) \ge (1-\alpha)\, g(\tfrac{1+\alpha}{1-\alpha}) + o(1)$$

Hence, theorem 4.1 holds not only for $w(x) = |x|^a$ but also for all w for which $g(\lambda)$ is convex and continuous. It will be, for example, if

(i) $w(x)$ is convex and satisfies (3.7), (3.8);

(ii) $$w(x) = \begin{cases} 0, & |x| \le w_0 \\ 1, & |x| > w_0 . \end{cases}$$

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