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AN EXAMPLE OF SINGULAR STATISTICAL EXPERIMENTS  
ADMITTING LOCAL EXPONENTIAL APPROXIMATION

by

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SUMMARY. In this paper the authors study the asymptotic minimax properties of statistical estimates constructed from independent observations with a density having jumps to zero.

1. Introduction. Statement of problem. Conditions.

Consider a sequence of indentially distributed independent random observations

$$X_1, X_2, \dots$$

with common distribution  $P_\theta$  depending on unknown parameter  $\theta$ . We shall suppose that  $X_j$  are real random variables and that the distributions  $P_\theta$  are continuous with respect to Lebesgue measure. We denote by  $f(x;\theta)$  the density of  $P_\theta$  relative to Lebesgue measure.

In the "regular case" of smooth dependent of  $f(x;\theta)$  on  $\theta$  the joint density  $\prod_{j=1}^n f(X_j;\theta)$ , after proper normalization, may be approximated when

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<sup>1</sup>The paper was written when the second author was at the Department of Statistics, University of North Carolina at Chapel Hill.

$n \rightarrow \infty$  by normal family (see [1], Appendix). This idea of local asymptotic normality was developed in important papers [2], [3] of L. LeCam. It was brilliantly explored in J. Hajek's paper [1].

In this paper we wish to investigate a singular case of a discontinuous density  $f$ , when however the joint density  $\prod_{j=1}^n f(X_j; \theta)$  admits the approximation by a nonnormal but also by a simple exponential family. Such a more general concept of local asymptotic exponentiality was also introduced by LeCam. See, in particular, the Example 3 of part 6 of [4] which is very close to the theme of this paper.

We now give precise statements of the restrictions which will be imposed on the density function  $f(x; \theta)$ .

I. The function  $f(x; \theta)$  is defined and measurable on the closed rectangle  $F' \times \Theta^c$  where the parametric set  $\Theta$  is an open interval on the real line. For any points  $\theta, \theta' \in \Theta, \theta \neq \theta'$ ,

$$\int_{-\infty}^{\infty} |f(x; \theta) - f(x; \theta')| dx > 0 .$$

II. The function  $f(x; \theta)$  is absolutely continuous in  $\theta$  for fixed  $x$  in each of regions  $x < x_1(\theta), x_1(\theta) < x < x_2(\theta), \dots, x > x_2(\theta)$ , where  $x_1(\theta), \dots, x_2(\theta)$  are monotone differentiable pairwise nonintersecting curves defined for  $\theta \in \Theta^c$  and  $0 < |x_j'(\theta)| < \infty, \text{sign } x_j'(\theta) = \text{sign } x_k'(\theta)$ .

III. The following limits exist uniformly in  $\theta$  for  $\theta$  in any compact subset of  $\Theta$  :

$$\lim_{x \rightarrow x_k(\theta)} f(x; \theta) = p_k(\theta), \quad \lim_{x \rightarrow x_k(\theta)} f(x; \theta) = q_k(\theta), \quad k = 1, \dots, r,$$

where the functions  $p_k(\theta), q_k(\theta)$  are continuous on  $\Theta$  and

$$\sum |p_k(\theta) - q_k(\theta)| > 0 .$$

IV. Let  $f'(x; \theta)$  denote the derivative of the absolutely continuous component of  $f(x; \theta)$ . The integrals  $\int_{-\infty}^{\infty} f'(x; \theta) dx$ ,  $\int_{-\infty}^{\infty} |f'(x; \theta)| dx$  are continuous function of  $\theta$ .

The asymptotic behavior of the estimates of the parameter  $\theta$  under the conditions stated was investigated in our paper [5]. Here we add one more condition

V. Either all  $q_k(\theta) \equiv 0$  or all  $p_k(\theta) \equiv 0$ .

The exponential distribution  $\exp\{-(x-\theta)\}$ ,  $x > \theta$ , gives an example of density which satisfies all conditions I - V. The uniform distribution on the interval  $[\theta - \frac{1}{2}, \theta + \frac{1}{2}]$  satisfies the conditions I - IV but not V.

We prove in part 2 that under the conditions I - V the normalized likelihood ratio

$$Z_n(u) = \frac{\prod_1^n f(X_j; \theta + u/n)}{\prod_1^n f(X_j; \theta)}$$

admits the good approximation by an exponential family (see Th. 2.1).

In part 3 we investigate general statistical experiments which admit the approximation by one sided exponential families. Our results are analogous to those which Hájek discovered for locally asymptotic normal families (see [1], [8]).

In part 4 we return to the case of independent identically distributed observations and establish some asymptotic minimax properties of Bayes estimates among all sequential estimates  $[\{t_n\}, \sigma]$  with  $E_\theta \sigma \leq n$ .

Here and below  $P_\theta\{\cdot\}$  and  $E_\theta\{\cdot\}$  denote probability and expectation generated by all sequence of observations when  $\theta$  is the true value of the parameter.

We use also below the following notation. If  $A$  is an event, then  $\chi(A)$  always denotes the indicator of  $A$ . For example, if  $\xi$  is a random variable,

$$\chi\{\xi > x\} = \begin{cases} 1, & \xi > x \\ 0, & \xi \leq x. \end{cases}$$

## 2. Asymptotic exponentiality of normalized likelihood ratio.

Define random function  $Z_n(u)$  by

$$Z_n(u) = \frac{n \prod f(X_j; \theta + u/n)}{\prod f(X_j; \theta)},$$

where  $\theta$  is a "true" value of parameter. We investigate here the asymptotic behavior of  $Z_n(u)$  when  $n \rightarrow \infty$ . We will consider below only the case  $\text{sign } x'_k(\theta) > 0$ ,  $q_k(\theta) \equiv 0$ . Three other cases reduce to this one by mappings  $(x, \theta) \rightarrow (x, -\theta)$ ,  $(x, \theta) \rightarrow (-x, -\theta)$ .

Define random moments  $\tau_n$  as

$$\tau_n = \inf\{u: x_k(\theta + u/n) > x_j > x_k(\theta) \text{ for some } k = 1, \dots, r, j = 1, \dots, n\}.$$

Let further  $p(\theta) = p = \sum x'_k(\theta) p_k(\theta)$  and let the random function

$$\tilde{Z}_n(u) = \begin{cases} e^{pu}, & u < \tau_n, \\ 0 & u > \tau_n. \end{cases}$$

**Theorem 2.1.** Assume that conditions I to V are satisfied (and  $\text{sign } x'_k > 0$ ,  $q_k = 0$ ). Then the following relations hold

$$(2.1) \quad \lim_{n \rightarrow \infty} P\{\tau_n > u\} = \begin{cases} e^{-pu}, & u \geq 0 \\ 1, & u \leq 0 \end{cases}$$

$$(2.2) \quad Z_n(u) - \tilde{Z}_n(u) \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{in probability ;}$$

and moreover

$$(2.3) \quad \sup_{|u| \leq H} E_\theta |Z_n(u) - \tilde{Z}_n(u)| \xrightarrow[n \rightarrow \infty]{} 0, \quad 0 < H < \infty$$

Proof. The (2.1) follows from the Poisson approximation of binomial distribution. The proof of (2.3) is broken down into a few lemmas.

Lemma 2.1. If  $\{\xi_n\}$  is a sequence of non-negative random variables converging in distribution to a random variable  $\xi$  then

$$(i) \quad \liminf_n E|\xi_n| \geq E|\xi|$$

$$(ii) \quad E|\xi_n| \rightarrow E|\xi| < \infty \iff |\xi_n| \text{ are uniformly integrable}$$

For proof see [6], p. 183. Define now the random processes  $\eta_j(u)$  and random variables  $\ell_j, T_j$  by

$$\eta_j(u) = \frac{1}{u} \left[ \frac{f(X_j; \theta+u)}{f(X_j; \theta)} - 1 \right], \quad \ell_j = \frac{f'(X_j; \theta)}{f(X_j; \theta)},$$

$$T_j = \inf\{v: x_k(\theta+v) > X_j > x_k(\theta) \text{ for some } k = 1, \dots, r\}.$$

Evidently,  $\tau_n = n \min(T_1, \dots, T_n)$ .

Lemma 2.2. Under conditions I - V

$$(2.4) \quad \lim_{u \rightarrow 0} E_\theta |\chi(T_j > u) \eta_j(u) - \ell_j| = 0$$

Proof. Evidently,  $\lim_{u \rightarrow 0} \chi(T_j > u) \eta_j(u) = \ell_j$  with  $P_\theta$ -probability 1.

Further,

$$\begin{aligned} E_\theta \chi(T_j > u) \eta_j(u) &\leq E_\theta \{ \chi(T_j > u) \frac{1}{u} \int_\theta^{\theta+u} |f'_z(X_j; z)| dz \} \\ &\leq \frac{1}{u} \int_\theta^{\theta+u} dz \int_{-\infty}^{\infty} |f'(x; z)| dx \rightarrow \int_{-\infty}^{\infty} |f'(x; \theta)| dx = E|\ell_j|. \end{aligned}$$

Because of Lemma 2.1 (i), it follows that

$$\lim_u E_\theta \{ \chi(T_j > u) | \eta_j(u) | \} = \int |f'(x; \theta)| dx = E_\theta | \ell_j | .$$

According to (ii) of Lemma 2.1, the  $\chi(T_j > u) | \eta_j(u) |$  are uniformly integrable and therefore:

$$E_\theta | \chi(T_j > u) \eta_j(u) - \ell_j | \rightarrow 0 .$$

Lemma 2.3. If the conditions I - V are satisfied, then for every  $\delta > 0$

$$P_\theta \{ T_j > u, | \eta_j(u) | > \delta/|u| \} = o(|u|) , \quad u \rightarrow 0 ,$$

$$P_{\theta+u} \{ T_j > u, | \eta_j(u) | > \delta/|u|, f(X_j; \theta) \neq 0 \} = o(|u|) , \quad u \rightarrow 0 .$$

Proof. Since (2.4), it follows that

$$\begin{aligned} P_\theta \{ T_j > u, | \eta_j(u) | > \delta/|u| \} &\leq P_\theta \{ T_j > u, | \eta_j(u) - \ell_j | > \delta/2|u| \} + \\ &+ P_\theta \{ | \ell_j | > \delta/2|u| \} \leq \frac{2|u|}{\delta} E_\theta \{ \chi(T_j > u) | \eta_j(u) - \ell_j | \} + \\ &+ \frac{2|u|}{\delta} E_\theta \{ \chi(| \ell_j | > \delta/2|u|) | \ell_j | \} = o(|u|) . \end{aligned}$$

As for the second assertion, if  $|f(x; \theta+u)/f(x; \theta) - 1| > \delta$  then

$$f(x; \theta+u) < \frac{1}{\delta_1} |f(x; \theta+u) - f(x; \theta)|, \quad \delta_1 > \min(\delta/2, 1/2) .$$

Hence

$$\begin{aligned} P_{\theta+u} \{ T_j > u, | \eta_j(u) | > \delta/|u|, f(X_j; \theta) \neq 0 \} &\leq \\ &\leq \frac{|u|}{\delta_1} E_\theta \{ \chi(T_j > u, | \eta_j(u) | > \delta/|u|) | \eta_j(u) | \} \leq \\ &\leq \frac{|u|}{\delta_1} E_\theta \{ \chi(T_j > u) | \eta_j(u) - \ell_j | \} + \\ &+ \frac{|u|}{\delta_1} E_\theta \{ \chi(| \eta_j(u) | > \delta/|u|) | \ell_j | \} = o(|u|) . \end{aligned}$$

Lemma 2.4. Let the conditions I - V are fulfilled, then

$$E_{\theta} \ell_j = \int_{-\infty}^{\infty} f'(x; \theta) dx = \sum_1^r x'_k(\theta) p_k(\theta) = p.$$

Proof. The direct calculation.

The next two lemmas give the necessary estimates of the deviation of  $Z_n(u)$  from  $\tilde{Z}_n(u)$ .

Lemma 2.5. Under conditions I - V for every  $0 < H < \infty$

$$(2.6) \quad \sup_{|u| \leq H} E_{\theta} \{ |Z_n(u) - e^{pu}| \chi(\tau_n > u) \} \rightarrow 0, n \rightarrow \infty.$$

Proof. Fix a small positive number  $\delta < \frac{1}{2}$ . If all differences

$$\left| \frac{f(X_j; \theta + u/n)}{f(X_j; \theta)} - 1 \right| < \delta, j = 1, 2, \dots, n,$$

then

$$\begin{aligned} |\ell_n Z_n(u) - pu| &\leq \frac{|u|}{n} \left| \sum_1^n (\ell_j p) \right| + \frac{|u|}{n} \sum_1^n |\eta_j(u/n) - \ell_j| + \\ &\quad + 2\delta \frac{|u|}{n} \sum_1^n |\eta_j(u/n)|. \end{aligned}$$

Hence, uniformly in  $|u| \leq H$ .

$$\begin{aligned} (2.7) \quad E_{\theta} \{ |Z_n(u) - e^{pu}| \chi(\tau_n > u, \frac{1}{n} \sum_1^n \left| \frac{f(X_j; \theta + u/n)}{f(X_j; \theta)} - 1 \right| < \delta) \} &\leq \\ &\leq |u| e^{pu} E_{\theta} \left\{ \frac{1}{n} \sum_1^n |\ell_j - E_{\theta} \ell_j| \right\} + \\ &\quad + |u| E_{\theta} \{ |\eta_1(u/n) - \ell_1| \chi(T_1 > u/n) \} + 2\delta |u| E_{\theta} |\eta_1(u/n)| \leq \\ &\leq B\delta + o(1). \end{aligned}$$

We used here the law of large numbers to estimate the first righthand term, and lemma 2.2 to estimate the second and third ones.



Further, by (2.5)

$$\begin{aligned}
 & E_{\theta} \{ |Z_n(u) - e^{pu}| \chi(\tau_n > u, \frac{1}{u} \{ \frac{n}{f(X_j; \theta+u/n)} - 1 \} > \delta) \} \leq \\
 & \leq n P_{\theta+u/n} \{ |\eta_1(u/n)| > \delta n/|u|, T_1 > u/n, f(X_1; \theta) \neq 0 \} + \\
 (2.8) \quad & + e^{-pu} n P_{\theta} \{ |\eta_1(u/n)| > \delta n/|u|, T_1 > u/n \} = o(1), n \rightarrow \infty
 \end{aligned}$$

uniformly in  $|u| \leq H$ . The inequalities (2.7), (2.8) give us the desired result (2.6).

Lemma 2.6. Under the conditions I - V

$$\sup_{|u| \leq H} E_{\theta} \{ |Z_n(u) - \tilde{Z}_n(u)| \chi(\tau_n < u) \} = o(1), n \rightarrow \infty.$$

Proof. We have

$$E_{\theta} \{ |Z_n(u) - \tilde{Z}_n(u)| \chi(\tau_n < u) \} = E_{\theta} \{ Z_n(u) \chi(\tau_n < u) \} \leq P_{\theta+u/n} \{ \tau_n < u \}.$$

By definition of  $\tau_n$  the last probability is

$$\begin{aligned}
 & P_{\theta+u/n} \{ \bigcup_{j=1}^n \bigcup_{k=1}^r \{ x_k(\theta + u/n) > X_j > x_k(\theta) \} \} \leq \\
 & \leq n \sum_{k=1}^r \int_{x_k(\theta)}^{x_k(\theta+u/n)} f(x; \theta+u/n) dx
 \end{aligned}$$

Then since  $\lim f(x; \theta+u/n) = 0$ ,  $x \rightarrow x_k(\theta+u/n)$  uniformly in  $u$  (conditions III and IV)

$$\int_{x_k(\theta)}^{x_k(\theta+u/n)} f(x; \theta+u/n) dx \leq o(1) |x_k(\theta + u/n) - x_k(\theta)| = o(1/n).$$

This last inequality proves the lemma. The assertions (2.2) and (2.3) are evident consequences of lemmas 2.5, 2.6 and theorem 1 is proved.

3. Locally asymptotically one sided exponential families of distributions. Limiting distributions of regular estimates; local asymptotic minimax property of Bayesian estimates.

Let us consider (following Hájek [1]) a sequence of statistical experiments  $\{X^n, A^n, P_\theta^n\}$ ,  $n \geq 1$ , where  $\theta$  runs through an open set  $\Theta \subseteq R^1$ . Let  $\alpha(n) \uparrow \infty$  be a sequence of normalizing factors. Take a point  $\theta \in \Theta$  and assume it to be the true value of parameter. Denote  $Z_n(u)$  the Radon-Nikodym derivative of the absolutely continuous part of  $P_{\theta+u/\alpha(n)}^n$  with respect to  $P_\theta^n$ . For the sake of brevity we shall write below  $P_\theta$  instead  $P_\theta^n$  and  $P_{\theta,u}$  instead  $P_{\theta+u/\alpha(n)}^n$ . Note that the definition  $Z_n(u)$  is just the same as in Section 2.

Assumption 3.1. Assume that either

$$(3.1) \quad Z_n(u) = \begin{cases} e^{pu} + o_p(1), & u < \tau_n \\ o_p(1), & u > \tau_n \end{cases}$$

or

$$(3.2) \quad Z_n(u) = \begin{cases} e^{-pu} + o_p(1), & u > -\epsilon_n \\ o_p(1), & u < -\epsilon_n \end{cases}$$

where  $p$  is a positive number, the positive random variables  $\tau_n$  are  $A^n$  measurable and satisfy  $\lim_n P_\theta\{\tau_n > u\} = e^{-pu}$ , and  $o_p(1)$  denotes random variables which go to zero in probability when  $n \rightarrow \infty$ .

Theorem 2.1 of Section 2 gives us an example of statistical experiments satisfying Assumption 3.1. We shall consider below (without special warning) the case (3.1) of Assumption 3.1 only. Denote

$$\tilde{Z}_n(u) = \begin{cases} e^{pu}, & u < \tau_n \\ 0, & u > \tau_n. \end{cases}$$

Define now the measure  $\tilde{P}_{\theta,u}$  on  $A^n$  by the formula

$$\tilde{P}_{\theta,u}(A) = \int_A \tilde{Z}_n(u) dP_\theta, \quad A \in A^n.$$

Theorem 3.1. Under Assumption 3.1 for every  $u > 0$

$$(3.3) \quad \int_{X^n} |dP_{\theta,u} - d\tilde{P}_{\theta,u}| \rightarrow 0, \quad n \rightarrow \infty.$$

Proof. For every fixed  $u$  both sequences  $\{Z_n(u)\}$ ,  $\{\tilde{Z}_n(u)\}$  tends in distribution to the random variable

$$Z(u) = \begin{cases} e^{pu}, & u < \tau, \\ 0, & u > \tau, \end{cases}$$

where  $P\{\tau > y\} = e^{-py}$ . Further, for  $u \geq 0$

$$\lim E_\theta Z_n(u) = \lim E_\theta \tilde{Z}_n(u) = 1 = E_\theta Z(u),$$

and by Lemma 2.1 the random variables  $Z_n(u)$ ,  $\tilde{Z}_n(u)$  are uniformly integrable with respect to  $P_\theta$ , so that

$$(3.4) \quad E_\theta |Z_n(u) - \tilde{Z}_n(u)| \rightarrow 0, \quad n \rightarrow \infty.$$

Next, let  $P_{\theta,u}^{(r)}$ ,  $P_{\theta,u}^{(s)}$  denote the regular and the singular parts of  $P_{\theta,u}$  with respect to  $P_\theta$ . Then for  $u > 0$

$$P_{\theta,u}^{(s)}\{X^n\} \leq 1 - P_{\theta,u}^{(r)}\{\tau_n > u\} = 1 - E_\theta\{Z_n(u) \cdot \chi(\tau_n > u)\} \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore,

$$\int_{X^n} |dP_{\theta,u} - d\tilde{P}_{\theta,u}| \leq E_\theta |Z_n(u) - \tilde{Z}_n(u)| + P_{\theta,u}^{(s)}\{X^n\} \rightarrow 0, \quad n \rightarrow \infty.$$

Theorem 3.2. Let us assume that (3.1) holds. Consider a sequence  $\{t_n\}$  of estimates of parameter  $\theta$  satisfying

(3.5)  $P_{\theta, u} \{ \alpha(n)(t_n - \theta - u/\alpha(n)) < y \} \rightarrow F(y)$  for every  $u \in R'$

in continuity points of some distribution function  $F(y)$ ,  $y \in R'$ .

Then we have

$$F = H_p * G,$$

where  $H_p$  is the exponential distribution on  $(0, \infty)$  with parameter  $p$ , i.e.  $1 - H_p(y) = e^{-py}$ ,  $y > 0$ , and  $G$  is a certain distribution function in  $R^1$ .

Proof. Let  $f(s)$  denotes the characteristic function of  $F$  and rewrite (3.5) in the form

$$E_{\theta, u} \{ \exp\{is\zeta_n - isu\} \} \rightarrow f(s),$$

where  $\zeta_n = \alpha(n)(t_n - \theta)$ . We have from this and (3.3)

$$f(s) = e^{pu + ius} \int_{X^n} e^{is\zeta_n} \chi(\tau_n > u) dP_\theta + o(1), \quad n \rightarrow \infty.$$

Multiplying both part of the last equality on  $e^{-\lambda u}$ ,  $\lambda = -\mu + iv$ ,  $\mu > 0$ , and integrating relative to  $u$ , we find

$$\begin{aligned} \frac{f(s)}{p+\lambda-is} &= \frac{1}{\lambda} \int_{X^n} e^{is\zeta_n} (1 - e^{-\lambda t_n}) dP_\theta + o(1) = \\ &= \frac{f(s)}{\lambda} - \frac{1}{\lambda} \int_{X^n} e^{is\zeta_n - \lambda t_n} dP_\theta + o(1). \end{aligned}$$

Now we can choose  $v = s$ ,  $\mu = 0$  and then we have

$$f(s) = \frac{p}{is-p} \lim \int_{X^n} e^{is(\zeta_n - t_n)} dP_\theta = \frac{p}{is-p} \int_{-\infty}^{\infty} e^{isy} dG(y) = \frac{p}{is-p} \cdot g(s),$$

where  $g(s)$  is the characteristic function of  $G$ . Because  $p/is-p$  is the characteristic function of  $H_p$  the theorem is proved.

Suppose that the loss resulting from replacement of the true vaule of the parameter  $\theta$  by its estimate  $t_n$  is assumed to be  $w(\alpha(n)(t_n - \theta))$ ,

when  $w(\cdot)$  is a given function. The mean loss, the reisk function, is then

$$(3.6) \quad E_{\theta}^{(n)} w(\alpha(n)(t_n - \theta)) .$$

We shall assume below that the function  $w$  in (3.6) satisfies the following condition:

$$(3.7) \quad w(y) = w(|y|) ; w(y) \uparrow , y \geq 0 ; w(y) \geq 0 , w \neq \text{const} .$$

$$(3.8) \quad \int_0^{\infty} w(y) e^{-\epsilon y} dy < \infty , \quad \epsilon > 0 .$$

Theorem 3.3. Under assumption 3.1, any sequence of estimates  $\{t_n\}$  for  $\theta$  satisfies (for the  $w$  of (3.7), (3.8))

$$(3.9) \quad \lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \sup_{|\theta - u| < \delta} E^{(n)} \{w(\alpha(n)(t_n - u))\} \geq \min_u P \int_{-\infty}^0 w(y-u) e^{py} dy .$$

Proof. Denote through  $w_a(y)$  a truncated version of  $w$ :

$$w_a(y) = \min\{w(y), a\} .$$

We have for any  $b < \delta \cdot \alpha(n)$

$$\begin{aligned} \sup_{|u-\theta| < \delta} E_u^{(n)} \{w(\alpha(n)(t_n - u))\} &\leq \frac{\alpha(n)}{b} \int_{\theta}^{\theta+b/\alpha(n)} E_u^{(n)} \{w(\alpha(n)(t_n - u))\} du = \\ &= \frac{1}{b} \int_0^b E_{\theta, u} \{w(\zeta_n - u)\} du , \end{aligned}$$

where we denote  $\alpha(n)(t_n - \theta) = \zeta_n$ . Further, by Theorem 3.1 for  $n \rightarrow \infty$

$$\begin{aligned} \frac{1}{b} \int_0^b E_{\theta, u} \{w(\zeta_n - u)\} du &\geq \frac{1}{b} \int_0^b E_{\theta, u} \{w_a(\zeta_n - u)\} du = \\ &= \frac{1}{b} E_{\theta} \int_0^b w_a(\zeta_n - u) \tilde{Z}_n(u) du + o(1) = \\ &= \frac{1}{b} E_{\theta} \int_0^{\min(b, \tau_n)} w_a(\zeta_n - u) e^{pu} du + o(1) \geq \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{b} E_{\theta} \{ \chi(\tau_n < b) \int_{-\infty}^{\tau_n} w_a(\tau_n - u) e^{pu} du \} - \\
&- a/bp + o(1) \geq \min_y \int_{-\infty}^0 w_a(y - u) e^{pu} du \cdot \\
&\cdot \frac{1}{b} E_{\theta} \{ \chi(\tau_n < b) e^{p\tau_n} \} - a/bp + o(1) = \\
&= \min_y p \int_{-\infty}^0 w_a(y - u) e^{pu} du - a/bp + o(1) .
\end{aligned}$$

Hence, putting here  $b = a^2$  we have

$$\begin{aligned}
&\liminf_{n \rightarrow \infty} \sup_{|\theta - u| < \delta} E_u^{(n)} \{ w(\alpha(n)(t_n - u)) \} \geq \\
&\geq \min_y p \int_{-\infty}^0 w_a(y - u) e^{pu} du - 1/ap .
\end{aligned}$$

Finally, under conditions (3.7), (3.8)

$$\lim_{a \rightarrow \infty} \min_y p \int_{-\infty}^0 w_a(y - u) e^{pu} du = \min_y \int_{-\infty}^0 w(y - u) e^{pu} du$$

and Theorem 3.3 is proved.

Return to the case of independent identically distributed observations satisfying conditions I - V. We have noted already that under conditions I - V the assumption 3.1 is always fulfilled with  $\alpha(n) = n$ . Define now the sequence of estimates  $\hat{t}_n(w) = \hat{t}_n$  in such a way that

$$\int w(\hat{t}_n - u) \prod_{j=1}^n f(X_j; u) du = \min_y \int w(y - u) \prod_{j=1}^n f(X_j; u) du$$

It is easy to deduce from [5] that under a wide conditions upon  $f$  and  $w$  (but a little more restrictive than I - V and (3.7), (3.8)) the estimates  $\hat{t}_n$  have the following properties:

(i) the limit distribution of  $n(\hat{t}_n - \theta)$  coincides with the distribution of  $\tau + y_w$  when  $y_w$  is the point of minimum of  $\int_{-\infty}^0 w(y - u) e^{pu} du$  and  $P\{\tau > y\} = e^{-py}$ ,  $y > 0$ .

$$(ii) \quad \lim_n E_{\theta} w(n(\hat{t}_n - \theta)) = p \int_{-\infty}^0 w(y_w - u) e^{pu} du .$$

So from the point of view of Theorem 3.2, 3.3 the estimates  $\hat{t}_n(w)$  are asymptotically "good" estimates with respect to the loss function  $w$ .

#### 4. The case of sequential estimation.

We continue to consider the sequence  $\{X_j\}$  of independent identically distributed observations satisfying the conditions I - V. Assume we are given: 1) a stopping time  $\sigma$ ; 2) a sequence of statistics  $\{t_n\}$ ,  $t_n = t_n(X_1, \dots, X_n)$ . As an estimate of parameter  $\theta$  we use the random variable  $t_\sigma(X_1, \dots, X_\sigma)$ . We call a pair  $d = [\{t_n\}, \sigma]$  a sequential estimation plan. We want to prove here that from the point of view of Theorem 3.3 sequential schemes are not better than fixed sample schemes.

Theorem 4.1. Denote  $\mathcal{D}_n$  the collection of all sequential plans  $d = [\{t_n\}, \sigma]$  with  $E_\theta \sigma \leq n$ ,  $\theta \in \Theta$ . Under the conditions I - V

$$(4.1) \quad q_a = \frac{\lim_{\delta \rightarrow 0}}{\delta} \liminf_{n \rightarrow \infty} \sup_{\mathcal{D}_n} \sup_{|\theta - u| \leq \delta} n^a E_u |t_\sigma - u|^a \geq \inf_y p \cdot \int_{-\infty}^0 |y - u|^a e^{pu} du, \quad a > 0.$$

Proof. We will at first show that we need consider only such plans  $d \in \mathcal{D}_n$  for which

$$(4.2) \quad P\{\sigma \geq \epsilon n\} = 1, \quad \epsilon > 0.$$

Lemma 4.1. For any  $\alpha > 0$  there exist a positive number  $\epsilon = \epsilon(\alpha) > 0$  and a sequence of plans  $d_n = [\{t_k^{(n)}\}, \sigma(n)]$  such that  $P_u\{\sigma_n \geq \epsilon n\} = 1$ ,  $u \in \Theta$  and

$$\frac{\lim_{\delta} \lim_n}{\delta} \sup_{|\theta - u| \leq \delta} n^a E_u |t_{\delta(n)}^{(n)} - u|^a \leq q_a + \alpha.$$

The proof of Lemma 4.1 coincides exactly with the proof of Lemma 2.5 of [7] and we omit it.

Lemma 4.2. The following relation holds

$$(4.2) \quad E_{\theta} \{ \tilde{Z}_n(u) | X_1, \dots, X_k \} = \tilde{Z}_k(u \cdot \frac{k}{n}) (1 + \rho(k, n, u)), \quad k \leq n,$$

where

$$\sup_{0 < u < H} \max_{1 \leq k \leq n} |\rho(k, n, u)| \rightarrow 0, \quad n \rightarrow \infty.$$

Proof. By definition of  $\tilde{Z}_n(u)$ ,

$$\begin{aligned} E_{\theta} \{ Z_n(u) | X_1, \dots, X_k \} &= e^{pu} E_{\theta} \{ \chi(\tau_n > u) | X_1, \dots, X_k \} = \\ &= e^{pu} \prod_{j=1}^k \chi(T_j > u/n) \prod_{j=k+1}^n E_{\theta} \chi(T_j > u/n) = \\ &= e^{pu} \chi(\tau_k > uk/n) (1 - P\{\cup_{j=1}^r \{x_k(\theta + u/n) > X_j > x_k(\theta)\}\})^{n-k} = \\ &= \tilde{Z}_k(uk/n) (1 + \rho(k, n, u)). \end{aligned}$$

The lemma is proved. Let now  $d \in \mathcal{D}_n$ . We may (and will) suppose by lemma 4.1 that  $d$  satisfies (4.2). Let us fix also two positive numbers  $N > \varepsilon$  and  $b > N$ . We specify the choice of the numbers  $\varepsilon, N, b$  later. Using Lemma 4.2, we obtain after a slight modification of the first half of (3.10).

$$\begin{aligned} n^a \sup_{|\theta - u| \leq \delta} E_u |t_{\sigma} - u|^a &\geq \frac{1}{b} \int_0^b E_{\theta + u/Nn} |n(t_{\sigma} - \theta) - u/N|^a du \geq \\ &\geq \frac{1}{b} E_{\theta} \sum_{\varepsilon n}^{Nn} \chi(\sigma = k) \int_0^b |n(t_{\sigma} - \theta) - u/N|^a \tilde{Z}_{nN}(u) du + \\ (4.3) \quad &+ o(1) = \frac{1}{b} E_{\theta} \sum_{\varepsilon n}^{Nn} \chi(\sigma = k) \int_0^b |n(t_{\sigma} - \theta) - u/N|^a \cdot \\ &\cdot \tilde{Z}_k(u \frac{k}{nN}) du + o(1) \geq \min_x p \int_{-\infty}^x |x - u| e^{pu} du \frac{N}{pb} \sum_{k = \varepsilon n}^{Nn} E_{\theta} \left(\frac{n}{k}\right)^{a+1} \chi(\sigma = k) \cdot \\ &\chi(\tau_k < \frac{kb}{Nn}) e^{p\tau_k} + o(1) + o(1/b). \end{aligned}$$



Hence, to prove the theorem it is sufficient to show that for every  $\alpha > 0$  it is possible to choose  $N, b, \varepsilon$  in such a way that

$$(4.4) \quad \sum_{k=n\varepsilon}^{Nn} \left(\frac{n}{k}\right)^{\alpha} \psi_k \geq 1 - \alpha, \quad \psi_k = \frac{N}{pb} \cdot E_{\theta} \frac{n}{k} \chi(\sigma=k) \chi(\tau_k < \frac{kb}{Nn}) e^{p\tau_k}.$$

To estimate the left side of (4.4) we use the following results.

Lemma 4.3. For any  $\alpha > 0$  there exists such a choice of numbers

$0 < \varepsilon < N < b$  in (4.3) that

$$(4.5) \quad \sum \psi_k \geq 1 - \alpha \quad \sum k\psi_k \leq n(1-\alpha)$$

Lemma 4.5. Let us suppose we are given positive numbers  $\psi_k \geq 0$  which satisfy conditions (4.5) and let  $g(u)$  be a convex decreasing function of  $u > 0$ . Then

$$(4.6) \quad \sum g(k/n) \psi_k \geq (1-\alpha) g\left(\frac{1+\alpha}{1-\alpha}\right)$$

and hence  $(g(u) = u^{-\alpha})$

$$(4.7) \quad \sum (n/k)^{\alpha} \psi_k \geq (1-\alpha) \left(\frac{1+\alpha}{1-\alpha}\right)^{\alpha}$$

The inequality (4.4) and hence the assertion of the theorem is a simple consequence of the relations (4.5) and (4.6). The last of them follows with ease from Jensen's inequality. Namely,

$$\begin{aligned} \sum g(k/n) \psi_k &\geq (1-\alpha) \sum g(k/n) \cdot \frac{\psi_k}{\sum \psi_j} \geq \\ &\geq (1-\alpha) g\left(\frac{1}{n} \sum \frac{k\psi_k}{\sum \psi_j}\right) \geq (1-\alpha) g\left(\frac{1+\alpha}{1-\alpha}\right) \end{aligned}$$

and (4.6) is proved.

To prove the first inequality (4.5) we define the number  $\beta = b - \sqrt{b}$  and note that because of lemma 4.1 for all sufficiently large  $n$

$$\begin{aligned}
\sup_{0 \leq u \leq \beta} P_{\theta+u/Nn} \{ \tau_{\sigma} > b\sigma/Nn \} &= \sup_{0 \leq u \leq \beta} P_{\theta+u/Nn} \{ \bigcap_{j=1}^{\sigma} \{ T_j > b/Nn \} \} \leq \\
&\leq \sup_{0 \leq u \leq \beta} P_{\theta+u/Nn} \{ \bigcap_{j=1}^{[en]} \{ T_j > b/Nn \} \} = \\
&= \sup_{0 \leq u \leq \beta} (P_{\theta+u/Nn} \{ T_1 > b/Nn \})^{[en]} \leq \exp\{-pe\sqrt{b}/2N\}.
\end{aligned}$$

Using this last inequality and Tchebichev's inequality, we obtain

$$\inf_{0 \leq u \leq \beta} P_{\theta+u/Nn} \{ \sigma \leq Nn, \tau_{\sigma} \leq b\sigma/Nn \} \geq 1 - \exp\{-pe\sqrt{b}/2N\} - 1/N.$$

Applying once more the arguments we used to derive (4.3), we find

$$\begin{aligned}
\inf_{0 \leq u \leq \beta} P_{\theta+u/Nn} \{ \sigma \leq Nn, \tau_{\sigma} \leq b\sigma/Nn \} &\leq \frac{1}{\beta} \int_0^{\beta} P_{\theta+u/Nn} \{ \sigma \leq Nn, \tau_{\sigma} \leq b\sigma/Nn \} du = \\
&= \frac{1}{\beta} E_{\theta} \left\{ \sum_{en}^{Nn} \chi(\sigma = k) \chi(\tau_k \leq bk/Nn) \int_0^{\beta} \tilde{Z}_{Nn}(u) du \right\} + \\
&+ o(1) \leq \frac{1}{\beta} E_{\theta} \left\{ \sum_{en}^{Nn} \chi(\sigma = k) \chi(\tau_k \leq bk/Nn) \cdot \right. \\
&\cdot \left. \int_0^{\beta} \tilde{Z}_k(uk/Nn) du \right\} + o(1) \leq \frac{N}{p\beta} \sum_{en}^{Nn} \frac{n}{k} E_{\theta} \{ \chi(\sigma = k) \cdot \\
&\cdot \chi(\tau_k \leq bk/Nn) e^{p\tau_k} \} + o(1).
\end{aligned}$$

Hence,

$$(4.8) \quad \sum \psi_k \geq (1 - \exp\{-pe\sqrt{b}/2N\} - 1/N)(1 - 1/\sqrt{b}) + o(1).$$

The estimate of  $\sum k\psi_k$  may be obtained in the same way. At first, since  $E_{\theta+u/Nn} \sigma \leq n$  we have

$$\sup_u E_{\theta+u/Nn} \{ \sigma \cdot \chi(\tau_k \leq bk/Nn) \} \leq n.$$

Therefore

$$\begin{aligned}
n &\geq \frac{1}{\beta} \int_0^{\beta} E_{\theta+u/Nn} \{ \sigma \chi(\tau_{\sigma} \leq b\sigma/Nn) \} du \geq \\
&\geq \frac{Nn}{bp} \sum_{en}^{Nn} E_{\theta} \{ \chi(\tau_k \leq bk/Nn) \cdot \chi(\sigma = k) e^{p\tau_k} u \} \\
&- \frac{Nn}{bp} \sum_{ep}^{Nn} E_{\theta} \{ \chi(\tau_k \leq bk/Nn) \cdot \chi(\sigma = k) \} + o(1) \geq \sum k\psi_k - nN/bp + o(1)
\end{aligned}$$

and

$$\sum k\psi_k \leq n(1 + N/bp) + o(1).$$

The inequalities (4.8) and (4.9) prove (4.5) because we can simultaneously make  $\exp\{-\varepsilon\sqrt{b}/2N\}$ ,  $N/b$ ,  $1/N$  as small as we want.

Remark. The theorem 4.1 is an analogue of the theorem 3.3 for the case  $w(x) = |x|^a$ ,  $a > 0$ . In fact, we proved a little more. Indeed, for a function  $w$  satisfying (3.7), (3.8) define

$$g_w(\lambda) = g(\lambda) = \min_x \int_{-\infty}^0 w(\lambda^{-1}(x-v)) e^{pv} dv$$

Then, like (4.3),

$$\sup_{|\theta-u| \leq \delta} E_u w(n^a(t_\sigma - u)) \geq \sum \psi_k g(k/n).$$

If we suppose that  $g(\lambda)$  is convex and continuous at  $\lambda = 1$  then by lemma 4.5 we find for every  $\alpha > 0$

$$\sum \psi_k g(k/n) \geq (1-\alpha) g\left(\frac{1+\alpha}{1-\alpha}\right) + o(1).$$

Hence, theorem 4.1 holds not only for  $w(x) = |x|^a$  but also for all  $w$  for which  $g(\lambda)$  is convex and continuous. It will be, for example, if

(i)  $w(x)$  is convex and satisfies (3.7), (3.8);

$$(ii) w(x) = \begin{cases} 0, & |x| \leq w_0 \\ 1, & |x| > w_0. \end{cases}$$

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