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A LIMIT THEOREM FOR THINNING OF POINT PROCESSES

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ABSTRACT

Let ξ and η be point processes and let $p\in[0,1]$. We say that ξ is a p-thinning of η, if it is obtained from η by deleting the atoms independently with probability 1-p. It is shown that, if $p_1,p_2,\ldots\in(0,1]$ with $p_n\to 0$ and if ξ_n is a p_n-thinning of some η_n for each $n\in N$, then ξ_n converges in distribution in the vague topology if and only if $p_n\eta_n$ does. Furthermore, denoting the limits by ξ and η respectively, we have that ξ is a subordinated (or doubly stochastic) Poisson process directed by η. These results include some limit theorems by Rényi and others on thinnings of a fixed process as well as a characterization by Mecke of the class of subordinated Poisson processes.

KEY WORDS AND PHRASES: Thinning of Point Processes; Subordinated Poisson Processes; Convergence in Distribution; Vague Topology; Random Measures.

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Let S be a locally compact second countable Hausdorff space and let $\mathcal{B}$ be the σ-ring of bounded Borel sets in S. By a *random measure* ξ on S we shall mean a mapping of some probability space $(\Omega,\mathcal{A},P)$ into the space $\mathcal{M}$ of Radon measures on $(S,\mathcal{B})$ such that ξB is a random variable for each $B\in\mathcal{B}$. When ξ is a.s. integer valued, it will also be called a *point process*.

If ξ and η are point processes while p is a real number in [0,1], we shall say that ξ is a p-*thinning of* η, provided it is obtained from η by deleting the unit atoms independently with probability 1-p. (Here an atom of size $k \geq 2$ is considered as the superposition of k unit atoms.) Furthermore, if η is an arbitrary random measure, we shall say that ξ is a *subordinated Poisson process directed by* η (abbreviated SP(η)) if, given η, ξ is conditionally distributed as a Poisson process with intensity measure η. The existence of such thinnings and SP-processes is easily verified, using Harris' existence theorem in the formulation of Jagers (1972).

The limiting behavior as $p\to 0$ of p-thinnings of a fixed point process has been considered by Rényi (1956), Nawrotzki (1962), Belyaev (1963) and Goldman (1967) (see also [5], [8] and [11]). In the present note, we shall prove a general limit theorem containing the results of these authors as well as the following interesting characterization of the class of SP-processes, due to Mecke (1968) (see also [7], page 34): A point process is SP if and only if it can be obtained as a p-thinning for each $p\in(0,1]$.

Before proceeding, we introduce some further terminology and notation. Let us write $\mathcal{F}$ for the class of bounded measurable functions on S with compact support, and let $\mathcal{F}_c$ be its sub-class of continuous functions. *Vague* convergence $\mu_n \xrightarrow{v} \mu$ of measures in M means that $\mu_n f \to \mu f$ for any $f \in F_c$. Here $\mu f = \int f(s)\mu(ds)$. Convergence in distribution of random elements (cf. [2]) is written $\xrightarrow{d}$. For random measures, the underlying topology in M is taken to be the vague one (see [4] and [7]). We are now able to state our main result.

Theorem. Let $p_1, p_2, \ldots \in (0,1]$ with $p_n \to 0$, and suppose that ξ_n for each $n \in N$ is a p_n-thinning of some point process η_n. Then $\xi_n \xrightarrow{d}$ some ξ if and only if $p_n \eta_n \xrightarrow{d}$ some η, and in this case, ξ is SP(η).

Proof. We shall need the facts that, if ξ is SP(η), then

$$Ee^{-\xi f} = E\exp\{-\eta(1-e^{-f})\}\ , \qquad f \in F\ , \tag{1}$$

while if ξ is a p-thinning of η,

$$Ee^{-\xi f} = E\exp\{\eta \log(1 - p(1-e^{-f}))\}\ , \quad f \in F\ . \tag{2}$$

These formulae were established by Mecke (1968) for S = R, and their proofs for general S are similar. To prove the assertions of the theorem, let us first suppose that $p_n \eta_n \xrightarrow{d} \eta$. We have to show that $\xi_n \xrightarrow{d} \xi$ where ξ is SP(η), or equivalently, by (1), (2) and [4], that for any $f \in F_c$,

$$E\exp\{\eta_n \log(1 - p_n(1-e^{-f}))\} \to E\exp\{-\eta(1-e^{-f})\}\ . \tag{3}$$

But this will follow by Theorem 5.1 in [2], if we can show that

$$-\eta_n \log(1 - p_n g) \xrightarrow{d} \eta g\ , \tag{4}$$

where $g = 1-e^{-f}$. By Theorem 5.5 in [2], it suffices to prove (4) for non-random η_n and η satisfying $p_n \eta_n \xrightarrow{v} \eta$, and since g has bounded support, we may even assume the $p_n \eta_n$ and η to be probability measures. If σ_n and σ are random elements in S with distributions $p_n \eta_n$ and η respectively, we may then write (4) in the form

$$-p_n^{-1}\ E\ \log(1 - p_n g(\sigma_n)) \to E\ g(\sigma),$$

and since g is bounded and continuous, this will follow by Theorem 5.1 in [2], provided we can show that

$$(5) \qquad -p_n^{-1} \log(1 - p_n g(\sigma_n)) \stackrel{d}{\to} g(\sigma) \quad .$$

By another application of Theorem 5.5 in [2], we may assume that the σ_n and σ are non-random elements in S satisfying $\sigma_n \to \sigma$, in which case (5) is a well-known result from calculus. This completes the proof of the sufficiency and the last assertion.

Conversely, suppose that $\xi_n \stackrel{d}{\to} \xi$. To prove that $p_n\eta_n \stackrel{d}{\to}$ some η , it suffices to show that the sequence $\{p_n\eta_n\}$ is tight. In fact, assuming tightness, any sequence $N' \subset N$ must contain some further subsequence N'' such that $p_n\eta_n \stackrel{d}{\to}$ some η'' as $n \to \infty$ through N''. By the part of the theorem already proved, it follows that ξ is $SP(\eta'')$, and so, by (1), the distribution of η'' is unique. The convergence of $p_n\eta_n$ now follows by Theorem 2.3 in [2].

To prove tightness of $\{p_n\eta_n\}$, it suffices by Lemma 1.2 in [6] to show that $\{p_n\eta_n B\}$ is tight for any $B \in \mathcal{B}$ with $\xi\partial B = 0$ a.s. But for such a B we have $\xi_n B \stackrel{d}{\to} \xi B$ by Theorem 1.1 in [6], or equivalently, in terms of Laplace transforms

$$E\ \exp\{\eta_n B \log(1 - p_n(1-e^{-t}))\} \to E\ \exp\{-\eta B(1-e^{-t})\} \ , \quad t \geq 0.$$

Here the right-hand side is continuous in t, so there exist for any $\varepsilon > 0$ some $u = 1-e^{-t} \in (0,1)$ and $n_0 \in N$ such that

$$E \exp\{\eta_n B \log(1 - p_n u)\} > 1 - \varepsilon/2, \quad n > n_0 .$$

Writing $n_1 = \max\{n: p_n > 1/2\}$, it follows by Čebyšev's inequality and an elementary estimate that, for $n > n_0 \vee n_1$,

$$\begin{aligned} P\{p_n \eta_n B > u^{-1} \log 2\} &\le P\{-\eta_n B \log(1-p_n u) > \log 2\} \\ &= P\{1 - \exp[\eta_n B \log(1-p_n u)] > 1/2\} \\ &\le 2\, E\{1 - \exp[\eta_n B \log(1-p_n u)]\} \le \varepsilon. \end{aligned}$$

This shows that $\{p_n \eta_n B\}$ is tight, and hence completes the proof.

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