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ZAKAI'S CLASS OF BANDLIMITED FUNCTIONS AND PROCESSES: ITS CHARACTERIZATION AND PROPERTIES

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ABSTRACT

This paper characterizes Zakai's class [3] of bandlimited functions and processes in terms of conventionally bandlimited functions and processes. This characterization was first conjectured by Zakai and is used here to derive sharper sampling representations and to study further properties of functions and processes bandlimited in the sense of Zakai.

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I. Introduction

The concept of a bandlimited function (and a bandlimited process) has been extended by Zakai [3] to a class of functions (and processes) which do not have a Fourier integral representation. In this paper Zakai's class of bandlimited functions is characterized by a representation in terms of conventionally bandlimited functions (Theorem 1). This characterization was conjectured by Zakai and results in a sharper sampling theorem for the entire class (Theorem 2). It is shown that this class of bandlimited functions can be defined by using a host of reproducing kernels other than the one used by Zakai (Theorem 4). Also, the properties of the class of conventionally bandlimited functions as a subset of Zakai's class of bandlimited functions are studied (Theorem 5).

Similar results are obtained for Zakai's class of bandlimited processes: a sharper sampling theorem (Theorem 6), a characterization by a representation in terms of conventionally bandlimited harmonizable processes (Theorem 7), and the relationship between conventionally bandlimited stationary or harmonizable processes and processes bandlimited in Zakai's sense (Theorem 8).

II. Bandlimited Functions

The following notation is used throughout this paper. m is the Lebesgue measure on the real line, μ is the finite measure on the Borel set of the real line defined by $[\frac{d\mu}{dm}](t) = \frac{1}{1+t^2}$, and $L_2(\mu)$ is the Hilbert space of Borel measurable complex-valued functions on the real line satisfying $\int_{-\infty}^{\infty} |f(t)|^2 d\mu(t) < \infty$. For $W, \delta > 0$ let

$$(1)\qquad H(\lambda) = H(\lambda;W,\delta) = \begin{cases} 1 & \text{for } |\lambda| \le W \\ 1 - \frac{|\lambda|-W}{\delta} & \text{for } W < |\lambda| \le W+\delta \\ 0 & \text{for } W+\delta < |\lambda| \end{cases}$$

and denote its inverse Fourier transform by

$$(1a)\qquad h(t) = h(t;W,\delta) = \frac{2}{\pi\delta^2} \sin(W + \frac{\delta}{2})t \sin \frac{\delta}{2} t \ .$$

Zakai [3] defines the class $B(W,\delta)$ of functions "bandlimited to (W,δ)" as the set of all functions f in $L_2(\mu)$ satisfying for almost all t

$$(2)\qquad f(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = (f*h)(t) \quad .$$

$B(W,\delta)$ is then a subspace of $L_2(\mu)$ and every function in $B(W,\delta)$ is equal almost everywhere to a continuous function (the right hand side of (2)). As in [3] only these continuous modifications will be considered in this paper.

Denote by $CB(W)$ the class of functions in $L_2(m)$ which are "conventionally bandlimited to W", i.e. $CB(W) = \{f \in L_2(m): F(\lambda) = 0 \text{ for } |\lambda| > W\}$ where F denotes the Fourier transform of f. Then $CB(W)$ is a subspace of $L_2(m)$ and $CB(W) \subset B(W,\delta)$ for all $\delta > 0$. Thus Zakai's concept of bandlimited functions generalizes the conventional one.

The following properties of functions in $B(W,\delta)$ were obtained by Zakai.

LEMMA 1 [3]. (a) If $f \in B(W,\delta)$ then $f(t) = f(0) + tg(t)$ where $g \in CB(W+\delta)$.

(b) If $g \in CB(A)$ then $f(t) = c + tg(t) \in B(W,\delta)$ for all $W \ge A$ and $\delta > 0$.

It was conjectured by Zakai that in Lemma 1a, $g \in CB(W)$. This is proved in the following theorem which thus provides the characterization of $B(W,\delta)$.

THEOREM 1. $f \in B(W,\delta)$ if and only if

$$f(t) = f(0) + tg(t) \tag{3}$$

where $g \in CB(W)$.

Proof. In view of (2) and the fact that h is a real-valued function, it suffices to prove the theorem for real-valued f.

(a) Let $f \in B(W,\delta)$. By repeated convolutions of f with h, (2) implies

$$f = f * h_n \ , \quad n=1,2,\ldots$$

where h_n is the (n-1)st fold convolution of h with itself, i.e., h_n is the inverse Fourier transform of $H_n(\lambda) = H^n(\lambda)$. Now by Lemma 1 a, $f(t) = f(0) + tg(t)$, $g \in CB(W+\delta)$. Substituting f in $f * h_n$, and noting that $\int h_n(t)dt = H_n(0) = 1$ implies $(f(0) * h_n)(t) = f(0)$, we have for all t and $n = 1,2,\ldots$

$$tg(t) = \int_{-\infty}^{\infty} (t-\tau)g(t-\tau)h_n(\tau)d\tau \ .$$

It can be seen from (1) that H_n is absolutely continuous with derivative $H_n' \in L_2(m)$ and thus [1, Theorem 61] that $th_n(t) \in L_2(m)$. Hence by Parseval's theorem and $g \in CB(W+\delta)$ we have for all t and $n=1,2,\ldots$

$$\frac{t}{2\pi} \int_{-(W+\delta)}^{W+\delta} G(\lambda)e^{i\lambda t}d\lambda = \frac{1}{2\pi} \int_{-(W+\delta)}^{W+\delta} G(\lambda)e^{i\lambda t}\{tH_n(\lambda) - iH_n'(\lambda)\}\, d\lambda$$

where G is the Fourier transform of g. Since $H_n(\lambda) = 1$ for $\lambda \in [-W,W]$, $n=1,2,\ldots$, it can be seen that the integrals over $[-W,W]$ on the right and left hand side are identical and hence for all t and $n=1,2,\ldots$

(4) $$a(t) \triangleq \int_E G(\lambda)e^{i\lambda t}\{t[H_n(\lambda) - 1] - iH_n'(\lambda)\}d\lambda = 0$$

where $E = [-W-\delta,-W] \cup [W,W+\delta]$. We will show that (4) implies $G(\lambda) = 0$ a.e. on E and hence $g \in CB(W)$ which proves the "only if" part of the theorem.

Let $s(t)$ be any complex-valued function on the real line such that $s(t) \in L_1(m)$ and $ts(t) \in L_1(m)$, and let $S(\lambda)$ be its Fourier transform. Then since $G[H_n-1]$ and GH_n' are in $L_2(m) \cap L_1(m)$, Fubini's theorem implies that for all $n = 1,2,\ldots$

$$0 = \int_{-\infty}^{\infty} s(t)a^*(t)dt = i\int_E G^*(\lambda)\{S'(\lambda)[H_n(\lambda) - 1] + S(\lambda)H_n'(\lambda)\}d\lambda$$

which can be written in the form

(5) $$\int_E G^*(\lambda)\,\frac{d}{d\lambda}\{S(\lambda)[H_n(\lambda) - 1]\}d\lambda = 0 \ .$$

Now choose $s(t)$ to be real, symmetric and such that $s(t) \in L_1(m)$, $ts(t) \in L_1(m)$ and $S(\lambda) = 1$ for $\lambda \in E$. This is certainly possible and in fact $S(\lambda)$ can be taken to be real, symmetric and infinitely differentiable function with support $[-(W+\delta+\varepsilon), (W+\delta+\varepsilon)]$, $\varepsilon > 0$. For such a function $s(t)$, and using the decomposition of $G = G_1 + iG_2$ into its real and imaginary part, we have from (5) that for all $n=1,2,\ldots$ and $i=1,2$,

$$\int_E G_i(\lambda)H_n'(\lambda)d\lambda = 0 \ .$$

Note that for $i = 1$ the integrand is an odd function since G_1 is even and H_n' is odd and thus the integral is equal to zero automatically. For $i = 2$ the integrand is an even function, since G_2 and H_n' are odd, and hence for $n = 1,2,\ldots$

$$\int_W^{W+\delta} G_2(\lambda)H_n'(\lambda)d\lambda = 0 .$$

Since $H_n(\lambda) = H^n(\lambda)$ and $H(\lambda) = 1 - \frac{\lambda-W}{\delta}$ on $[W,W+\delta]$, the change of variable $u = H(\lambda)$ gives that for $n = 1,2,\ldots$

$$\int_0^1 G_2(W+\delta-\delta u)u^{n-1}du = 0 .$$

Since $G_2(W+\delta-\delta u)\epsilon L_2([0,1],m)$ and the set of functions $\{u^k\}_{k=0}^{\infty}$ is complete [2, Theorem 11.2.1] in $L_2([0,1],m)$, it follows that $G_2(W+\delta-\delta u) = 0$ a.e. on $[0,1]$ and thus

$$G_2(\lambda) = 0 \text{ a.e. for } \lambda\epsilon E .$$

Next choose $s(t)$ such that $s(t)\epsilon L_1(m)$, $ts(t)\epsilon L_1(m)$ and $S(\lambda) = \operatorname{sgn} \lambda$ for $\lambda\epsilon E$. This can be done as follows: Let $\Phi(\lambda)$ be a real, symmetric, infinitely differentiable function with support $[-\frac{\delta}{2} - \varepsilon, \frac{\delta}{2} + \varepsilon]$, for some $\varepsilon > 0$, such that $\Phi(\lambda) = 1$ on $[-\frac{\delta}{2}, \frac{\delta}{2}]$; such functions are known to exist. Now define S by

$$S(\lambda) = \Phi(\lambda-W-\frac{\delta}{2}) - \Phi(\lambda+W+\frac{\delta}{2}) .$$

Then $S(\lambda)$ is a real, odd, infinitely differentiable function with support $[-(W+\delta+\varepsilon), (W+\delta+\varepsilon)]$ such that $S(\lambda) = \operatorname{sgn} \lambda$ for $\lambda\epsilon E$. It is then clear that $s(t)\epsilon L_1(m)$, $ts(t)\epsilon L_1(m)$, and it follows from (5), since $G_2 = 0$ a.e. on E and G_1, H_n are even, that for $n = 1,2,\ldots$

$$\int_W^{W+\delta} G_1(\lambda)[H_n(\lambda) - 1]'d\lambda = 0$$

or equivalently

$$\int_W^{W+\delta} G_1(\lambda)H_n'(\lambda)d\lambda = 0 .$$

As for G_2, it now follows that

$$G_1(\lambda) = 0 \quad \text{a.e. for } \lambda \in E .$$

Thus $G(\lambda) = 0$ a.e. on E and hence $G(\lambda) = 0$ a.e. for $|\lambda| > W$, i.e. $g \in CB(W)$.

(b) Conversely, let $f(t) = f(0) + tg(t)$ where $g \in CB(W)$. We show that $f \in B(W,\delta)$ for all $\delta > 0$. Indeed, it is clear that $f \in L_2(\mu)$ and as in part (a) we have

$$(f*h)(t) = f(0) \int_{-\infty}^{\infty} h(\tau)d\tau + \int_{-\infty}^{\infty} (t-\tau)g(t-\tau)h(\tau)d\tau$$

$$= f(0) + \frac{1}{2\pi} \int_{-W}^{W} G(\lambda)e^{it\lambda}\{tH(\lambda) - iH'(\lambda)\}d\lambda$$

$$= f(0) + \frac{t}{2\pi} \int_{-W}^{W} G(\lambda)e^{it\lambda}d\lambda = f(0) + tg(t) = f(t). \qquad \square$$

It follows from Theorem 1 that the class $B(W;\delta)$ is in fact *independent* of $\delta > 0$ and we shall therefore denote it by $B(W)$ and call functions in $B(W)$ "bandlimited to W."

Zakai [3] defines the bandwidth $W_0(f)$ of a function $f \in B(W)$ as the smallest A such that $[f(t) - f(0)]/t$ is conventionally bandlimited to A. Note that $W_0(f)$ depends on f and it is reasonable to define the bandwidth W_0 of the class $B(W)$ by

$$W_0 = \sup_{f \in B(W)} W_0(f) .$$

As a consequence of Theorem 1 we have $W_0(f) \le W$ and $W_0 = W$; whereas from [3] one can only conclude $W_0 \le W + \delta$. This determination of the bandwidth of the class $B(W)$ results in a sampling representation (Theorem 2) with slower rates than those in [3].

We note that if $f \epsilon B(W)$, then f can be extended to the complex plane $z = t + i\sigma$ via (2) as in [3], i.e. $f(z) = \int_{-\infty}^{\infty} f(\tau)h(z-\tau)d\tau$, and $f(z)$ is then entire. Theorem 1 offers an alternative method of extension, i.e.,

$$f(z) = f(0) + \frac{z}{2\pi}\int_{-W}^{W} G(\lambda)e^{i\lambda z}d\lambda$$

from which an exponential bound for $f(z)$ can be easily obtained which is independent of δ in contrast to the bound obtained in [3] via the convolution integral.

A direct consequence of the characterization of $B(W)$ provided in Theorem 1 is the following sampling theorem whose proof is carried out as in [3] and is thus omitted.

THEOREM 2. For all $f \epsilon B(W)$ and $0 < \tau < \frac{\pi}{W}$ we have

$$f(z) = \sum_{n=-\infty}^{\infty} f(n\tau) \frac{\sin[(\pi/\tau)(z-n\tau)]}{(\pi/\tau)(z-n\tau)}$$

and the convergence is uniform in any bounded region of the z-plane.

Another consequence of Theorem 1 is that if $f \epsilon B(W)$ then f is reproduced via a convolution integral $f = f * \phi$, for a variety of kernels ϕ whose Fourier transforms are essentially arbitrary outside $[-W,W]$. In fact we have the following characterization of such kernels: Let K be the class of all complex-valued functions ϕ on the real line such that

$$\phi(t) \epsilon L_1(m) \quad \text{and} \quad t\phi(t) \epsilon L_1(m) \cup L_2(m) .$$

These are the weakest properties of ϕ such that the convolution $f * \phi$ with $f \in B(W)$ be well defined.

THEOREM 3. Let $\phi \in K$. Then $f = f * \phi$ for all $f \in B(W)$ if and only if the Fourier transform Φ of ϕ satisfies

$$\Phi(\lambda) = 1 \quad \text{for} \quad \lambda \in [-W,W] .$$

<u>Proof</u>. (a) Assume that $\phi \in K$ and $f = f * \phi$ for all $f \in B(W)$. By Theorem 1, $f(t) = f(0) + tg(t)$ where $g \in CB(W)$ and thus

$$(f*\phi)(t) = f(0)\int_{-\infty}^{\infty} \phi(\tau)d\tau + t\int_{-\infty}^{\infty} g(t-\tau)\phi(\tau)d\tau - \int_{-\infty}^{\infty} g(t-\tau)\tau\phi(\tau)d\tau .$$

All integrals are well defined; the first since $\phi \in L_1(m)$, the second is an $L_2(m)$ function as a convolution of $g \in L_2(m)$ with $\phi \in L_1(m)$, and the third is either a function in $L_2(\mu)$ as a convolution of $g \in L_2(m)$ with $\tau\phi(\tau) \in L_1(m)$ or a convolution of $g \in L_2(m)$ and $\tau\phi(\tau) \in L_2(m)$. We then have

$$(f*\phi)(t) = f(0)\Phi(0) + \frac{1}{2\pi}\int_{-W}^{W} G(\lambda)e^{i\lambda t}\{t\Phi^*(\lambda) - i[\Phi'(\lambda)]^*\}d\lambda .$$

Since $(f*\phi)(t) = f(t) = f(0) + \frac{t}{2\pi}\int_{-W}^{W} G(\lambda)e^{it\lambda}d\lambda$, we finally have

$$f(0)[1-\Phi(0)] + \frac{1}{2\pi}\int_{-W}^{W} G(\lambda)e^{it\lambda}\{t[1-\Phi^*(\lambda)] + i[\Phi'(\lambda)]^*\}d\lambda = 0 \tag{6}$$

for all t and all $f \in B(W)$, i.e., for all complex numbers $f(0)$ and $G \in L_2([-W,W],m)$. For $G = 0$ we obtain $\Phi(0) = 1$. Then the second term in (6) is equal to 0 for all t and all $G \in L_2([-W,W],m)$. It follows that for all t

$$t[1-\Phi^*(\lambda)] + i[\Phi'(\lambda)]^* = 0 \quad \text{a.e. on } [-W,W]$$

and thus $\Phi(\lambda) = 1$ a.e. on $[-W,W]$. Since Φ is continuous, $\Phi(\lambda) = 1$ for $\lambda \epsilon [-W,W]$.

(b) The sufficiency follows from the fact that for $\phi \epsilon K$ and $f \epsilon B(W)$, $f - f*\phi$ is given by the left hand side of (6), and the latter is equal to 0 when $\Phi(\lambda) = 1$ for $\lambda \epsilon [-W,W]$. □

The method of the proof of Theorem 1 suggests that kernels other than (1) can be used to define the class $B(W)$. It appears that the basic property that H must satisfy is $H(\lambda) = 1$ on $[-W,W]$, and that the behavior of H outside $[-W,W]$ is essentially arbitrary. In the following theorem we give such a class of kernels. Denote by H the class of functions h defined as the inverse Fourier transform of real, symmetric, twice continuously differentiable functions H with support $[-W-\delta, W+\delta]$, for some $\delta > 0$, and such that $H(\lambda) = 1$ for $\lambda \epsilon [-W,W]$. Then

$$h(z) = \frac{1}{2\pi} \int_{-(W+\delta)}^{W+\delta} H(\lambda) e^{i\lambda z} d\lambda \quad , \quad z = t+i\sigma,$$

is entire and for some finite constant C,

$$|h(z)| \leq C \frac{e^{(W+\delta)|\sigma|}}{1+t^2} \quad . \tag{7}$$

This is seen as follows. Integrating by parts we have

$$(-iz)^2 h(z) = \frac{1}{2\pi} \int_{-(W+\delta)}^{W+\delta} H^{(2)}(\lambda) e^{i\lambda t} d\lambda$$

and hence

$$(1+|z|^2)|h(z)| \le e^{(W+\delta)|\sigma|}\{\frac{1}{2\pi}\int_{-(W+\delta)}^{W+\delta}(|H(\lambda)| + |H^{(2)}(\lambda)|)d\lambda\}.$$

For $h \in H$ define the class $B(W;h)$ by

$$B(W;h) = \{f \in L_2(\mu) : f = f*h\} .$$

Then the class $B(W;h)$ has properties similar to those of the class $B(W)$ and under an additional assumption on h we have the following

THEOREM 4. If $h \in H$ and $H(\lambda)$ is strictly decreasing on $[W,W+\delta]$, then $B(W;h) = B(W)$.

Proof. (a) We first show that $B(W;h) \subset B(W)$. Let $f \in B(W;h)$. As in [3, Lemma 2 and Theorem 1], it follows from (7) that $f(t) = f(0) + tg(t)$ where $g \in CB(W+\delta)$. In view of the definition of $B(W;h)$ and the fact that h is real, it suffices to consider real f and prove that $f \in B(W)$. Now as in part (a) of the proof of Theorem 1 we obtain that for $n = 1,2,\ldots$ and $i = 1,2$

$$\int_W^{W+\delta} G_i(\lambda)[H^n(\lambda)]'d\lambda = 0$$

where $G = G_1 + iG_2$ is the Fourier transform of G. Since $H(\lambda)$ is strictly monotone decreasing on $[W,W+\delta]$ with $H(W) = 1$ and $H(W+\delta) = 0$, the change of variable $u = H(\lambda)$ gives that for $i = 1,2$, and $n = 1,2,\ldots$

$$\int_0^1 G_i[H^{-1}(u)]u^{n-1}du = 0 .$$

Note that $\int_0^1 G_i^2[H^{-1}(u)]du = -\int_W^{W+\delta} G_i^2(\lambda)H'(\lambda)d\lambda < \infty$ since H' is continuous and hence bounded on $[W,W+\delta]$, and $G_i \in L_2([-W-\delta, W+\delta],m)$. Hence

$G_i[H^{-1}(u)]\epsilon L_2([0,1],m)$, and since the set $\{u^k\}_{k=0}^{\infty}$ is complete in $L_2([0,1],m)$, it follows that $G_i[H^{-1}(u)] = 0$ a.e. on $[0,1]$ and thus $G_i(\lambda) = 0$ a.e. on $[W,W+\delta]$. Thus $G = 0$ a.e. on $[-W-\delta,-W] \cup [W,W+\delta]$, $g\epsilon CB(W)$ and hence $f\epsilon B(W)$.

(b) The inclusion $B(W) \subset B(W;h)$ is proved as in part (b) of the proof of Theorem 1. □

The class H of functions used in Theorem 4 does not include the function h given by (1a) which was originally used in defining the class $B(W)$. However, the essential property needed for the proof of Theorem 4 is the inequality (7) which both (1a) and functions in H satisfy. Therefore it should be pointed out that Theorem 4 is valid with the class H replaced by the class H' of functions h defined as the inverse Fourier transform of a real symmetric function H having bounded derivative, with $H(\lambda) = 1$ for $\lambda\epsilon[-W,W]$ and such that the inequality (7) is satisfied.

It was noted earlier that if $f\epsilon L_2(m)$ and $f\epsilon CB(W)$ then $f\epsilon B(W)$. The question arises whether the only functions in $B(W)$ that are in $L_2(m)$ are those in $CB(W)$. It is also of interest to study $CB(W)$ as a subset of $B(W)$ and to characterize functions $f\epsilon CB(W)$ in terms of the representation of Theorem 1. These questions are answered in the following

THEOREM 5. (i) If $f\epsilon L_2(m)$, then $f\epsilon B(W)$ if and only if $f\epsilon CB(W)$. Also $CB(W)$ is dense in $B(W)$ with respect to the metric of $L_2(\mu)$.

(ii) Let $f\epsilon B(W)$ have the representation of Theorem 1, $f(t) = f(0) + tg(t)$, where $g\epsilon CB(W)$ with Fourier transform G. Then $f\epsilon CB(W)$ if and only if for some $\Phi\epsilon L_2([-W,W],m)$

$$G(\lambda) = \begin{cases} \int_{-W}^{\lambda} \Phi(u)du & \text{a.e. on } (-W,0) \\ \int_{\lambda}^{W} \Phi(u)du & \text{a.e. on } (0,W) \end{cases}$$

and

$$f(0) = \frac{1}{2\pi i} \int_{-W}^{W} \Phi(u)du .$$

Proof. (i) Let $f \in L_2(m)$. Then $f = f*h$ is equivalent to $F(\lambda) = F(\lambda)H(\lambda)$ a.e. where F is the Fourier transform of f and H is given by (1). Hence $f \in B(W)$ is equivalent to $F(\lambda) = 0$ a.e. outside $[-(W+\delta), W+\delta]$ for all $\delta > 0$ and thus to $F(\lambda) = 0$ a.e. outside $[-W,W]$, i.e., $f \in CB(W)$. It follows that $B(W) \cap L_2(m) = CB(W)$.

Next we will show that the closure of $CB(W)$ in $L_2(\mu)$ is equal to $B(W)$. It suffices to show that $f \in B(W)$ and $f \perp CB(W)$ implies $f = 0$. Assume that $f \in B(W)$ and $f \perp CB(W)$. Then $f \perp q$ for all $q \in CB(W)$, i.e.,

$$0 = \int_{-\infty}^{\infty} f(t)q^*(t) \frac{dt}{1+t^2} .$$

Since $q(t) = \frac{1}{2\pi} \int_{-W}^{W} Q(\lambda)e^{it\lambda}d\lambda$, it follows from Fubini's theorem that for all $Q \in L_2([-W,W],m)$ we have

$$0 = \frac{1}{2\pi} \int_{-W}^{W} Q^*(\lambda)\left(\int_{-\infty}^{\infty} f(t)e^{-it\lambda} \frac{dt}{1+t^2}\right) d\lambda .$$

Now $f \in L_2(\mu)$ implies $\frac{f(t)}{1+t^2} \in L_1(m) \cap L_2(m)$; hence the function inside the parentheses is continuous and in $L_2(m)$ and it follows that

$$\int_{-\infty}^{\infty} f(t)e^{-it\lambda} \frac{dt}{1+t^2} = 0 \quad \text{for all } \lambda \in [-W,W].$$

Since $f\epsilon B(W)$, then by Theorem 1, $f(t) = f(0) + tg(t)$ where $g\epsilon CB(W)$ with Fourier transform G. It follows that for all $\lambda\epsilon[-W,W]$

$$f(0)\int_{-\infty}^{\infty} \frac{1}{1+t^2} e^{-it\lambda}dt + \int_{-\infty}^{\infty} \frac{t}{1+t^2} g(t)e^{-it\lambda}dt = 0 .$$

Since $\frac{t}{1+t^2} \epsilon L_2(m)$ with Fourier transform $-i\pi \operatorname{sgn} \lambda\, e^{-|\lambda|}$, it follows by Parseval's theorem that

$$\int_{-W}^{W} e^{-|\lambda-u|}\operatorname{sgn}(\lambda-u)G(u)du = -i2\pi f(0)e^{-|\lambda|}, \qquad \lambda\epsilon[-W,W].$$

This integral equation can be written as

$$e^{-\lambda}\int_{-W}^{\lambda} e^{u}G(u)du - e^{\lambda}\int_{\lambda}^{W} e^{-u}G(u)du = ce^{-|\lambda|} , \qquad \lambda\epsilon[-W,W]$$

where we have set $c = -2i\pi f(0)$. Since all functions are absolutely continuous we obtain by differentiation

$$-e^{-\lambda}\int_{-W}^{\lambda} e^{u}G(u)du - e^{\lambda}\int_{\lambda}^{W} e^{-u}G(u)du + 2G(\lambda) = -ce^{-|\lambda|}\operatorname{sgn}(\lambda) \qquad \text{a.e. } \lambda\epsilon[-W,W].$$

It follows that on each of the intervals $(-W,0)$ and $(0,W)$, G is a.e. equal to an absolutely continuous function, and hence we can consider a modification of G which is absolutely continuous on $(-W,0)$ and $(0,W)$, and thus a.e. differentiable. By adding the last two equations we have

$$-2e^{\lambda}\int_{\lambda}^{W} e^{-u}G(u)du + 2G(\lambda) = ce^{-|\lambda|}(1-\operatorname{sgn} \lambda) \quad \text{a.e. } \lambda\epsilon[-W,W]$$

or equivalently

$$- \int_{\lambda}^{W} e^{-u} G(u)\,du + e^{-\lambda} G(\lambda) = \begin{cases} 0\,, & \lambda > 0 \\ c\,, & \lambda < 0 \end{cases} \qquad \text{a.e.} \quad \lambda \in [-W,W] \;.$$

By differentiation we have

$$e^{-\lambda} G'(\lambda) = 0 \quad \text{a.e.} \; \lambda \in [-W,W] \;.$$

Thus $G' = 0$ a.e. on $[-W,W]$, and since it is absolutely continuous on $(-W,0)$ and $(0,W)$, it follows that

$$G(\lambda) = \begin{cases} a & \lambda \in (-W,0) \\ b & \lambda \in (0,W) \end{cases} \;.$$

Now substituting G in the original integral equation we find (after some calculations)

$$[a + (e^{-W}-1)b - c]e^{\lambda} - (ae^{-W})e^{-\lambda} = 0 \qquad \text{for} \quad \lambda \in (-W,0)$$

$$(be^{-W})e^{\lambda} + [a(1-e^{-W}) - b - c]e^{-\lambda} = 0 \qquad \text{for} \quad \lambda \in (0,W) \;.$$

Hence $a = 0$, $b = 0$, $c = 0$. It follows that $G = 0$, hence $g = 0$, and $f(0) = 0$ and therefore $f = 0$, which completes the proof.

(ii) First assume that $f \in CB(W)$ with Fourier transform F. Then $tg(t) = f(t) - f(0)$ for all t can be written as

$$\frac{t}{2\pi} \int_{-W}^{W} G(\lambda) e^{it\lambda} d\lambda = \frac{1}{2\pi} \int_{-W}^{W} F(\lambda)(e^{it\lambda} - 1)dt \;.$$

Using $e^{it\lambda} - 1 = it \int_0^{\lambda} e^{itu} du = it \int_{-W}^{W} e^{itu} \chi_{(0,\lambda)}(u)du$ and Fubini's theorem we have that for all $t \neq 0$

$$\int_{-W}^{W} G(\lambda) e^{it\lambda} d\lambda = i \int_{-W}^{W} \left(\int_{-W}^{W} F(\lambda) \chi_{(0,\lambda)}(u) d\lambda \right) e^{itu} du \;.$$

Since both integrals are continuous functions in t, equality holds for all t and thus

$$G(\lambda) = i\int_{-W}^{W} F(u)\chi_{(0,u)}(\lambda)du \quad \text{a.e. on } [-W,W]$$

from which the desired expression for G follows with $\Phi = iF$. We also have

$$\frac{1}{2\pi i}\int_{-W}^{W}\Phi(u)du = \frac{1}{2\pi}\int_{-W}^{W}F(u)du = f(0).$$

Conversely, assume that $f\epsilon B(W)$ with $f(0)$ and g (i.e. G) satisfying the expressions in part (ii) of the theorem. Then, as before, $G(\lambda) = \int_{-W}^{W}\Phi(u)\chi_{(0,u)}(\lambda)du$ a.e. on $[-W,W]$ and

$$tg(t) = \frac{t}{2\pi}\int_{-W}^{W}G(\lambda)e^{it\lambda}d\lambda = \frac{1}{2\pi i}\int_{-W}^{W}\Phi(u)(e^{iut}-1)du$$

and finally

$$f(t) = f(0) + tg(t) = \frac{1}{2\pi i}\int_{-W}^{W}\Phi(u)e^{iut}du\ ,$$

and thus $f\epsilon CB(W)$. □

It should be remarked that $B(W)^{\perp}$, the orthogonal complement of $B(W)$ in $L_2(\mu)$, can be characterized as follows: Let $CB(W)^{\perp}$ be the orthogonal complement of $CB(W)$ in $L_2(m)$; then it can be seen from the fact that $CB(W)$ is dense in $B(W)$ (Theorem 5 i) that $f\epsilon B(W)^{\perp}$ if and only if $\frac{f(t)}{1+t^2}\ \epsilon\ CB(W)^{\perp}$, i.e., if and only if the L_2-Fourier transform of $\frac{f(t)}{1+t^2}$ vanishes on $[-W,W]$.

III. Bandlimited Stochastic Processes

Let $X = \{X(t,\omega), -\infty < t < \infty\}$ be a second order, mean square continuous stochastic process on a probability space (Ω, F, P) with correlation function $R(t,s)$. Without loss of generality the process X may be assumed to be measurable. According to Zakai's definition [3] and the remark following Theorem 1, X is called "bandlimited to W" if

(8) $$\int_{-\infty}^{\infty} \frac{R(t,t)}{1+t^2}\, dt < \infty$$

and with probability one its sample functions are bandlimited to W ($X(\cdot,\omega) \in B(W)$ a.s.). It is shown in [3] that wide sense stationary and harmonizable processes that are "conventionally bandlimited to W," i.e., whose spectral measure is concentrated on $[-W,W]$ and $[-W,W]\times[-W,W]$ respectively, are bandlimited to W.

The characterization of processes bandlimited to W [3, Theorem 5] and their sampling representation [3, p. 154] can be stated as follows in view of Theorems 1 and 2.

THEOREM 6. (i) A second order process X is bandlimited to W if and only if its correlation function R is continuous, satisfies (8) and $R(t,\cdot)$ is bandlimited to W for all t.

(ii) If X is bandlimited to W and $0 < \tau < \frac{\pi}{W}$, then with probability one

$$X(t,\omega) = \sum_{n=-\infty}^{\infty} X(n\tau,\omega) \frac{\sin[(\pi/\tau)(t-n\tau)]}{(\pi/\tau)(t-n\tau)}$$

for all t.

Part (i) of Theorem 6 characterizes the correlation function of bandlimited processes. A characterization in terms of a representation of the process itself, similar to that of Theorem 1, can be obtained as follows. Zakai proves in [3, Theorem 6] that bandlimited processes have properties similar to those stated in Lemma 1 for bandlimited functions. Using Theorem 1, these properties can be strengthened to the following

THEOREM 7. X is bandlimited to W if and only if

$$X(t,\omega) = X(0,\omega) + tY(t,\omega)$$

for all t and almost all ω, where Y is a harmonizable process conventionally bandlimited to W with correlation function R_Y such that $\int_{-\infty}^{\infty} R_Y(t,t)dt < \infty$.

Finally it is shown that the concepts of "conventionally bandlimited" and "bandlimited" process coincide for the class of wide sense stationary and harmonizable processes; also the conventionally bandlimited processes are characterized in terms of the representation of Theorem 7.

THEOREM 8. (i) Let X be a harmonizable or a mean square continuous, wide sense stationary process. Then X is bandlimited to W if and only if it is conventionally bandlimited to W.

(ii) Let X be bandlimited to W and have the representation $X(t) = X(0) + tY(t)$ of Theorem 7. Then X is wide sense stationary conventionally bandlimited to W if and only if for some finite measure μ on the Borel subsets of $[-W,W]$,

$$E[Y(t)Y^*(s)] = \int_{-W}^{W} \frac{e^{it\lambda}-1}{t} \frac{e^{-is\lambda}-1}{s} d\mu(\lambda)$$

$$E[Y(t)X^*(0)] = \int_{-W}^{W} \frac{e^{it\lambda}-1}{t} d\mu(\lambda)$$

$$E[|X(0)|^2] = \mu\{[-W,W]\}$$

and then μ is the spectral measure of X. Similar necessary and sufficient conditions can be expressed for X to be harmonizable conventionally bandlimited to W.

Note that the first two expressions in (ii) of Theorem 8 can be written in an equivalent form exhibiting the harmonizable character of Y, i.e.,

$$E[Y(t)Y^*(s)] = \iint_{-W}^{W} e^{i(t\tau-s\sigma)}\phi(\tau,\sigma)d\tau d\sigma$$

$$E[Y(t)X^*(0)] = \int_{-W}^{W} e^{it\tau}\psi(\tau)d\tau$$

where ϕ and ψ are given by

$$\phi(\tau,\sigma) = \begin{cases} \mu\{[\max(\tau,\sigma),W]\} & \text{for } 0 \le \tau,\ \sigma \le W \\ \mu\{[-W,\min(\tau,\sigma))\} & \text{for } -W \le \tau,\ \sigma < 0 \\ 0 & \text{for } -W \le \tau < 0 \le \sigma \le W \\ & \text{and } -W \le \sigma < 0 \le \tau \le W \end{cases}$$

$$\psi(\tau) = \begin{cases} i\ \mu\{[\tau,W]\} & \text{for } 0 \le \tau \le W \\ i\ \mu\{[-W,\tau)\} & \text{for } -W \le \tau < 0 \end{cases} .$$

<u>Proof</u>. (i) We will give the proof for X a harmonizable process (when X is stationary the proof is even simpler). Then

$$R(t,s) = \iint_{-\infty}^{\infty} e^{i(tu-sv)} d\mu(u,v)$$

where μ is the two-dimensional spectral measure of X (μ is a finite, complex measure, nonnegative definite on Borel measurable rectangles). Assume first that X is bandlimited to W. From Theorem 6 i it follows that $R(t,\cdot)\epsilon B(W)$ for all t, and since $R(t,s) = R^*(s,t)$ then $R(\cdot,t)\epsilon B(W)$. Hence for all t,s,

$$R(t,s) = \iint_{-\infty}^{\infty} R(u,v)h(t-u)h(s-v)dudv$$

where h is given by (1a). Replacing R on the right hand side by its spectral representation and using Fubini's theorem, we obtain

$$R(t,s) = \iint_{-\infty}^{\infty} e^{i(t\tau-s\sigma)}H(\tau)H(\sigma)d\mu(\tau,\sigma)$$

where $H(\tau) = H(\tau;W,\delta)$ is given by (1). Since this is true for all $\delta > 0$ and $\lim_{\delta\to 0} H(\tau;W,\delta) = \chi_{[-W,W]}(\tau)$, it follows from the bounded convergence theorem that

$$R(t,s) = \int_{-W}^{W}\int_{-W}^{W} e^{i(t\tau-s\sigma)}d\mu(\tau,\sigma)$$

and thus X is conventionally bandlimited to W. Conversely, if X is conventionally bandlimited to W then it is easily checked that $R(t,s) = \{R(t,\cdot) * h(\cdot)\}(s)$ and (8) follows from $|R(t,t)| \le \mu\{[-W,W] \times [-W,W]\}$; hence X is bandlimited.

The proof of (ii) is straightforward and is thus omitted. □

References

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[3] M. Zakai, Band-limited functions and the sampling theorem, Information and Control, 8 (1965), pp. 143-158.