

¹Research sponsored in part by the Office of Naval Research under Contract N00014-67A-0321-0002, Task NR042214 with the University of North Carolina.

²Replaces report No. 908, "A limit theorem for thinning of point processes".

LIMITS OF COMPOUND AND THINNED POINT PROCESSES¹

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Institute of Statistics Mimeo Series No. 923²

May 1974

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Abstract

Let $\eta = \sum_j \delta_{T_j}$ be a point process on some space S and let $\beta, \beta_1, \beta_2, \dots$ be identically distributed non-negative random variables which are mutually independent and independent of η . We can then form the compound point process $\xi = \sum_j \beta_j \delta_{T_j}$ which is a random measure on S . The purpose of this paper is to study the limiting behavior of ξ as $\beta \xrightarrow{d} 0$. In the particular case when β takes the values 1 and 0 with probabilities p and $1-p$ respectively, ξ becomes a p -thinning of η and our theorems contain some classical results by Rényi and others on the thinnings of a fixed process, as well as a characterization by Mecke of the class of subordinated Poisson processes.

COMPOUND AND THINNED POINT PROCESSES; INFINITELY DIVISIBLE RANDOM MEASURES; SUBORDINATED POISSON PROCESSES; CONVERGENCE IN DISTRIBUTION; REGULARITY AND DIFFUSENESS

1. Introduction

Let S be a locally compact second countable Hausdorff space and let \mathcal{B} be the ring of bounded Borel sets in S . By a *random measure* ξ on S we shall mean a mapping of some probability space (Ω, \mathcal{A}, P)

Research sponsored in part by the Office of Naval Research under Contract N00014-67A-0321-0002, Task NRO42214 with the University of North Carolina

into the space $M = M(S)$ of Radon measures on (S, \mathcal{B}) such that ξB is a random variable for each $B \in \mathcal{B}$. When ξ is a.s. confined to the subspace $N \subset M$ of integer-valued measures, it will also be called a *point process*.

Let us write F for the class of measurable functions $S \rightarrow R_+$ and F_c for its subclass of continuous functions with bounded support. *Vague convergence*, $\mu_n \xrightarrow{V} \mu$, in M means that $\mu_n f \rightarrow \mu f$ for each $f \in F_c$ while *weak convergence*, $\mu_n \xrightarrow{W} \mu$, defined for bounded measures on M , means that $\mu_n \xrightarrow{V} \mu$ and $\mu_n S \rightarrow \mu S$. (Here $\mu f = \int f(s) \mu(ds)$.) Convergence in distribution [2] of random elements in topological spaces is written \xrightarrow{d} . For random measures, the underlying topology in M is taken to be the vague one. It is known [11] that a sequence $\{\xi_n\}$ of random measures is tight, and hence the sequence $\{P\xi_n^{-1}\}$ of corresponding distributions relatively compact [2], iff $\{\xi_n B\}$ is tight for each $B \in \mathcal{B}$. Furthermore [4], $\xi_n \xrightarrow{d} \xi$ iff

$$(\xi_n B_1, \dots, \xi_n B_k) \xrightarrow{d} (\xi B_1, \dots, \xi B_k), \quad B_1, \dots, B_k \in \mathcal{B}_\xi, \quad k \in \mathbb{N}, \quad (1.1)$$

where $\mathcal{B}_\xi = \{B \in \mathcal{B} : \xi \partial B = 0 \text{ a.s.}\}$, and also iff $\xi_n f \xrightarrow{d} \xi f$, $f \in F_c$.

In terms of L-transforms ($L =$ Laplace), defined for arbitrary random measures ξ by $L_\xi(f) = Ee^{-\xi f}$, $f \in F$, the latter criterion can be written (cf. [11])

$$L_{\xi_n}(f) \rightarrow L_\xi(f), \quad f \in F_c. \quad (1.2)$$

It may be shown that any point process η on S has the representation

$$\eta = \sum_{j=1}^{\nu} \delta_{T_j}, \quad (1.3)$$

where ν is a $\overline{\mathbb{Z}}_+$ -valued random variable while $\{T_j\}$ is a sequence of random elements in S without any limit point. (Here $\delta_s \in N$ is defined for $s \in S$ by $\delta_s B = 1_B(s)$, $B \in \mathcal{B}$, where 1_B denotes the indicator of B .) Now suppose that $\beta, \beta_1, \beta_2, \dots$ are identically distributed \mathbb{R}_+ -valued random variables which are mutually independent and independent of η , and define

$$\xi = \sum_{j=1}^{\nu} \beta_j \delta_{T_j} .$$

Since $\xi B = \sum_j \beta_j 1_B(T_j) < \infty$, $B \in \mathcal{B}$, it is easily seen that ξ is a random measure on S . We shall say that ξ is a *compound point process* determined by η and β , and we write $\xi \stackrel{d}{=} C(\eta, \beta)$ for brevity. In the particular case when β only takes the values 1 and 0, with probabilities p and $1-p$ respectively, ξ will be called a *p-thinning* of η .

By the assumed independence of η and $\{\beta_j\}$, the L-transform of ξ may be calculated by means of Fubini's theorem, i.e. we may first consider η as non-random, and then perform mixing with respect to its distribution. Writing $\phi = L_\beta$, we obtain for $f \in F$ and non-random

$$\begin{aligned} \eta = \mu &= \sum_j \delta_{t_j} \\ E e^{-\xi f} &= E \exp[-\sum_j \beta_j f(t_j)] = \Pi E \exp[-\beta_j f(t_j)] = \Pi \phi \circ f(t_j) \\ &= \exp \sum_j \log \phi \circ f(t_j) = \exp(\mu \log \phi \circ f) , \end{aligned}$$

and hence in general, by mixing,

$$L_\xi(f) = E e^{-\xi f} = E \exp(\eta \log \phi \circ f) = L_\eta(-\log \phi \circ f), \quad f \in F . \quad (1.4)$$

This shows in particular that $P\xi^{-1}$ does not depend on the representation

(1.3) (which is not unique, even apart from the order of terms). The above mixing procedure also provides an alternative way of *defining* a $\xi \stackrel{d}{=} C(\eta, \beta)$, (cf. [7] page 359; use 1.6.2 in [6] to check measurability). In the particular case when ξ is a p -thinning of η , (1.4) takes the form (cf. [8])

$$L_{\xi}(f) = L_{\eta}(-\log[1 - (1-p)e^{-f}]), \quad f \in F. \quad (1.5)$$

The main purpose of this paper is to study the limiting behavior as $\beta \stackrel{d}{\rightarrow} 0$ of compound point processes. We shall be able (in Section 3) to describe the class of possible limits, and under a mild regularity condition (which is not needed in the thinning case), necessary and sufficient conditions will be given for convergence in distribution to a specified member of this class. The results take a particularly simple form for thinnings. In this case, the class of limits consists of all subordinated Poisson (SP-) processes (often called doubly stochastic Poisson, or simply Cox processes). If, moreover, ξ_n is a p_n -thinning of η_n for each $n \in \mathbb{N}$ and if $p_n \stackrel{d}{\rightarrow} 0$, then $\xi_n \stackrel{d}{\rightarrow} SP(\eta)$ (the SP-process directed by η) iff $p_n \eta_n \stackrel{d}{\rightarrow} \eta$. This proposition contains as particular cases the classical thinning results by Rényi (1956), Nawrotzki (1962), Belyaev (1963) and Goldman (1967) (cf. Theorem 6.10.1 in [6]), who all consider p -thinnings of some fixed point process and change the "time" scale by the factor p^{-1} . It also contains the following interesting characterization by Mecke (1968) (cf. Theorem 5.6.12 in [6]) of the class of SP-processes: A point process is in SP iff it is a p -thinning for each $p \in (0,1]$.

A secondary result of some independent interest is Lemma 4 which indicates how the classical regularity conditions ensuring a point process to be simple (see e.g. Satz 1.3.5 in [6] and Theorem 2.5 in [4]) have analogues ensuring an arbitrary random measure to be diffuse. Even the converse proposition (Dobrushin's lemma) carries over to this context.

2. Preliminaries

In this section we shall introduce the class of limiting random measures and prove some auxiliary results. Throughout the paper we use g to denote the function $1 - e^{-x}$ on R_+ and write $R'_+ = R_+ \setminus \{0\}$. For any measure λ on R_+ or R'_+ we define a new measure $g\lambda$ on R_+ by $(g\lambda)(dx) = g(x)\lambda(dx)$.

Our first lemma is essentially contained in Theorem 3.1 of [4] (which needs correction: condition (i) must also hold with \liminf instead of \limsup).

Lemma 1. Let $\alpha \in R_+$ and $\lambda \in M(R'_+)$ with $\alpha + \lambda g = 1$, and define

$$\psi(t) = \alpha t + \int_{R'_+} (1 - e^{-tx}) \lambda(dx), \quad t \in R_+. \quad (2.1)$$

Further suppose that β_1, β_2, \dots are R_+ -valued random variables such that $\beta_n \xrightarrow{d} 0$, and put

$$\phi_n = L_{\beta_n}, \quad c_n = Eg(\beta_n) = 1 - \phi_n(1), \quad n \in N.$$

Then $-c_n^{-1} \log \phi_n \rightarrow \psi$ uniformly on bounded intervals iff

$$c_n^{-1} g(P\beta_n^{-1}) \xrightarrow{w} \alpha \delta_0 + g\lambda.$$

Let α and λ be such as in the lemma and consider an arbitrary measure $\mu \in M$. Then the measures $\alpha\mu \in M$ and $\lambda \times \mu \in M(\mathbb{R}_+^1 \times S)$ may serve as the canonical measures of an infinitely divisible random measure ξ on S with (μ -homogeneous) independent increments (see e.g. [4]), the L-transform of which is given for $f \in F$ by

$$\begin{aligned} -\log Ee^{-\xi f} &= \alpha\mu f + \int_{\mathbb{R}_+^1 \times S} (1 - e^{-xf(t)}) (\lambda \times \mu)(dxdt) \\ &= \int_S \left\{ \alpha f(t) + \int_{\mathbb{R}_+^1} (1 - e^{-xf(t)}) \lambda(dx) \right\} \mu(dt) = \mu(\psi \circ f). \end{aligned}$$

Again we may consider $\mu = \eta$ as a random measure and mix with respect to its distribution (check the measurability by means of 1.6.2 in [6]), to obtain

$$L_\xi(f) = Ee^{-\xi f} = Ee^{-\eta(\psi \circ f)} = L_\eta(\psi \circ f), \quad f \in F. \quad (2.2)$$

The distribution of ξ being determined by $P\eta^{-1}$, α and λ , we shall write for brevity $\xi \stackrel{d}{=} S(\eta, \alpha, \lambda)$, (S for subordination). Note that we obtain $\psi = g$ and $\xi \stackrel{d}{=} SP(\eta)$ in the particular case when $\alpha = 0$ and $\lambda = \delta_1$. We shall need the following uniqueness result. For $C \in \mathcal{B}$, say that η is non-zero in C if $\eta C \stackrel{d}{\neq} 0$ and diffuse there is η has a.s. no atoms in C .

Lemma 2. If $\xi \stackrel{d}{=} S(\eta, \alpha, \lambda)$, where $\alpha + \lambda g = 1$ while η is known to be non-zero and diffuse in some region $C \in \mathcal{B}$, then $P\eta^{-1}$, α and λ are uniquely determined by $P\xi^{-1}$.

The proof of Lemma 2 is based on the following result which is obtained in the same way as Satz 5.6.9 in [6] by using 1.3.7 in place of 1.3.10.

Lemma 3. Let η be a diffuse random measure on S . Then $P\eta^{-1}$ is uniquely determined by the quantities $Ee^{-\eta B}$, $B \in \mathcal{B}$.

Proof of Lemma 2. By (2.2),

$$Ee^{-\xi B} = Ee^{-\eta B \psi(1)} = Ee^{-\eta B}, \quad B \in \mathcal{B}.$$

If we replace B here by $B \cap C$, it follows by Lemma 3 that $P\xi^{-1}$ determines the distribution of the restriction of η to C , and in particular $P(\eta C)^{-1}$ is unique. From (2.2) we further obtain

$$L_{\xi C}(t) = Ee^{-t\xi C} = Ee^{-\eta C \psi(t)} = L_{\eta C} \circ \psi(t), \quad t \in \mathbb{R}_+.$$

Since $\eta C \stackrel{d}{\neq} 0$, $L_{\eta C}$ has a unique inverse $L_{\eta C}^{-1}$ on $(0,1]$ and we obtain

$$\psi = L_{\eta C}^{-1} \circ L_{\xi C},$$

proving the uniqueness of ψ , and hence by Lemma 1 of α and λ .

Now it may be seen from (2.1) that ψ has a unique inverse ψ^{-1} on $[0, \|\psi\|)$, where $\|\cdot\|$ denotes the supremum norm. If $f \in F$ is such that $\|f\| < \|\psi\|$, we thus obtain from (2.2) $L_{\eta}(f) = L_{\xi}(\psi^{-1} \circ f)$, so

$$L_{\eta f}(t) = Ee^{-t\eta f} = L_{\xi}(\psi^{-1} \circ (tf)), \quad f \in F, \quad t \in [0, \frac{\|\psi\|}{\|f\|}).$$

By the uniqueness of analytic continuations it follows that $L_{\eta f}$ is unique, and hence so is $P\eta^{-1}$, since $f \in F$ was arbitrary.

To ensure diffuseness, we introduce regularity conditions as follows.

Let us say that η is *regular* in $C \in \mathcal{B}$ if there exists some array $\{C_{mj}\} \subset \mathcal{B}$ of finite partitions of C (one for each m) such that

$$\lim_{m \rightarrow \infty} \sum_j P\{\eta C_{mj} > \varepsilon\} = 0, \quad \varepsilon > 0.$$

More generally, if $\eta, \eta_1, \eta_2, \dots$ are random measures on S , we shall say that $\{\eta_n\}$ is η -regular in $C \in \mathcal{B}_\eta$ if there exists some array $\{C_{mj}\} \subset \mathcal{B}_\eta$ of partitions of C such that

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sum_j P\{\eta_n C_{mj} > \varepsilon\} = 0, \quad \varepsilon > 0.$$

Lemma 4. Let $\eta, \eta_1, \eta_2, \dots$ be random measures on S with $\eta_n \xrightarrow{d} \eta$, and suppose that η is regular in $C \in \mathcal{B}$ or that $\{\eta_n\}$ is η -regular in $C \in \mathcal{B}_\eta$. Then η is diffuse in C . The converses are also true if $E\eta C < \infty$.

Proof. The diffuseness of a regular η follows from the relation

$$P\{\max_{s \in C} \eta\{s\} > \varepsilon\} \leq P\{\max_j \eta C_{mj} > \varepsilon\} \leq \sum_j P\{\eta C_{mj} > \varepsilon\}.$$

We further obtain for η -regular $\{\eta_n\}$ with $\eta_n \xrightarrow{d} \eta$

$$\sum_j P\{\eta C_{mj} > \varepsilon\} \leq \sum_j \liminf_{n \rightarrow \infty} P\{\eta_n C_{mj} > \varepsilon\} \leq \limsup_{n \rightarrow \infty} \sum_j P\{\eta_n C_{mj} > \varepsilon\}, \quad \varepsilon > 0,$$

so η is regular and hence diffuse. Conversely, suppose that η is diffuse on C and $E\eta C < \infty$, and let us choose $\{C_{mj}\} \subset \mathcal{B}$ such that $\max_j |C_{mj}| \rightarrow 0$, ($|\cdot|$ denoting the diameter). Then clearly $\max_j \eta C_{mj} \rightarrow 0$ a.s., so

$$\eta C \geq \varepsilon \sum_j 1_{\{\eta C_{mj} > \varepsilon\}} \rightarrow 0 \quad \text{a.s.}, \quad \varepsilon > 0,$$

and hence by dominated convergence

$$\sum_j P\{\eta C_{mj} > \varepsilon\} = E \sum_j 1_{\{\eta C_{mj} > \varepsilon\}} \rightarrow 0, \quad \varepsilon > 0,$$

proving regularity of η . If we choose the C_{mj} in \mathcal{B}_η (which is always possible, cf. the Remark in [4] page 10), we further obtain

$$\limsup_{n \rightarrow \infty} \sum_j P\{\eta_n C_{mj} > \varepsilon\} \leq \sum_j \limsup_{n \rightarrow \infty} P\{\eta_n C_{mj} \geq \varepsilon\} \leq \sum_j P\{\eta C_{mj} \geq \varepsilon\},$$

and so $\{\eta_n\}$ is η -regular.

3. Main results

Recall the definitions of $C(\eta, \beta)$ and $S(\eta, \alpha, \lambda)$, given in Sections 1 and 2 respectively. We shall always assume that $\beta \stackrel{d}{\neq} 0$ and that $\alpha + \lambda g = 1$.

Theorem 1. For each $n \in \mathbb{N}$, let $\xi_n \stackrel{d}{=} C(\eta_n, \beta_n)$ and put $c_n = \text{Eg}(\beta_n)$. Suppose that $\beta_n \stackrel{d}{\neq} 0$. Then $\xi_n \stackrel{d}{\neq} 0$ iff $c_n \eta_n \stackrel{d}{\neq} 0$. Furthermore, the conditions

$$(i) \quad c_n \eta_n \stackrel{d}{\neq} \eta,$$

$$(ii) \quad c_n^{-1} g(P\beta_n^{-1}) \stackrel{w}{\neq} \alpha \delta_0 + g\lambda$$

imply that $\xi_n \stackrel{d}{\neq} S(\eta, \alpha, \lambda)$. Conversely, $\xi_n \stackrel{d}{\neq} \xi$ implies that $\xi \stackrel{d}{=} S(\eta, \alpha, \lambda)$, and if $\{c_n \eta_n\}$ is ξ -regular in some $C \in \mathcal{B}_\xi$ with $\xi C \stackrel{d}{\neq} 0$, then (i) and (ii) are satisfied for some η , α and λ .

No regularity condition is needed in the thinning case:

Theorem 2. For each $n \in \mathbb{N}$, let $p_n \in (0, 1]$ and let ξ_n be a p_n -thinning of some point process η_n . Suppose that $p_n \rightarrow 0$. Then $\xi_n \stackrel{d}{\neq}$ some ξ iff $p_n \eta_n \stackrel{d}{\neq}$ some η , and in this case $\xi \stackrel{d}{=} \text{SP}(\eta)$.

Corollary (Mecke). Let ξ be a point process on S . Then $\xi \stackrel{d}{=} \text{some SP}(\eta)$ iff ξ is distributed as a p -thinning for each $p \in (0,1]$.

Remark. As was pointed out in [5], the last result is essentially contained in Theorems 5.1 and 5.2 of that paper. It may be of some interest to observe that even Theorem 2 above (in particular cases, such as when $S = R_+$) can be obtained from results in [5], (viz. Theorems 2.3 and 4.2 there). We leave details to the reader

4. Proofs

Proof of Theorem 1. To prove that $\xi_n \xrightarrow{d} 0$ iff $c_n \eta_n \xrightarrow{d} 0$, note that by (1.4)

$$E e^{-t \xi_n B} = E \exp[\eta_n B \log \phi_n(t)], \quad t \in R_+, B \in \mathcal{B}, \quad n \in N, \quad (4.1)$$

where $\phi_n = L_{\beta_n}$. Using the elementary inequalities

$$1 - x \leq -\log x \leq 2(1 - x), \quad x \in [\frac{1}{2}, 1], \quad (4.2)$$

and the fact that $\phi_n \rightarrow 1$ since $\beta_n \xrightarrow{d} 0$, we get for sufficiently large $n \in N$

$$c_n = 1 - \phi_n(1) \leq -\log \phi_n(1) \leq 2(1 - \phi_n(1)) = 2c_n,$$

so by (4.1) with $t = 1$

$$E \exp(-2c_n \eta_n B) \leq E e^{-\xi_n B} \leq E \exp(-c_n \eta_n B), \quad B \in \mathcal{B}.$$

Letting $n \rightarrow \infty$, it is seen that $\xi_n B \xrightarrow{d} 0$ iff $c_n \eta_n B \xrightarrow{d} 0$, $B \in \mathcal{B}$,

and the assertion follows.

Let us next suppose that (i) and (ii) are satisfied. To show that in this case $\xi_n \xrightarrow{d} \xi \equiv S(n, \alpha, \lambda)$, it suffices to verify (1.2),

i.e. by (1.4) and (2.2) to show that, for any fixed $f \in F_c$,

$$E \exp(\eta_n \log \phi_n \circ f) \rightarrow E e^{-\eta(\psi \circ f)},$$

where ψ is defined by (2.1). Since the function e^{-x} is bounded and continuous on R_+ , it is enough by Theorem 5.2 in [2] to show that

$$-\eta_n \log \phi_n \circ f \xrightarrow{d} \eta(\psi \circ f),$$

and by Theorem 5.5 in [2] it suffices to prove this for non-random $\eta, \eta_1, \eta_2, \dots$ satisfying (i), or more generally, to show that for any $\mu, \mu_1, \mu_2, \dots \in M$ with $\mu_n \xrightarrow{V} \mu$

$$-c_n^{-1} \mu_n \log \phi_n \circ f \rightarrow \mu(\psi \circ f). \quad (4.3)$$

Since f has compact support and $\psi(0) = 0$, $\phi_1(0) = \phi_2(0) = \dots = 1$, we may assume that even $\mu_n \xrightarrow{V} \mu$, and after normalization (which is possible except in the trivial case $\mu S = 0$), that μ, μ_1, μ_2, \dots are probability measures. In this case, (4.3) may be written in the form

$$-c_n^{-1} E \log \phi_n \circ f(T_n) \rightarrow E \psi \circ f(T), \quad (4.4)$$

where T, T_1, T_2, \dots are random elements in S with distributions μ, μ_1, μ_2, \dots , hence satisfying $T_n \xrightarrow{d} T$. By (ii) and Lemma 1, the sequence $\{c_n^{-1} \log \phi_n\}$ is uniformly bounded on finite intervals, and f being bounded, it follows that $\{c_n^{-1} \log \phi_n \circ f\}$ is uniformly bounded. Hence, by Theorem 5.2 in [2], (4.4) is implied by

$$-c_n^{-1} \log \phi_n \circ f(T_n) \xrightarrow{d} \psi \circ f(T). \quad (4.5)$$

Applying Theorem 5.5 in [2] once more, it is seen that (4.5) needs verification only for non-random T, T_1, T_2, \dots with $T_n \rightarrow T$, and so, f being continuous, it remains to prove that

$$-c_n^{-1} \log \phi_n(x_n) \rightarrow \psi(x)$$

whenever $x, x_1, x_2, \dots \in R_+$ with $x_n \rightarrow x$. But by Lemma 1 this follows from (ii), and so the convergence $\xi_n \xrightarrow{d} \xi$ is established.

Suppose conversely that $\xi_n \xrightarrow{d}$ some ξ , and let $B \in \mathcal{B}_\xi$ be arbitrary. Then $\xi_n B \xrightarrow{d} \xi B$ (cf. (1.1)), so by (4.1)

$$E \exp[\eta_n B \log \phi_n(t)] \rightarrow E e^{-t\xi B}, \quad t \in R_+,$$

and since this limit tends to one as $t \rightarrow 0$, there exists for each $\epsilon > 0$ some $t \in (0,1)$ satisfying

$$E \exp[\eta_n B \log \phi_n(t)] \geq 1 - \epsilon/2, \quad n \in N. \quad (4.6)$$

Now it follows by (4.2) and the elementary inequality

$$1 - e^{-tx} \geq t(1 - e^{-x}), \quad t \in [0,1], \quad x \in R_+,$$

that

$$-\log \phi_n(t) \geq 1 - \phi_n(t) = E(1 - e^{-t\beta_n}) \geq tE(1 - e^{-\beta_n}) = tc_n, \quad n \in N,$$

and so by (4.6) and Čebyšev's inequality

$$\begin{aligned} P\{c_n \eta_n B > t^{-1} \log 2\} &= P\{tc_n \eta_n B > \log 2\} \leq P\{-\eta_n B \log \phi_n(t) > \log 2\} \\ &= P\{1 - \exp[\eta_n B \log \phi_n(t)] > \frac{1}{2}\} \leq 2E\{1 - \exp[\eta_n B \log \phi_n(t)]\} \leq \epsilon, \end{aligned}$$

proving tightness of $\{c_n \eta_n B\}$. Since $B \in \mathcal{B}_\xi$ was arbitrary, it follows that $\{c_n \eta_n\}$ is tight.

Let us now assume that $\xi_n \xrightarrow{d} \xi \neq 0$ and prove that then

$$\lim_{t \rightarrow 0} \limsup_{n \rightarrow \infty} c_n^{-1} (1 - \phi_n(t)) = 0. \quad (4.7)$$

Suppose on the contrary that the limit in (4.7) is greater than some $\epsilon > 0$. Then there exist arbitrarily small $t > 0$ such that, for some sequence $N' \subset N$,

$$c_n^{-1} (1 - \phi_n(t)) > \epsilon, \quad n \in N', \quad (4.8)$$

and since $\{c_n \eta_n\}$ is tight, we may choose some subsequence $N'' \subset N'$ for which $c_n \eta_n \xrightarrow{d}$ some η . According to the first assertion, $\xi \xrightarrow{d} 0$ implies $\eta \xrightarrow{d} 0$, so we may further choose some $B \in \mathcal{B}_\xi \cap \mathcal{B}_\eta$ with $\eta B \xrightarrow{d} 0$. By (4.1), (4.2) and (4.8),

$$E e^{-t \xi_n B} = E \exp[\eta_n B \log \phi_n(t)] \leq E \exp[-\eta_n B (1 - \phi_n(t))] \leq E \exp(-\epsilon c_n \eta_n B),$$

and since $\xi_n B \xrightarrow{d} \xi B$ and $c_n \eta_n B \xrightarrow{d} \eta B$ on N'' , it follows that $E e^{-t \xi B} \leq E e^{-\epsilon \eta B}$. But since t could be chosen arbitrarily small, this yields the contradiction $1 \leq E e^{-\epsilon \eta B} < 1$, proving that (4.7) is indeed true.

By Čebyšev's inequality we further obtain for any $r, t > 0$

$$c_n^{-1} P\{\beta_n > r\} \leq \frac{c_n^{-1} E(1 - e^{-\beta_n t})}{1 - e^{-rt}} = \frac{c_n^{-1} (1 - \phi_n(t))}{1 - e^{-rt}},$$

and letting in order $n \rightarrow \infty$, $r \rightarrow \infty$ and $t \rightarrow 0$, we get by (4.7)

$$\lim_{r \rightarrow \infty} \limsup_{n \rightarrow \infty} c_n^{-1} P\{\beta_n > r\} = 0,$$

which shows that the sequence $\{c_n^{-1} g(P\beta_n^{-1})\}$ of probability measures on R_+ is tight. In particular there exist some sequence $N' \subset N$ and some η, α and λ such that (i) and (ii) hold on N' , and we may conclude from the sufficiency part of the theorem that ξ has the asserted form. (In the case $\xi \stackrel{d}{=} 0$, we may take $\eta = 0$ and choose arbitrary α and λ .)

Let us finally assume that $\xi_n \stackrel{d}{\neq} \xi$, and that $\{c_n \eta_n\}$ is ξ -regular in some region $C \in \mathcal{B}_\xi$ with $\xi C \stackrel{d}{\neq} 0$. Since the sequences $\{c_n \eta_n\}$ and $\{c_n^{-1} g(P\beta_n^{-1})\}$ are both tight, any sequence $N' \subset N$ must contain some subsequence N'' such that (i) and (ii) hold on N'' for some η, α and λ . By the direct part of the theorem we have $\xi \stackrel{d}{=} S(\eta, \alpha, \lambda)$, and in particular $\mathcal{B}_\xi = \mathcal{B}_\eta$ by (2.2), proving that $\eta C \stackrel{d}{\neq} 0$ and that $\{c_n \eta_n\}$ is η -regular on C . Using Lemma 4, it follows that η is diffuse in C , and therefore $P\eta^{-1}$, α and λ are unique by Lemma 2. This proves (i) and (ii) for the original sequence (cf. Theorem 2.3 in [2]), and the proof is complete.

Proof of Theorem 2. The preceding proof applies with $c_n \equiv p_n$, $\alpha = 0$ and $\lambda = \delta_1$, (and it may even be simplified in the present case, since (ii) is automatically satisfied). No regularity assumption is needed, since if $\xi \stackrel{d}{=} SP(\eta)$, then $P\eta^{-1}$ is uniquely determined by $P\xi^{-1}$, (cf. [6] page 316 or the proof of Lemma 2 above).

Proof of the Corollary. Let $\xi \stackrel{d}{=} \text{some } SP(\eta)$ and let $p \in (0, 1]$ be arbitrary. If $\eta_p \stackrel{d}{=} SP(p^{-1}\eta)$ and if ξ_p is a p -thinning of η_p ,

we get by (1.5) and (2.2) for any $f \in F$

$$\begin{aligned} L_{\xi_p}(f) &= L_{\eta_p}(-\log[1 - p(1-e^{-f})]) = L_{p^{-1}\eta}(1 - [1 - p(1 - e^{-f})]) \\ &= L_{p^{-1}\eta}(p(1-e^{-f})) = L_{\eta}(1-e^{-f}) = L_{\xi}(f) , \end{aligned}$$

and so $\xi \stackrel{d}{=} \xi_p$. This proves that ξ is a p -thinning for each $p \in (0,1]$.
Conversely, suppose that ξ has this property. Applying Theorem 2 with $p_n = n^{-1}$ and $\xi_n = \xi$, $n \in \mathbb{N}$, it is seen that ξ must be an SP-process.

References

- [1] BELYAEV, YU.K. (1963) Limit theorems for dissipative flows. *Theor. Probab. Appl.* 8, 165 - 173.
- [2] BILLINGSLEY, P. (1968) Convergence of probability measures. Wiley, New York.
- [3] GOLDMAN, J. R. (1967) Stochastic point processes: limit theorems. *Ann. Math. Statist.* 38, 771 - 779.
- [4] KALLENBERG, O. (1973) Characterization and convergence of random measures and point processes. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 27, 9 - 21.
- [5] KALLENBERG, O. (1973) Canonical representations and convergence criteria for processes with interchangeable increments. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 27, 23 - 36.
- [6] KERSTAN, J., MATTHES, K., and MECKE, J. (1974) Unbegrenzt teilbare Punktprozesse. Akademie-Verlag, Berlin.
- [7] LOEVE, M. (1963) Probability theory, 3rd ed. Van Nostrand, Princeton.
- [8] MECKE, J. (1968) Eine charakteristische Eigenschaft der doppelt stochastischen Poissonschen Prozesse. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 11, 74 - 81.
- [9] NAWROTZKI, K. (1962) Ein Grenzwertsatz für homogene zufällige Punktfolgen (Verallgemeinerung eines Satzes von A. Rényi). *Math. Nachr.* 24, 201 - 217.
- [10] RÉNYI, A. (1956) A characterization of Poisson processes. (In Hungarian with summaries in Russian and English.) *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 1, 519 - 527.
- [11] WALDENFELS, W. v. (1968) Charakteristische Funktionale zufälliger Maße. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* 10, 279 - 283.