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WEAK CONVERGENCE OF HIGH LEVEL EXCEEDANCES BY A STATIONARY SEQUENCE

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SUMMARY

In this paper we consider a stationary sequence $\{\xi_n : n = 1, 2 \dots\}$ satisfying weak dependence restrictions ((2.1) and (2.2) below). For each n the point process N_n is defined to consist of the *exceedances* of a certain level u_n (i.e. the instants j for which $\xi_j > u_n$). It is shown that the point processes N_n converge weakly (as random elements of the natural metric space to which they belong) to a Poisson process. This gives, in particular, generalizations of results of [6, Section 5]. The arguments use results of [6] and a general convergence theorem for point processes (based on convergence of probabilities of no events), which gives a simplifying and clarifying viewpoint.

1. INTRODUCTION

The asymptotic Poisson nature of the upcrossings of a high level by a stationary normal process has been known for a considerable time (cf. [2] and references therein). This result has been especially useful, in demonstrating that the maximum of such a normal process in a given time, has the asymptotic double exponential extreme value distribution. Conversely it is possible to show (as in [1]) from this asymptotic distribution of the maximum, that the

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upcrossings of n high level by this stationary normal process, are approximately Poisson.

Because of these facts, one is led to suspect that the asymptotic Poisson nature of high level crossings is related closely to normality of the sequence, or at least to the occurrence of the double exponential asymptotic distribution of the maximum. This is not the case, however. In fact the Poisson property of high level upcrossings occurs in rather general circumstances and in combination with any of the three "extreme value distributions" according to the basic distributional properties of the process.

Some properties of this kind have been discussed for stationary sequences in [6]. As noted there, it is more natural, for a stationary *sequence* $\{\xi_n : n = 1, 2 \ldots\}$ to consider *exceedances* of a level u (i.e. instants n for which $\xi_n > u$) rather than strict *upcrossings* (i.e. instants n for which $\xi_{n-1} < u < \xi_n$), it being easily seen that these are asymptotically "equivalent" notions.

It was shown in [6] that, under wide distributional conditions, the number L_n of exceedances of a suitably chosen level u_n in the time interval $(1, 2 \ldots n)$ has a limiting Poisson distribution. We shall show here that this result is true not only for intervals, but (based on an obvious normalization) for (almost) arbitrary Borel sets, and that convergence of joint probabilities of this type also occurs. These facts will be shown very simply from the asymptotic distribution of the maximum [6, Theorem 3.1] by means of a general point process convergence theorem. It will, in fact, be shown that, after a suitable change of time scale, the *point processes* consisting of exceedances of u_n , converge weakly (as random elements of a certain metric space) to a Poisson process on the real line. This provides a clarifying viewpoint as well as simplification of the arguments and generalization of the results of Section 5 of [6].

The conditions used apply most simply and naturally to the discrete parameter case considered here (as in [6]). A later paper is planned to deal with the corresponding (but more complex) situation in continuous time.

2. ASSUMPTIONS, NOTATION, AND BASIC RESULTS

We assume that $\{\xi_n : n = 1, 2 \ldots$ is a stationary sequence with finite -dimensional distribution functions $F_{j_1 \ldots j_n}(x_1 \ldots x_n)$ and write

$F_{j_1 \ldots j_n}(u) = F_{j_1 \ldots j_n}(u, u \ldots u)$. Two dependence conditions referred to as $D(u_n)$, $D'(u_n)$ in [6] are also relevant here. In fact the form of $D(u_n)$ given here will be very slightly weaker than that used in [6] (but still sufficient for the proofs and results given there). Specifically if $\{u_n\}$ is a sequence of real numbers we shall say that $\{\xi_n\}$ satisfies the condition $D(u_n)$ if for any integers

$$1 \leq i_1 < i_2 \ldots < i_p < j_1 \ldots < j_q \leq n \ , \ j_1 - i_p \geq \ell \ .$$

we have

$$(2.1) \quad |F_{i_1 \ldots i_p j_1 \ldots j_q}(u) - F_{i_1 \ldots i_p}(u)\, F_{j_1 \ldots j_q}(u)| \leq \alpha_{n,\ell}$$

where $\alpha_{n,\ell}$ is non-increasing in ℓ and where $\lim_{n\to\infty} \alpha_{n,k_n} = 0$ for some sequence $k_n \to \infty$, and such that $k_n/n \to 0$. (Note that if (2.1) holds for some $\alpha_{n,\ell}$ it is clearly possible to take $\alpha_{n,\ell}$ to be non-increasing in ℓ).

Further the condition $D'(u_n)$ is said to hold if

$$(2.2)\quad \limsup_{n\to\infty}\left[n \sum_{j=2}^{n} P\{\xi_i > u_{nk}, \xi_j > u_{nk}\}\right] = o(1/k) \quad \text{as} \quad k \to \infty .$$

A discussion of these conditions may be found in [6].

We are particularly concerned with sequences $\{u_n = u_n(\tau)\}$ satisfying

$$(2.3)\qquad 1 - F_1(u_n) = P\{\xi_1 > u_n\} \sim \tau/n \quad \text{as} \quad n \to \infty .$$

The question of definition of such sequences is mentioned in [6]. Note that if $\{u_n(1)\}$ is defined satisfying (2.3) for $\tau = 1$ then we may define $u_n(\tau) = u_{[n/\tau]}(1)$, where $[x]$ (here and below) denotes the integer part of x.

It will be useful to give Theorem 3.1 of [6] the following slightly more general form

THEOREM 2.1

Let $\{\xi_n\}$ be a stationary sequence and suppose that $D(u_n(\alpha\tau))$, $D'(u_n(\alpha\tau))$ are satisfied for some $\alpha > 0$, $\tau > 0$, where $u_n(\circ)$ is defined to satisfy (2.3). Then

$$P\{M_{[\alpha n]} \le u_n(\tau)\} \to e^{-\alpha\tau} \quad \text{as} \quad n \to \infty .$$

PROOF: It follows at once from Theorem 3.1 of [6] that

$$P\{M_{[\alpha n]} \le u_{[\alpha n]}(\alpha\tau)\} \to e^{-\alpha\tau}$$

and hence it is only necessary to show that

$$(2.4) \qquad P\{M_{[\alpha n]} \le u_n(\tau)\} - P\{M_{[\alpha n]} \le u_{[\alpha n]}(\alpha\tau)\} \to 0 .$$

If $u_n(\tau) > u_{[\alpha n]}(\alpha\tau)$ the left hand side of (2.4) is

$$P\{u_{[\alpha n]}(\alpha\tau) < M_{[\alpha n]} \le u_n(\tau)\} \le P\left\{\bigcup_{j=1}^{[\alpha n]} \{u_{[\alpha n]}(\alpha\tau) \le \xi_j \le u_n(\tau)\}\right\}$$

$$\le \alpha n\{F(u_n(\tau)) - F(u_{[\alpha n]}(\alpha\tau))\} .$$

If $u_n(\tau) < u_{[\alpha n]}(\alpha\tau)$ the same result holds with reversed signs. Hence the left hand side of (2.4) is in all cases dominated in modulus by

$$\begin{aligned} \alpha n|F(u_n(\tau)) - F(u_{[\alpha n]}(\alpha\tau))| &= \alpha n|\{1 - F(u_{[\alpha n]}(\alpha\tau))\} - \{1 - F(u_n(\tau))\}| \\ &= \alpha n\left|\frac{\alpha\tau}{[\alpha n]}(1 + o(1)) - \frac{\tau}{n}(1 + o(1))\right| \\ &= o(1) \quad \text{as} \quad n \to \infty , \end{aligned}$$

as required.

3. CONVERGENCE OF EXCEEDANCES TO A POISSON PROCESS

Throughout this section we assume that $\{\xi_n\}$ is a stationary sequence and that $D(u_n)$, $D'(u_n)$ hold ($u_n = u_n(\tau)$ satisfying 2.3)), for all $\tau > 0$.

For each $n = 1, 2, \ldots$, define a discrete parameter process $\eta_n(t)$ for $t = j/n$ ($j = 1, 2, \ldots$,) by $\eta_n(j/n) = \xi_j$. Thus the η_n are obtained from $\{\xi_j\}$ simply by time scale changes. Let N_n be the point process consisting of the exceedances of $u_n(\tau)$ by η_n ($N_n(B)$ denoting the number of such exceedances in the Borel set B).

The point processes N_n may be properly regarded as random elements in either the space of integer-valued increasing step functions on $[0,\infty)$, or the space N of integer-valued Borel measures on $[0,\infty)$. In either case the space is metric under the "vague topology" (e.g., generated in N by the functions $\mu \to \int f d\mu$ for continuous f with bounded support - cf. [3,4]) and we may consider convergence in distribution of such random elements. ($\zeta_n \xrightarrow{d} \zeta$ will be used to indicate this convergence.) The following result is a special case of a theorem of Kallenberg ([4]Theorem 2.3) modified according to a remark of Kurtz [5].

THEOREM 3.1

Let ζ_n, $n = 1, 2, \ldots$ be point processes on the positive real line and let ζ be a point process without multiple points and such that $\zeta(\{a\}) = 0$ a.s. for every fixed real $a \geq 0$. If

$$\text{(i)}\quad P\{\zeta_n(B) = 0\} \to P\{\zeta(B) = 0\}$$

for all sets B of the form $\bigcup_1^r (a_i, b_i]$ $(a_1 < b_1 < a_2 < b_2 \ldots < a_r < b_r)$

$$\text{(ii)}\quad \limsup_n E\zeta_n(a,b] \leq E\zeta(a,b] \quad \text{for all finite } a < b,$$

then $\zeta_n \xrightarrow{d} \zeta$.

The main task is to verify Condition (i) in our case, where $\zeta_n = N_n$, and $\zeta = N$ is a Poisson process with parameter τ. (i) in fact has already been shown for intervals $B = (0,\alpha]$ in Theorem 2.1. We verify it for finite unions of intervals in the following short lemmas.

LEMMA 3.2

Under the conditions and notation stated at the start of this section, if $0 < a < b$, $B = (a,b]$

$$P\{N_n(B) = 0\} \to e^{-(b-a)\tau} \quad \text{as} \quad n \to \infty .$$

PROOF:
$$P\{N_n(B) = 0\} = P\{\xi_j \le u_n \ , \ a < j/n \le b\}$$
$$= P\{\xi_j \le u_n \ , \ [an] < j \le [bn]\}$$
$$= P\{M_{[bn]-[an]} \le u_n\}$$

by stationarity. But it is easily seen that

$$[(b - a)n] \le [bn] - [an] \le [(b - a)n] + 1 \le [(b + h - a)n]$$

for any fixed $h > 0$, when n is large enough, and hence then

$$P\{M_{[(b+h-a)n]} \le u_n\} \le P\{N_n(B) = 0\} \le P\{M_{[(b-a)n]} \le u_n\} .$$

By Theorem 2.1, the outside terms have the respective limits $e^{-\tau(b+h-a)}$, $e^{-\tau(b-a)}$, and since h is arbitrary we have the conclusion of the lemma.

LEMMA 3.3

Under the same conditions if $B = \bigcup_1^r (a_i,b_i]$ $a_1 < b_1 < a_2 < b_2 \cdots < a_r < b_r$, then

$$\lim_{n\to\infty} P\{N_n(B) = 0\} = e^{-\tau m(B)}$$

where $m(B) = \sum_1^r (b_i - a_i)$ is the Lebesgue measure of B .

PROOF: If $B_j = (a_j, b_j]$, $N_n(B_j) = 0$ is equivalent to $M(E_j) \le u_n$ where $E_j = ([na_j]+1, [na_j]+2, \ldots [nb_j])$ and $M(E_j) = \max(\xi_k : k \in E_j)$. Hence

$$P\{N_n(B) = 0\} = P\{\bigcap_{j=1}^{r} (M(E_j) \le u_n)\} =$$

$$= \prod_{j=1}^{r} P\{N_n(B_j) = 0\} + \left[P\{\bigcap_{j=1}^{r} (M(E_j) \le u_n)\} - \prod_{j=1}^{r} P\{M(E_j) \le u_n\} \right].$$

The first term converges, as $n \to \infty$, to $\prod_{j=1}^{r} e^{-\tau(b_j - a_j)} = e^{-\tau m(B)}$ by Lemma 3.2. On the other hand, by Lemma 2.3 of [6], the modulus of the remaining difference of terms does not exceed $(r-1)\alpha_{n[n\lambda]}$ where $\lambda = \min_{1 \le j \le r-1} (b_{j+1} - a_j)$. But by $D(u_n)$, $\alpha_{n,k_n} \to 0$ for some $k_n \to \infty$, $k_n/n \to 0$. Since $\alpha_{n,\ell}$ is non-increasing in ℓ, and eventually $[n\lambda] > k_n$, we have $\alpha_{n[n\lambda]} \to 0$ as $n \to \infty$, from which the result follows.

We may, finally, now obtain the main result.

THEOREM 3.4

Let $D(u_n)$, $D'(u_n)$ hold ($u_n = u_n(\tau)$ satisfying (2.3)) for all $\tau > 0$, for the stationary sequence $\{\xi_j\}$. Let $\eta_n(j/n) = \xi_j$, $j = 1, 2, \ldots$ $n = 1, 2, \ldots$, and let η_n be the point process consisting of the exceedances of $u_n(\tau)$ by N_n. Then $N_n \xrightarrow{d} N$ as $n \to \infty$, where N is a Poisson process with parameter τ.

PROOF: (i) of Theorem 3.1 holds by Lemma 3.3 since for such B,

$$P\{N_n(B) = 0\} \to e^{-\tau m(B)} = P\{N(B) = 0\}$$

(ii) is immediate since

$$\begin{aligned} E\{N_n(a,b]\} &= ([nb] - [na])(1 - F_1(u_n)) \\ &\sim n(b - a)(\tau/n) = \tau(b - a) \\ &= EN(a,b] . \end{aligned}$$

Hence the result holds.

COROLLARY

Under the conditions of the theorem if B is any Borel set whose boundary has Lebesgue measure zero $(m(\partial B) = 0)$ then, for any $r = 0, 1, 2\ldots$

$$P\{N_n(B) = r\} \to e^{-\tau m(B)}[\tau m(B)]^r \,/\, r!$$

The joint distribution of any finite number $N_n(B_1) \ldots N_n(B_k)$ corresponding to disjoint B_i (with $m\partial B_i = 0$ for each i) converges to the product of the corresponding Poisson probabilities

This follows at once since the random variables $N_n(B)$ converge in distribution to $N(B)$ (and $(N_n(B_1) \ldots N_n(B_k)) \xrightarrow{d} (N(B_1) \ldots N(B_k))$, if $N_n \xrightarrow{d} N$ (cf. [4]). ($N(\partial B) = 0$ a.s. if $m(\partial B) = 0$ since N is a Poisson process).

Thus we have a "full" weak Poisson limit for the exceedances of $u_n(\tau)$ under rather general dependence restrictions. It is easily seen (even by considering i.i.d. sequences) that this Poisson behaviour can occur together with any of the possible asymptotic extreme value distributions for the maximum.

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