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OF SECOND ORDER PROCESSES*

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SUMMARY

Necessary and sufficient conditions are given for almost all paths of a Gaussian process to be absolutely continuous with derivative in L_p . Also sufficient conditions for almost sure path absolute continuity of second order processes are derived, slightly generalizing those previously known.

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Sufficient conditions for a stochastic process to have absolutely continuous paths with probability one were first derived for weakly stationary processes in [6, pp. 536-7] and were later generalized to second order processes in [7, pp. 186-7]. For a Gaussian process it is known that its paths are absolutely continuous with probability zero or one [4], but necessary and sufficient conditions for the two alternatives were known only for stationary processes [2, 10], and thus also for processes with stationary increments, and for harmonizable processes [2]. Theorem 2 gives several equivalent necessary and sufficient conditions for the paths of a Gaussian process to be absolutely continuous with probability one. These conditions, which are slightly more general than those of [7], are shown in Theorem 1 to be sufficient for almost sure path absolute continuity of a second order process. The results of Theorems 1 and 2 extend to the case where almost all paths have $n - 1$ continuous derivatives with the $(n - 1)$ th derivative absolutely continuous.

Throughout this paper $T = [a, b]$ is a finite interval and $\xi = \{\xi(t, \omega), t \in T\}$ a real stochastic process of second order on the probability space (Ω, \mathcal{F}, P) with correlation function $R(t, s) = E(\xi_t \xi_s)$. $\xi(\cdot, \omega)$ denotes the path of the process ξ corresponding to $\omega \in \Omega$ and ξ_t denotes the random variable of the process corresponding to $t \in T$. We will relate the absolute continuity of the paths of ξ on T with the absolute continuity of its correlation function R on $T \times T$. Real valued absolutely continuous functions of two variables are defined and have properties similar to those of real valued absolutely continuous functions of one variable [9, Section 493]. Thus R is absolutely continuous on $T \times T$ if and only if there is a Lebesgue

integrable function r on $T \times T$ such that for all $t_1, t_2, s_1, s_2 \in T$,
 $\Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R = \int_{t_1}^{t_2} \int_{s_1}^{s_2} r(u,v) du dv$, where $\Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R = R(t_2, s_2) - R(t_2, s_1) - R(t_1, s_2) + R(t_1, s_1)$. By $\mathcal{R}(R)$ we denote the reproducing kernel Hilbert space of the nonnegative definite function R .

We will also make use of the map $T \rightarrow L_2(\Omega) = L_2(\Omega, F, P)$ defined by $t \rightarrow \xi_t$, which is a Hilbert space valued function defined on T . So let us introduce the following notation and properties that will be used in the sequel. Let H be a Hilbert space with norm $\|\cdot\|$. $L_p[T, H]$, $1 \leq p < \infty$, denotes the space of (equivalence classes of) measurable functions $f : T \rightarrow H$ such that $\int_T \|f(t)\|^p dt < \infty$. For every $f \in L_p[T, H]$, $1 \leq p < \infty$, the Bochner integral $\int_T f(t) dt$ is well defined. $W_{1,p}[T, H]$ is the set of all functions $f : T \rightarrow H$ of the form $f(t) = f(a) + \int_a^t g(u) du$ for some $g \in L_p[T, H]$. A function $f : T \rightarrow H$ is called absolutely continuous if for every $\epsilon > 0$ there is $\delta > 0$ such that for all finite collections $\{(a_i, b_i)\}_{i=1}^n$ of disjoint subintervals of T , $\sum_{i=1}^n |b_i - a_i| < \delta$ implies $\sum_{i=1}^n \|f(b_i) - f(a_i)\| < \epsilon$. It can be shown [1, appendice] that a function $f : T \rightarrow H$ is absolutely continuous if and only if $f \in W_{1,1}[T, H]$. Also $f \in W_{1,p}[T, H]$, $1 \leq p < \infty$, if and only if it is absolutely continuous, differentiable a.e. [Leb] on T , and $\frac{df}{dt} \in L_p[T, H]$. When H is the real line we have the familiar results for real functions and we will use the notation $L_p[T]$ and $W_{1,p}[T]$ for the corresponding spaces.

In Theorem 1 we give sufficient conditions for the paths of a second order process to be absolutely continuous with probability one. In Theorem 2

we will show that these conditions are also necessary for Gaussian processes.

THEOREM 1. If $\xi = \{\xi(t, \omega) , t \in T = [a, b]\}$ is a real separable stochastic process of second order on the probability space (Ω, \mathcal{F}, P) with correlation function $R(t, s) = E(\xi_t \xi_s)$, the following are equivalent.

(i) With probability one the paths of ξ are absolutely continuous and there is a measurable second order process $\eta = \{\eta(t, \omega) , t \in T\}$ such that

$$(1) \quad \int_a^b \sqrt{E(\eta_t^2)} dt < \infty$$

and

$$(2) \quad \xi(t, \omega) = \xi(a, \omega) + \int_a^t \eta(u, \omega) du \quad \text{for all } t \in T$$

with probability one.

(ii) The map $T \rightarrow L_2(\Omega)$ defined by $t \rightarrow \xi_t$ is absolutely continuous.

(iii) There is a measurable nonnegative definite function r on $T \times T$ with $R(r)$ separable and

$$(3) \quad \int_a^b \{r(u, u)\}^{\frac{1}{2}} du < \infty$$

such that for all $t_1, t_2, s_1, s_2 \in T$,

$$(4) \quad \Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R = \int_{t_1}^{t_2} \int_{s_1}^{s_2} r(u, v) du dv .$$

(iv) $R(t, \cdot)$ is absolutely continuous on T for every fixed $t \in T$, (R is absolutely continuous on $T \times T$) and there is a measurable subset T_0 of T with Lebesgue measure zero such that $\partial^2 R(t,s)/\partial t \partial s$ exists for all $t, s \in T - T_0$ and satisfies

$$(5) \quad \int_a^b \left(\frac{\partial^2 R(t,s)}{\partial t \partial s} \Big|_{t=s=u} \right)^{\frac{1}{2}} du < \infty .$$

In general, (i) is stronger than the absolute continuity of the paths of ξ with probability one (for an exceptional case see Theorem 2), and (iii) and (iv) are stronger than the absolute continuity of R on $T \times T$ (the stationary case is an exception here). Condition (iv) may in some cases be easier to verify than condition (iii). Trivial examples show that the presence of the zero Lebesgue measure set T_0 in (iv) is necessary. Also (iv) implies that for each $t \in T - T_0$, $\partial R(t, \cdot)/\partial t$ is absolutely continuous on T .

When ξ is harmonizable or weakly stationary with (two- and one-dimensional respectively) spectral distribution F , then condition (iv) is equivalent to $\iint_{-\infty}^{\infty} |\lambda \mu| dF(\lambda, \mu) < \infty$ and $\int_{-\infty}^{\infty} \lambda^2 dF(\lambda) < \infty$ respectively, and the fact that (iv) implies (i) is known [6, pp. 536-7; 7, pp. 186-7]. In the stationary case (iv) is also equivalent to the absolute continuity of R as a function of two variables on $T \times T$. If ξ is strictly stationary, it is clear that (1) of (i) may be dropped, but it seems that we still need to require η to be of second order.

The relationship between the almost sure path absolute continuity of ξ and the absolute continuity of the map $t \rightarrow \xi_t$ has also been considered in

[11] where, under certain conditions (not applicable here), it is shown that the former implies the latter.

PROOF. (i) implies (ii). Define the function $g : T \rightarrow L_2(\Omega)$ by $g(t) = \eta_t$. We will show that $g \in L_1[T, L_2(\Omega)]$. In order to show that g is measurable, since T is compact, it suffices to show that (a) for every $h \in L_2(\Omega)$, $E[hg(t)]$ is Lebesgue measurable, and (b) there is a countable subset C of $L_2(\Omega)$ such that $g(t) \in \bar{C}$, the closure of C , for all $t \in T$ [5, p. 92]. (a) follows from $E[hg(t)] = E(h\eta_t)$ and the product measurability of η . Since η is second order and measurable it follows that $H(\eta)$, the closure in $L_2(\Omega)$ of the linear space generated by $\{\eta_t, t \in T\}$, is separable [3, Theorem 1]. Then any countable dense subset C of $H(\eta)$ satisfies (b). Thus g is measurable and $g \in L_1[T, L_2(\Omega)]$ follows from (1).

Now define the second order process $X(t, \omega) = \int_a^t \eta(u, \omega) du$ for all $t \in T$ with probability one, and the function $Y : T \rightarrow L_2(\Omega)$ by the Bochner integral $Y(t) = \int_a^t g(u) du$. By using the basic properties of the Bochner integral, the measurability of the processes η and X , and Fubini's theorem justified by (1) we have for all fixed $t \in T$,

$$E(X_t^2) = \int_a^t \int_a^t E(\eta_u \eta_v) du dv$$

$$E[X_t Y(t)] = \int_a^t E(X_t g(u)) du = \int_a^t \int_a^t E(\eta_u \eta_v) du dv$$

$$E[Y^2(t)] = \int_a^t E(g(u) Y(t)) du = \int_a^t \int_a^t E(\eta_u \eta_v) du dv .$$

Hence for all fixed $t \in T$, $E[X_t - Y(t)]^2 = 0$ and thus by (2), $\xi_t = \xi_a + X_t = \xi_a + Y(t) = \xi_a + \int_a^t g(u)du$ in $L_2(\Omega)$. Since $g \in L_1[T, L_2(\Omega)]$, it follows that the map $t \rightarrow \xi_t$ belongs to $W_{1,1}[T, L_2(\Omega)]$ and thus it is absolutely continuous [1, p. 145].

(ii) implies (i). Since the function $t \rightarrow \xi_t$ is absolutely continuous, there is a $g \in L_1[T, L_2(\Omega)]$ such that

$$\xi_t = \xi_a + \int_a^t g(u)du \quad \text{for all } t \in T$$

the equality being in $L_2(\Omega)$. Since g is measurable, there is a measurable subset N of T with Lebesgue measure zero such that $g(T - N)$ is separable. Now define the process $\zeta = \{\zeta(t, \omega), t \in T\}$ by

$$\zeta_t = \begin{cases} g(t), & t \in T - N \\ 0, & t \in N \end{cases}.$$

Then ζ is of second order, $\{\zeta_t, t \in T\}$ is separable as a subset of $L_2(\Omega)$, and hence so is $H(\zeta)$, the closure of the linear space generated by it. Let $Q(t, s) = E[g(t)g(s)]$ and $K(t, s) = E(\zeta_t \zeta_s)$, $t, s \in T$. Since g is measurable, it is weakly measurable and thus for all fixed $t \in T$, $Q(t, \cdot)$ is measurable. Since for fixed $t \in T$, $K(t, \cdot)$ and $Q(t, \cdot)$ agree on $T - N$, it follows that $K(t, \cdot)$ is also measurable. An inspection of the proof of Theorem 1 of [3] shows that $H(\zeta)$ separable and $K(t, \cdot)$ measurable for all $t \in T$ imply that ζ has a measurable modification, denoted by η . Alternatively one can easily show the product measurability of K from the

separability of $R(K)$ (which is isomorphic to $H(\zeta)$) and the measurability of $K(t, \cdot)$, and then apply Theorem 1 of [3] directly. Now for all $t \in T - N$, $E(\eta_t^2) = E(\zeta_t^2) = E[g^2(t)]$ and $g \in L_1[T, L_2(\Omega)]$ implies (1). This in turn implies

$$E \left[\int_a^t |\eta(u, \omega)| du \right] = \int_a^t E(|\eta_t|) dt \leq \int_a^t \sqrt{E(\eta_t^2)} dt < \infty$$

and thus $\eta(\cdot, \omega) \in L_1[T]$ with probability one, i.e. for all $\omega \in \Omega - \Omega_0$ with $P(\Omega_0) = 0$. Now define the process $X = \{X(t, \omega), t \in T\}$ by

$$X(t, \omega) = \begin{cases} \xi(a, \omega) + \int_a^t \eta(u, \omega) du & t \in T, \omega \in \Omega - \Omega_0 \\ 0 & t \in T, \omega \in \Omega_0 \end{cases} .$$

Clearly the paths of X are absolutely continuous with probability one. Also it is easily seen, as in the previous part of the proof, that for all fixed $t \in T$, $E(\xi_t - X_t)^2 = 0$ and thus $P\{\omega \in \Omega : \xi(t, \omega) = X(t, \omega)\} = 1$. If S is a countable dense subset of T which is a separating set for ξ , we have $P\{\omega \in \Omega : \xi(t, \omega) = X(t, \omega), t \in S\} = 1$ and since X has continuous paths with probability one it follows that $P\{\omega \in \Omega : \xi(t, \omega) = X(t, \omega), t \in T\} = 1$ and thus (i) is satisfied.

(i) implies (iii). This is obvious, with $r(u, v) = E(\eta_u \eta_v)$, when we note that the measurability of the second order process η implies that r is measurable and $R(r)$ is separable [3, Theorem 1].

(iii) implies (ii). From (4) we have that for all $t, s \in T$,

$$E[(\xi_t - \xi_a)(\xi_s - \xi_a)] = \int_a^t \int_a^s r(u,v) du dv .$$

There exists a separable Hilbert space H and a function $f : T \rightarrow H$ such that $\langle f(u), f(v) \rangle_H = r(u,v)$ and $\{f(t), t \in T\}$ is complete in H ; for instance take $H = \mathcal{R}(r)$ and $f(t) = r(t, \cdot)$. Every $h \in H$ is the limit of a sequence of linear combinations from $\{f(t), t \in T\}$: $h = \lim_n h_n$, $h_n = \sum_{i=1}^N a_{n,i} f(t_{n,i})$. Thus for all $t \in T$, $\langle h, f(t) \rangle_H = \lim_n \sum_{i=1}^N a_{n,i} r(t_{n,i}, t)$ and since r is measurable, f is weakly measurable and also measurable since H is separable. Then (3) implies that $f \in L_1[T, L_2(\Omega)]$ and we have for all $t, s \in T$

$$\langle \xi_t - \xi_a, \xi_s - \xi_a \rangle_{L_2(\Omega)} = \langle \int_a^t f(u) du, \int_a^s f(v) dv \rangle_H .$$

It follows that there is an isomorphism A between the closure in H of the linear space generated by $\{\int_a^t f(u) du, t \in T\}$ and the closure in $L_2(\Omega)$ of the linear space generated by $\{\xi_t - \xi_a, t \in T\}$, such that

$$\xi_t - \xi_a = A \int_a^t f(u) du \quad \text{for all } t \in T .$$

Define $g : T \rightarrow L_2(\Omega)$ by $g(t) = Af(t)$. Then $g \in L_1[T, L_2(\Omega)]$ and

$$A \int_a^t f(u) du = \int_a^t g(u) du \quad [8, p. 83].$$

It follows that $\xi_t = \xi_a + \int_a^t g(u) du$, $t \in T$, and (ii) is satisfied.

(iv) implies (iii). If we define r by $r(t,s) = \partial^2 R(t,s)/\partial t \partial s$ for $t,s \in T - T_0$ and $r(t,s) = 0$ elsewhere, everything in (iii) is obvious except perhaps that $R(r)$ is separable. This is shown as follows. Since $\partial^2 R(t,s)/\partial t \partial s$ exists for $t = s \in T - T_0$, the mean square derivative $\hat{\xi}$ of ξ exists on $T - T_0$. Define the process ζ by $\zeta_t = \hat{\xi}_t$, $t \in T - T_0$, and $\zeta_t = 0$, $t \in T_0$. Then $E(\zeta_t \zeta_s) = r(t,s)$, $t,s \in T$ and $H(\zeta) \subset H(\xi)$ where $H(\zeta)$ is the closure in $L_2(\Omega)$ of the linear space generated by $\{\zeta_t, t \in T\}$ and similarly for $H(\xi)$. Now $R(R)$ and $H(\xi)$ are isomorphic, and so are $R(r)$ and $H(\zeta)$, and thus the separability of $R(R)$ implies that of $R(r)$.

(iii) implies (iv). This will be shown in the proof of Theorem 2.

THEOREM 2. If $\xi = \{\xi(t,\omega), t \in T = [a,b]\}$ is a real separable Gaussian process on the probability space (Ω, F, P) with correlation function R , each of the conditions (ii), (iii) and (iv) of Theorem 1 is necessary and sufficient for the paths of ξ to be absolutely continuous with probability one.

For harmonizable and stationary Gaussian processes the equivalence of almost sure path absolute continuity to (iv) was shown in [2, 10].

PROOF. It suffices to show that if the paths of ξ are absolutely continuous with probability one then (i) of Theorem 1 is satisfied; in fact it will also be shown that (iv) is also satisfied proving thus that (iii) implies (iv).

Since ξ has with probability one continuous paths it is product measurable [12, p. 122]. If $T_d(\omega)$ is the set of points in T where the

path $\xi(\cdot, \omega)$ is differentiable, the almost sure path absolute continuity of ξ implies $\text{Leb}\{T - T_d(\omega)\} = 0$ a.s. . Also, if T_d is the set of points in T where the paths of ξ are differentiable with probability one, it is shown in [2, Theorem 3(ii)] that $\text{Leb}\{T_d(\omega) \Delta T_d\} = 0$ a.s. . It follows that $\text{Leb}\{T - T_d\} = 0$ and thus we will take $T_0 = T - T_d$. For every $t \in T_d = T - T_0$ the paths of ξ are differentiable with probability one, hence ξ is mean square differentiable at t (since almost sure convergence of a sequence of Gaussian random variable implies convergence in the mean square) and thus $\partial^2 R(t,s)/\partial t \partial s$ exists for all $t, s \in T_d$. Now let $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 0$ be such that for all $\omega \in \Omega - \Omega_0$, $\xi(\cdot, \omega)$ is absolutely continuous and define $\zeta = \{\zeta(t, \omega) , t \in T\}$ by

$$\zeta(t, \omega) = \begin{cases} \limsup_{n \rightarrow \infty} n[\xi(t + \frac{1}{n}, \omega) - \xi(t, \omega)] & t \in [a, b) , \omega \in \Omega - \Omega_0 \\ 0 & t \in [a, b) , \omega \in \Omega_0 \quad \text{and} \quad t = b , \omega \in \Omega . \end{cases}$$

Then ζ is product measurable and for all $\omega \in \Omega - \Omega_0$, we have $\zeta(t, \omega) = \xi'(t, \omega)$ for $t \in T_d(\omega) - \{b\}$, where $\xi'(\cdot, \omega)$ denotes the path derivative of $\xi(\cdot, \omega)$. Also for all $t \in T_d - \{b\}$, $\zeta(t, \omega) = \xi'(t, \omega)$ a.s. . Now define $\eta = \{\eta(t, \omega) , t \in T\}$ by

$$\eta(t, \omega) = \begin{cases} \zeta(t, \omega) & t \in T_d , \omega \in \Omega \\ 0 & t \in T_0 , \omega \in \Omega . \end{cases}$$

It is clear that η is product measurable and also Gaussian, since for all $t \in [a, b) - T_0$, $\eta_t = \lim_{n \rightarrow \infty} n(\xi_{t+\frac{1}{n}} - \xi_t)$ a.s. and ξ is Gaussian. Also

for all $\omega \in \Omega - \Omega_0$,

$$\eta(\circ, \omega) = \zeta(\circ, \omega) = \xi'(\circ, \omega) \quad \text{a.e. [Leb] on } T$$

and since $\xi(\circ, \omega)$ is absolutely continuous, $\xi'(\circ, \omega) \in L_1[T]$. It follows that with probability one $\eta(\circ, \omega) \in L_1[T]$ and hence by a result in [13, p. 391], $\int_a^b E^{\frac{1}{2}}(\eta_u^2) du < \infty$, i.e. (1) is satisfied. (5) follows if we note that for all $u \in T_d - \{b\}$,

$$E(\eta_u^2) = E(\zeta_u^2) = E(\xi'_u{}^2) = \frac{\partial^2 R(t,s)}{\partial t \partial s} \Big|_{t=s=u}.$$

We also have

$$\xi(t, \omega) = \xi(a, \omega) + \int_a^t \eta(u, \omega) du, \quad t \in [a, b], \quad \omega \in \Omega - \Omega_0.$$

Now an application of Fubini's theorem justified by (1) gives

$$R(t,s) = R(t,a) + \int_a^s E(\xi_t \eta_u) du \quad t, s \in T$$

$$\Delta_{t_1}^{t_2} \Delta_{s_1}^{s_2} R = \int_{t_1}^{t_2} \int_{s_1}^{s_2} E(\eta_u \eta_v) du dv \quad t_1, t_2, s_1, s_2 \in T$$

and since by (1) the functions inside the integrals are Lebesgue integrable, it follows that $R(t, \circ)$ is absolutely continuous on T for every fixed $t \in T$ and that R is absolutely continuous on $T \times T$. Thus (i) and (iv)

are satisfied.

The following results are straightforward generalizations of Theorems 1 and 2 and thus their proofs are omitted. The only additional result needed is that if the process ξ is real measurable and Gaussian and $1 \leq p < \infty$,

$\int_T |\xi(t, \omega)|^p dt < \infty$ with probability one if and only if $\int_T \{R(t, t)\}^{p/2} dt < \infty$ [13, p. 391]. We also need to introduce the space $W_{n,p}[T, \mathbb{H}]$, n a positive integer and $1 \leq p < \infty$, of all functions $f : T \rightarrow \mathbb{H}$ which are $(n - 1)$ -times continuously differentiable on T and $f^{(n-1)} \in W_{1,p}[T, \mathbb{H}]$ (the derivatives are meant in the strong sense). When \mathbb{H} is the real line we have the space, denoted by $W_{n,p}[T]$, of all $(n - 1)$ -times continuously differentiable real functions on T with $f^{(n-1)}$ absolutely continuous with derivative in $L_p[T]$.

THEOREM 3. If $\xi = \{\xi(t, \omega), t \in T = [a, b]\}$ is a real separable stochastic process of second order on the probability space (Ω, \mathcal{F}, P) with correlation function R and if n is a positive integer, the following are equivalent.

- (i) $\xi(\cdot, \omega) \in W_{n,1}[T]$ and $\xi^{(n-1)}$ satisfies (i) of Theorem 1.
- (ii) The map $T \rightarrow L_2(\Omega)$ defined by $t \rightarrow \xi_t$ belongs to $W_{n,1}[T, L_2(\Omega)]$.
- (iii) $\partial^{2(n-1)} R(t, s) / \partial t^{n-1} \partial s^{n-1}$ exists on $T \times T$ and satisfies (iii) of Theorem 1.
- (iv) $\partial^{2(n-1)} R(t, s) / \partial t^{n-1} \partial s^{n-1}$ exists on $T \times T$ and satisfies (iv) of Theorem 1.

Again when ξ is harmonizable or stationary, (iv) is equivalent to $\iint_{-\infty}^{\infty} |\lambda|^n |\mu|^n dF(\lambda, \mu) < \infty$ and $\int_{-\infty}^{\infty} \lambda^{2n} dF(\lambda) < \infty$ respectively.

THEOREM 4. If $\xi = \{\xi(t, \omega) , t \in T = [a, b]\}$ is a real separable Gaussian process on the probability space (Ω, \mathcal{F}, P) with correlation function R , n is a positive integer and $1 \leq p < \infty$, the following are equivalent.

- (i) $\xi(\cdot, \omega) \in W_{n,p}[T]$ with probability one.
- (ii) The map $T \rightarrow L_2(\Omega)$ defined by $t \rightarrow \xi_t$ belongs to $W_{n,p}[T, L_2(\Omega)]$.
- (iii) and (iv) are as in Theorem 3 with $1/2$ replaced by $p/2$ in (3) and (5).

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