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NOTES ON CHARACTERISTIC FUNCTIONS - I: ON DENSITIES SMALL AT INFINITY*

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1. Introduction

Let $p(x)$ be a probability density function (p.d.f.) and $p^{\dagger}(\theta)$ the corresponding characteristic function (c.f.). Suppose that, for some positive integer k,

$$\int_0^{\infty} \theta^k |p^{\dagger}(\theta)| d\theta < \infty .$$

Then the first k derivatives of $p(x)$ will exist, be bounded, and tend to zero as $x \to \infty$. This suggests that if, for some positive and non-decreasing function $M(\theta)$ of $\theta > 0$,

$$\int_0^{\infty} M(\theta) |p^{\dagger}(\theta)| d\theta < \infty , \tag{1.1}$$

then the more rapidly growing $M(\theta)$ happens to be, the "smoother" $p(x)$ will be. Indeed, as is easily seen, if (1.1) holds with $M(\theta) = e^{\alpha\theta}$, say, for some fixed real $\alpha > 0$, then $p(z)$ will be an analytic function of complex z in the strip $-\alpha < Iz < \alpha$.

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In proving certain theorems of renewal theory, probability densities with compact support and a very high degree of "smoothness" are a useful tool (see [2]). In any case it is of interest to know to what extent a p.d.f. with compact support is restricted in its smoothness. A famous theorem of Paley and Wiener settles this issue fairly easily, and we shall prove the following.

<u>THEOREM 1.</u> *A necessary and sufficient condition for the existence of a p.d.f.* p(x) *with compact support, such that (1.1) holds is that*

$$\int_0^\infty \frac{|\log M(\theta)|}{1+\theta^2}\, d\theta < \infty . \tag{1.2}$$

It might be noted that one can easily arrange p(x) to be an even function which vanishes identically when $|x| \geq 1$, should this prove convenient.

An intriguing question centers on situations where the integral (1.2) diverges. For example, suppose

$$M(\theta) \sim \exp\{\theta/\log\theta\} , \text{ as } \theta \to \infty .$$

Although no associated p(x) with compact support exists for such a case, is it nevertheless possible that there are densities which are extremely small outside a compact interval or, more correctly, small at infinity? The interpretation of the phrase "extremely small" will be made clear in the detailed statements below.

We shall as a first step, in section 4, deduce the following theorem from the aforementioned theorem of Paley and Wiener.

THEOREM 2. *Let* $\phi(x)$ *be a real, non-null and non-negative function belonging to* $L_2(-\infty, +\infty)$ *and such that*

$$\int_{-\infty}^{+\infty} \frac{|\log \phi(x)|}{1+|x|^3}\, dx < \infty .$$

Choose any convenient fixed $\lambda > 0$ *and for large positive* r *define*

$$a(r) = \int_{|t|>r} \frac{\log[1/\phi(t)]}{|t|^3}\, dt$$

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and

$$b(r) = \frac{1}{\pi} \int_{\lambda \le |t| \le r} \frac{\log[1/\phi(t)]}{t^2}\, dt .$$

Then: (i) If $x^2\phi(x)$ *belongs to* $L_1(-\infty, +\infty)$ *, there exists a* $g(x)$ *in* $L_1(-\infty, +\infty)$ *with a Fourier transform* $g^{\dagger}(\theta)$ *such that* $|g^{\dagger}(\theta)| \equiv \phi(\theta)$ *, and, as* $x \to \infty$

$$g\big(b(x)\big) = O\big(a(x)\big) .$$

(ii) If $x^2\phi(x)$ *belongs to* $L_2(-\infty, \infty)$ *there exists a* $g(x)$ *in* $L_2(-\infty, \infty)$ *, whose Fourier transform* $g^\dagger(\theta)$ *is such that* $|g^\dagger(\theta)| \equiv \phi(\theta)$ *, and*

$$\int_{b(r)}^{\infty} |g(x)|^2 dx = O\big(a(r)\big) .$$

It might be noted here that should $x^2\phi(x)$ belong to $L_1 \cap L_2$ then the functions $g(x)$ arising in (i) and (ii) above can be taken to be the *same* function. This claim will be clear from the proof of the theorem.

Theorem 2 gives us some idea of what can be achieved in situations where the conditions for the classical Theorem XII of Paley and Wiener [1] fail to hold. Some special examples are illuminating.

Example 1. Suppose $\phi(t) = e^{-c|t|}$ for some constant $c > 0$. Then both cases (i) and (ii) apply. We can take $\lambda = 1$ and find $b(r) = (c/\pi)\log r$ and $a(r) = c/r$. Thus we can find a $g(x)$ with $|g^\dagger(t)| \equiv e^{-c|t|}$ and

$$|g(x)| = O\left(e^{-\frac{\pi x}{c}}\right)$$

$$\int_x^{\infty} |g(t)|^2 dt = O\left(e^{-\frac{\pi x}{c}}\right) .$$

Example 2. Suppose $\log\phi(t) = -(c|t|)/\log|t|$, for $t \geq e$. We choose $\lambda = e$ and get $b(r) = (c/\pi)\log\log r$ and $a(r) \sim (r \log r)^{-1}$ as $r \to \infty$. Again (i) and (ii) apply. For (i), say, we find

$$|g(x)| = O\left(\exp\left(-\frac{\pi x}{c} - e^{\frac{\pi x}{c}}\right)\right) .$$

The second example indicates that when $\phi(t)$ decreases at rates less rapid than exponentially, the function $g(t)$ can indeed be made to be "exceedingly small" as $t \to +\infty$. It might be noted that should

$$\int_{-\infty}^{+\infty} \frac{|\log \phi(t)|}{1+t^2}\, dt < \infty ,$$

then $b(r)$ is bounded above by some finite constant k, say, and we obtain the conclusion already in the Paley - Wiener theorem, namely, that $g(t)$ can be identically zero for all $t \geq k$.

Finally, in section 5, we shall use Theorem 2 to treat the original "smoothness" question of probability densities which are small at infinity. We shall prove:

<u>THEOREM 3.</u> *Let* $M(\theta)$ *be a real, non-decreasing function of* $\theta \geq 0$, $M(0)>0$, *and*

$$\int_{1}^{\infty} \frac{|\log M(\theta)|}{\theta^3}\, d\theta < \infty .$$

Define

$$a_2(x) = \int_{x}^{\infty} \frac{\log M(4t)}{t^3}\, dt ,$$

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and, for fixed $\lambda > 0$,

$$b_2(x) = \rho + \frac{2}{\pi}\int_\lambda^x \frac{\log M(4t)}{t^2}\,dt ,$$

where

$$\rho = \frac{2}{\pi}\int_\lambda^\infty \frac{\log(1+16t^2)}{t^2}\,dt .$$

Then if

[3a] $$1/M(|x|) \in L_1 ,$$

or

[3b] $$1/M(|x|) \in L_2 ,$$

there exists an even p.d.f. $p(x)$ *with a c.f.* $p^{\dagger}(\theta)$ *such that*

(1.3) $$\int_{-\infty}^{+\infty} M(|\theta|)\,|p^{\dagger}(\theta)|\,d\theta < \infty .$$

If (3a) holds then

(1.4) $$p\big(b_2(x)\big) = O\big(\{a_2(x)\}^2\big) ,$$

and if (3b) holds, then

$$\int_{b_2(x)}^{\infty} p(x)dx = O\big(a_2(x)\big) \quad (1.5)$$

By way of illustration, consider an example similar to Example 2, above, but for ease suppose $\log M(4x) = cx/\log x$. Then the density $p(x)$ turns out, by (1.4), to be $O\left(\exp\left(-\frac{\pi|x|}{c} - e^{\frac{\pi|x-\rho|}{c}}\right)\right)$ as $|x| \to \infty$. This example displays the main general finding in this paper. If $M(\theta)$ grows "just too fast" for (1.2), but not as fast as an exponential, then a suitably smooth p.d.f. $p(x)$ can be found which, while not having compact support, is extremely small $\left(\text{e.g. } O(e^{-e^{|x|}})\right)$ at infinity.

2. Proof of Theorem 1

The necessity part of the theorem is easily disposed of. Since $M(\theta)$ is positive and non-decreasing, (1.1) implies that $p^{\dagger}(\theta) \in L_1(-\infty, +\infty)$. Thus $p(x)$ is necessarily bounded, and, having compact support, therefore belongs also to $L_2(-\infty, +\infty)$. Thus $p^{\dagger}(\theta)$ must also belong to $L_2(-\infty, +\infty)$, and we can appeal to Theorem XII of Paley and Wiener [1] to infer that

$$\int_0^{\infty} \frac{|\log|p^{\dagger}(\theta)||}{1+\theta^2} d\theta < \infty .$$

This implies that

$$\int_0^{\infty} \frac{|\log\{|p^{\dagger}(\theta)|(1+\theta^2)\}|}{1+\theta^2} d\theta < \infty . \quad (2.1)$$

However, Jensen's inequality, gives

$$\frac{2}{\pi}\int_0^\infty \frac{\log\{M(\theta)\,|p^\dagger(\theta)|\,(1+\theta^2)\}}{1+\theta^2}d\theta \tag{2.2}$$

$$\leq \log\left\{\frac{2}{\pi}\int_0^\infty M(\theta)\ |p^\dagger(\theta)|\ d\theta\right\}$$

$$< \infty ,$$

since (1.1) is assumed to hold. If we combine (2.1) and (2.2) we discover

$$\int_0^\infty \frac{\log M(\theta)}{1+\theta^2}\,d\theta < \infty .$$

But $M(\theta) \geq M(0) > 0$ for all $\theta > 0$, so $|\log M(\theta)| \leq \max\{|\log M(0)|, \log M(\theta)\}$, and (1.2) follows.

Now suppose (1.2) to be given. Then (1.2) also holds with $M(\theta)$ replaced by $M(\theta)(1+\theta^2)$. Let us set, for $-\infty < \theta < \infty$,

$$\phi(\theta) = \frac{1}{(1+16\theta^2)M(4|\theta|)} .$$

Since $M(|\theta|) \geq M(0) > 0$, it follows that $\phi(\theta) \in L_2(-\infty, +\infty)$ and we can again appeal to Theorem XII of Paley and Wiener [1] to deduce the existence of a function $g(x)$ in $L_2(-\infty, +\infty)$ with the properties:

(i) $g(x) \equiv 0$ for $x \geq 1$

(ii) the transform $g^\dagger(\theta)$ of $g(x)$ is such that $|g^\dagger(\theta)| \equiv \phi(\theta)$.

But it is plain that $\phi(\theta)$, and hence $g^\dagger(\theta)$, is in $L_1(-\infty, +\infty)$. Thus $g(x)$ must be equal almost everywhere to a bounded and continuous function. We can thus assume, with no loss of generality, by a suitable finite "shift" of $g(x)$ if necessary, that $g(x)$ is non-zero in some interval $(-\varepsilon, +\varepsilon)$ for some $\varepsilon > 0$.

Now define $u(x) = |g(x)|^2$. Then $u(x)$ belongs to $L_1(-\infty, +\infty)$ and has a Fourier Transform

$$u^\dagger(\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} g^\dagger(\alpha)\,\overline{g^\dagger(\alpha - \theta)}\,dx\,. \tag{2.3}$$

Thus

$$|u^\dagger(\theta)| \le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi(x)\phi(x - \theta)dx\,.$$

Suppose $\theta \ge 0$. Then

$$|u^\dagger(\theta)| \le \frac{1}{2\pi}\int_{-\infty}^{\frac{1}{2}\theta} \phi(\alpha)\phi(\alpha - \theta)d\alpha$$

$$+\frac{1}{2\pi}\int_{\frac{1}{2}\theta}^{\infty} \phi(\alpha)\phi(\alpha - \theta)d\alpha\,.$$

In $-\infty < \alpha \le \frac{1}{2}\theta$, $\phi(\alpha - \theta) \le \phi(\frac{1}{2}\theta)$; in $\frac{1}{2}\theta \le \alpha < \infty$, $\phi(\alpha) \le \phi(\frac{1}{2}\theta)$. Thus, if we write $I = \int_{-\infty}^{+\infty} \phi(x)dx$, then

$$|u^\dagger(\theta)| \quad \frac{I}{\pi}\phi\left(\frac{\theta}{2}\right)\,. \tag{2.4}$$

It should be clear that (2.4) will also be true if $\theta < 0$.

Consider

$$v(x) = u(x)u(-x) .$$

The following statements are true of $v(x)$: it is non-null, non-negative, bounded, and identically zero outside the interval $(-1, +1)$. Moreover it has a Fourier transform

$$v^{\dagger}(\theta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u^{\dagger}(\alpha)u^{\dagger}(\alpha - \theta)d\alpha .$$

An argument similar to that applied to (2.3), using (2.4), will then show that, for some finite constant A ,

$$|v^{\dagger}(\theta)| \leq A\phi(\theta/4)$$

$$= \frac{A}{(1+\theta^2)M(|\theta|)} .$$

Thus

$$\int M(|\theta|)|v^{\dagger}(\theta)|d\theta < \infty . \qquad (2.5)$$

The sufficiency part of Theorem 1 follows from (2.5) if we take $p(x)$ to be some suitable multiple of $v(x)$.

3. Some Lemmas

In this section we establish certain lemmas needed in the proof of Theorem 2.

Suppose $\phi(x)$ belongs to $L_2(-\infty, +\infty)$ and takes only non-negative values. Let us define A to be the set of real points x where

$$(1 + x^4)\phi(x) \geq 1 \ , \tag{3.1}$$

and let B be the complementary set of reals where (3.1) fails to hold. Then if $x \in A$ and $\phi(x) \leq 1$,

$$|\log \phi(x)| \leq \log(1 + x^4) \ ;$$

while if $x \in A$ and $\phi(x) > 1$,

$$|\log \phi(x)| \leq \phi(x) \ .$$

Thus

$$\int_A \frac{|\log \phi(x)|}{1+x^2}\, dx \leq \int_{-\infty}^{+\infty} \frac{\log(1+x^4)}{1+x^2}\, dx + \int_{-\infty}^{+\infty} \frac{\phi(x)}{1+x^2}\, dx \ .$$

Both the integrals on the right converge, the second one because $\phi(x) \in L_2$. Thus

$$\int_A \frac{|\log \phi(x)|}{1+x^2}\, dx < \infty \ . \tag{3.2}$$

Let us therefore define

$$\psi(x) = \phi(x) \quad \text{if} \quad x \in A$$

$$= \frac{1}{1+x^4} \quad \text{if} \quad x \in B .$$

Then, for all real x, $\psi(x) \geq \phi(x)$. But

$$0 \leq \psi(x) \leq \phi(x) + \frac{1}{1+x^4} , \tag{3.3}$$

so that $\psi(x) \in L_2$ also. Indeed, if we were given that $\phi(x) \in L_1 \cap L_2$ then (3.3) also shows that $\psi(x) \in L_1 \cap L_2$. On the other hand,

$$\int_{-\infty}^{+\infty} \frac{|\log\psi(x)|}{1+x^2}\,dx \leq \int_A \frac{|\log\phi(x)|}{1+x^2}\,dx + \int_{-\infty}^{+\infty} \frac{\log(1+x^4)}{1+x^2}\,dx < \infty , \quad \text{by (3.2)}.$$

We have therefore proved the following.

<u>Lemma 3.1</u> *If* $\phi(x)$ *is any non-negative function in* L_2 *then there exists another function* $\psi(x)$ *in* L_2 *such that* $\psi(x) \geq \phi(x)$ *for all* x *and*

$$\int_{-\infty}^{+\infty} \frac{|\log\psi(x)|}{1+x^2}\,dx < \infty .$$

Furthermore, if $\phi(x) \in L_1 \cap L_2$ *then* $\psi(x) \in L_1 \cap L_2$ *also, and if* $\int_{-\infty}^{+\infty} x^2\phi(x)dx < \infty$ *then* $\int_{-\infty}^{+\infty} x^2\psi(x)dx < \infty$.

Note that the final comment in this enunciation follows immediately from the inequalities (3.3).

We shall also need the following lemma which helps us in assessing the relative order of magnitudes of certain integrals that we shall encounter.

<u>Lemma 3.2</u> *If* $\phi(x)$ *is any real valued, non-negative function in* L_2 *then for all large real* r

$$\int_r^{\infty} \frac{\log[1/\phi(x)]}{x^3}\,dx > \frac{\log r}{4r^2}. \tag{3.4}$$

If $\int_{-\infty}^{+\infty} x^2\phi(x)dx < \infty$, *(implying* $\phi(x) \in L_1 \cap L_x$ *) then (3.4) shows that for any fixed* ζ, $0 < \zeta < 1$.

$$\int_{\zeta r}^{\infty} \phi(x)dx = o\left\{\int_r^{\infty} \frac{\log[1/\phi(x)]}{x^3}\,dx\right\}. \tag{3.5}$$

<u>Proof.</u> Since $\log(1/y)$ is convex in $(0,\infty)$, it follows from Jensen's inequality that

$$2r^2\int_r^{\infty} \frac{\log[\{1/\phi(x)\}^2]}{x^3}\,dx > \log\frac{1}{\gamma},$$

where

$$\gamma = 2r^2\int_r^{\infty} \frac{[\phi(x)]^2}{x^3}\,dx \le \frac{2}{r}\int_r^{\infty} [\phi(x)]^2dx = \frac{\delta(r)}{r}, \text{ say,}$$

and $\delta(r) \to 0$ as $r \to \infty$. Thus

$$\int_r^\infty \frac{\log\{1/\phi(x)\}}{x^3}\,dx > \frac{\log[r/\delta(r)]}{4r^2} ,$$

and this is a stronger result than the claimed one, (3.4).

To prove the latter part of the lemma we merely note that

$$\int_{\zeta r}^\infty \phi(x)dx \le \frac{1}{\zeta^2 r^2}\int_{\zeta r}^\infty x^2\phi(x)dx = 0\left(\frac{1}{r^2}\right) , \text{ as } r \to \infty ,$$

provided $\int_{-\infty}^{+\infty} x^2\phi(x)dx < \infty$. This completes the proof of the lemma.

4. Proof of Theorem 2

Let $\phi(x) \in L_2/-\infty, +\infty)$ be non-negative, non-null, and such that

$$\text{(4.1)} \qquad \int_{-\infty}^{+\infty} \frac{|\log\phi(x)|}{1+|x|^3}\,dx < \infty ,$$

although

$$\text{(4.2)} \qquad \int_{-\infty}^{+\infty} \frac{|\log\phi(x)|}{1+x^2}\,dx = \infty .$$

By Lemma 3.1 we can find a real function $\psi(x)$ in $L_2(-\infty, +\infty)$ such that $\psi(x) \ge \phi(x)$, all x , and

$$\text{(4.3)} \qquad \int_{-\infty}^{+\infty} \frac{|\log\psi(x)|}{1+x^2}\,dx < \infty .$$

For large $r > 0$ define

$$\phi_r(x) = \phi(x) \quad \text{for} \quad |x| \le r$$
$$= \psi(x) \quad \text{for} \quad |x| > r .$$

Then, for $z = x + iy,\ y > 0$, define the harmonic function

(4.4) $$U_r(x,y) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{y \log\phi_r(t)}{(x-t)^2+y^2}\,dt .$$

Let $V_r(x,y)$ be the function conjugate to $U_r(x,y)$ and set

(4.5) $$h_r(z) = \exp\{U_r(x,y) + iV_r(x,y)\} .$$

Evidently

$$\int_{-\infty}^{+\infty} \frac{|\log\phi_r(x)|}{1+x^2}\,dx < \infty .$$

Thus the following four results follow from Paley & Wiener [1, pp. 17,18]: -

(a) $$\lim_{y\downarrow 0} U_r(x,y) = \log\phi_r(x)$$

(b) $$\lim_{y\downarrow 0} |h_r(x + iy)| = \phi_r(x)$$

(c) $$\int_{-\infty}^{+\infty} |h_r(x + iy)|^2 dx \le \int_{-\infty}^{+\infty} [\phi_r(x)]^2 dx$$

(d) there exists $k_r(\theta) \in L_2(-\infty, +\infty)$ such that $k_r(\theta) \equiv 0$ for $\theta \ge 0$, and if we define

$$K_r(x) = \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{-A}^{0} e^{-ix\theta} k_r(\theta)\,d\theta$$

then

$$\underset{y \downarrow 0}{\text{l.i.m.}}\ h_r(x+iy) = K_r(x) .$$

Thus $|K_r(x)| = \phi_r(x)$ almost everywhere.

Next we note that for $s > r$,

$$U_r(x,y) - U_s(x,y) = \frac{1}{\pi}\int_{r<|t|<s} \frac{y \log[\psi(t)/\phi(t)]}{(x-t)^2+y^2}\,dt = \mathcal{R} f_{rs}(z) \text{ , say, where}$$

(4.6) $$f_{rs}(z) = \frac{1}{\pi i}\int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{t-z}\,dt$$

is analytic everywhere except for the points on the segment $\mathcal{I}z = 0$, $r < \mathcal{R}z < s$. We infer that

$$V_r(x,y) - V_s(x,y) = \mathcal{I} f_{rs}(z) = -\frac{1}{\pi}\int_{r<|t|<s} \frac{(t-x)\log[\psi(t)/\phi(t)]}{(t-x)^2+y^2}\,dt .$$

Further, if $|x| < r$, we have

$$\lim_{y \downarrow 0} V_r(x,y) - V_s(x,y) = V_{rs}(x) \text{ , say,}$$

where

(4.7) $$V_{rs}(x) = -\frac{1}{\pi}\int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{(t-x)}\,dt .$$

Fix ζ, $0 < \zeta < 1$, and assume that $x \le \zeta r$. Then, using in (4.7) the simple identity

$$\frac{1}{t-x} = \frac{1}{t} + \frac{x}{t^2} + \frac{x^2}{t^2(t-x)}$$

we find

(4.8) $$V_{rs}(x) = -\frac{1}{\pi}\int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{t}\,dt - \frac{x}{\pi}\int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{t^2}\,dt + \rho_{rs}(x) \text{ , say,}$$

where

(4.9) $$|\rho_{rs}(x)| < \frac{x^2}{(1-\zeta)}\int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{|t|^3}\,dt \ .$$

Let us now set, for any convenient choice of a fixed $\lambda > 0$,

(4.10) $$\alpha(r) = \frac{1}{\pi}\int_{\lambda\le|t|\le r} \frac{\log[\psi(t)/\phi(t)]}{t}\,dt$$

and

(4.11) $$\beta(r) = \frac{1}{\pi}\int_{\lambda\le|t|\le r} \frac{\log[\psi(t)/\phi(t)]}{t^2}\,dt \ .$$

Then, from (4.8), (4.10), (4.11) we derive

(4.12) $$V_{rs}(x) = [\alpha(r) - \alpha(s)] + x[\beta(r) - \beta(s)] + \rho_{rs}(x) \ .$$

Let us next set

$$G_r(x) = e^{-i\alpha(r)-ix\beta(r)}K_r(x) \quad . \tag{4.13}$$

Then $|G_r(x)| = \phi_r(x)$ almost everywhere, and if $g_r(\theta)$ is the Fourier Transform of $G_r(x)$ in the sense that (a.e.):

$$G_r(x) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{2\pi}\int_{-A}^{+A} e^{-i\theta x} g_r(\theta)d\theta$$

and

$$g_r(\theta) = \underset{A\to\infty}{\text{l.i.m.}} \int_{-A}^{+A} e^{i\theta x} G_r(x)dx \ ,$$

then (4.13) shows that (a.e.)

$$g_r(\theta) = e^{-i\alpha(r)} k_r\big(\theta - \beta(r)\big) \quad . \tag{4.14}$$

From (3.3), for $|x| < \zeta r$,

$$|U_r(x,y) - U_s(x,y)| \le \frac{1}{\pi}\int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{(1-\zeta)t}\,dt = I(r,s) \ , \text{ say.} \tag{4.15}$$

Let us, for typographical ease, write

$$E(r,s,x) = \frac{e^{-i\alpha(r)-i\beta(r)x}}{e^{-i\alpha(s)-i\beta(s)x}} \ .$$

Then define

$$m_{rs}(x,y) = \left\{\frac{h_r(x+iy)}{h_s(x+iy)}\right\} E(r,s,x) - 1 \ . \tag{4.16}$$

From (4.5) we have

(4.17) $$1 + m_{rs}(x,y) = e^{\lambda_{rs}(x,y)}\ , \text{ say,}$$

where

$$\mathcal{R}\lambda_{rs}(x,y) = U_r(x,y) - U_s(x,y)\ .$$

Thus, by (4.14), for all $y > 0$ and all $|x| \le \zeta r$,

(4.18) $$|m_{rs}(x,y)| \leqq 1 + e^{I(r,s)} < \infty\ .$$

Furthermore, for all $|x| \le \zeta r < r$, we have from (a) that

(4.19) $$\lim_{y\downarrow 0} U_r(x,y) - U_s(x,y) = 0\ ;$$

that is,

$$\lim_{y\downarrow 0} \mathcal{R}\lambda_{rs}(x,y) = 0\ .$$

Additionally, (4.16) and (4.5) show

$$\mathcal{I}\lambda_{rs}(x,y) = V_r(x,y) - V_s(x,y) - [\alpha(r) - \alpha(s)] - x[\beta(r) - \beta(s)]\ ,$$

and so, using (4.12),

(4.20) $$\lim_{y\downarrow 0} \mathcal{I}\lambda_{rs}(x,y) = V_{rs}(x) - [\alpha(r)-\alpha(s)] - x[\beta(r)-\beta(s)] = \rho_{rs}(x)\ .$$

Thus (4.17), (4.19), and (4.20) show that, as $y \downarrow 0$,

$$m_{rs}(x,y) \to e^{i\rho_{rs}(x)} - 1 . \tag{4.21}$$

We know from (d) that

$$\lim_{y\downarrow 0} \int_{-\zeta r}^{\zeta r} |h_r(x + iy) - K_r(x)|^2 dx = 0 , \tag{4.22}$$

and, since

$$\int_{-\zeta r}^{\zeta r} |h_r(x + iy) - K_r(x)| dx \le \sqrt{\left\{2\zeta r \int_{-\zeta r}^{\zeta r} |h_r(x + iy) - K_r(x)|^2 dx\right\}} ,$$

we also have

$$\lim_{y\downarrow 0} \int_{-\zeta r}^{\zeta r} |h_r(x + iy) - K_r(x)| dx = 0 . \tag{4.23}$$

From (4.13),

$$\begin{aligned}\int_{-\zeta r}^{\zeta r} |G_r(x) - G_s(x)| dx &= \int_{-\zeta r}^{\zeta r} |E(r,s,x)K_r(x) - K_s(x)| dx \\ &\le \int_{-\zeta r}^{\zeta r} |K_s(x) - h_s(x + iy)| dx \\ &\quad + \int_{-\zeta r}^{\zeta r} |K_r(x) - h_r(x + iy)| dx \\ &\quad + \int_{-\zeta r}^{\zeta r} |h_s(x + iy)|\, |m_{rs}(x,y)| dx .\end{aligned}$$

By (4.23) we can replace the first two terms on the right of this inequality by $\varepsilon(y)$, a bounded function which tends to zero as $y \downarrow 0$, for r and s fixed (recall $s > r$). The remaining term is

$$\leq \int_{-\zeta r}^{\zeta r} |K_s(x)|\, |m_{rs}(x,y)|dx + \int_{-\zeta r}^{\zeta r} |h_s(x+iy) - K_s(x)|\, |m_{rs}(x,y)|dx .$$

Thus, if we use (4.18),

$$\begin{aligned}\int_{-\zeta r}^{\zeta r} |G_r(x) - G_s(x)|dx &\leq \varepsilon(y) + [1 + e^{I(r,s)}] \int_{-\zeta r}^{\zeta r} |K_s(x) - h_s(x+iy)|dx \\ &\qquad + \int_{-\zeta r}^{\zeta r} |m_{rs}(x,y)|\, |K_s(x)|dx , \\ &\leq \varepsilon(y) + \int_{-\zeta r}^{\zeta r} |m_{rs}(x,y)|\phi(x)dx ,\end{aligned}$$

where $\varepsilon(y)$ in the last inequality includes the first two terms on the right of the preceeding inequality. Then, by dominated convergence and a further appeal to (4.18), we have from (4.21)

$$\int_{-\zeta r}^{\zeta r} |G_r(x) - G_s(x)|dx \leq \int_{-\zeta r}^{\zeta r} |e^{i\rho_{rs}(x)} - 1|\phi(x)dx \leq A \int_{-\zeta r}^{\zeta r} |\rho_{rs}(x)|\phi(x)dx ,$$

for some absolute constant A. The last integral, in turn is, by (4.9),

$$\leq \frac{A}{\pi(1-\zeta)} \int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{|t|^3}\, dt \times \int_{-\zeta r}^{\zeta r} x^2\phi(x)dx .$$

Let us set

$$S = \int_{-\infty}^{+\infty} x^2 \phi(x)dx ,$$

and let us proceed on the assumption that S is finite. Then

$$\int_{-\zeta r}^{\zeta r} |G_r(x) - G_s(x)|dx \le \frac{AS}{\pi(1-\zeta)} \int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{|t|^3} dt .$$

But $|G_r(x)| = \phi_r(x)$ a.e., so we may infer that $G_r(x)$ is in L_1 and that

$$\int_{-\infty}^{+\infty} |G_r(x) - G(x)|dx \le \frac{AS}{\pi(1-\zeta)} \int_{r<|t|<s} \frac{\log[\psi(t)/\phi(t)]}{|t|^3} dt$$

$$+ \int_{|t|>\zeta r} \phi_r(x)dx + \int_{|t|>\zeta r} \phi_s(x)dx .$$

Thus it transpires that $\{G_r(x)\}$ is a fundamental sequence of L_1-functions and hence there exists $G(x) \in L_1$ such that (using the definitions of $\phi_r(x)$ and $\phi_s(x)$),

$$(4.24)\quad \int_{-\infty}^{+\infty} |G_r(x) - G(x)|dx \le \frac{AS}{\pi(1-\zeta)} \int_{|t|>r} \frac{\log[\psi(t)/\phi(t)]}{|t|^3} dt + 2\int_{\zeta r<|t|<r} \phi(t)dt + \int_{|t|>r} [\phi(t) + \psi(t)]dt \ .$$

The terms involving ψ on the right of this inequality give

$$(4.25)\quad -\frac{AS}{\pi(1-\zeta)} \int_{|t|>r} \frac{\log[1/\psi(t)]}{|t|^3} dt + \int_{-r}^{r} \psi(t)dt \ .$$

If we appeal to Lemma 3.2, applied to $\psi(x)$, then (3.4) and (3.5) show (4.25) to be ultimately negative. The same lemma applied to the terms involving ϕ on the right of (4.24) then shows that we can have a simplier inequality

$$(4.26)\quad \int_{-\infty}^{+\infty} |G_r(x) - G(x)|dx \le A' \int_{|t|>r} \frac{\log[1/\phi(t)]}{|t|^3} dt \ ,$$

for all large r, where A' is some new constant.

Let $g(\theta)$ be the Fourier Transform of $G(x)$. Then, since $G_r(x) - G(x)$ is in L_1 , we have that for almost all θ ,

$$g_r(\theta) - g(\theta) = \int_{-\infty}^{+\infty} e^{ix\theta}[G_r(x) - G(x)]dx \ .$$

Therefore, from (4.26), for all large r ,

$$|g_r(\theta) - g(\theta)| \le A' \int_{|t|>r} \frac{\log[1/\phi(t)]}{|t|^3} dt \ .$$

However, (4.14) shows that $g_r(\theta) \equiv 0$ when $\theta \geq \beta(r)$. Thus we discover that

(4.27) $$g(\beta(r)) \leq A'' \int_{|t|>r} \frac{\log[1/\phi(t)]}{|t|^3}\, dt$$

Note that, by (4.11), and (3.4) applied to $\psi(x)$, for all large r we shall have

$$\beta(r) \leq \frac{1}{\pi} \int_1^r \frac{\log[1/\phi(t)]}{t^2} = b(r) .$$

We then have from (4.27) the more convenient result

$$g(b(r)) = O(a(r)) ,$$

where

$$a(r) = \int_{|t|>r} \frac{\log[1/\phi(t)]}{|t|^3}\, dt .$$

This proves one part of Theorem 2 , the part based on the assumption that $x^2\phi(x) \in L_1$. To prove the other part, on the assumption $x^2\phi(x) \in L_2$, we proceed as follows. For any function f , say, in $L_2(-\zeta r, \zeta r)$ write

$$||f|| = \sqrt{\int_{-\zeta r}^{\zeta r} |f(x)|^2 dx} .$$

Then let us note that

$$E(r,s,x)h_r(x + iy) - h_s(x + iy) = h_s(x + iy)m_{rs}(x,y) ,$$

$$E(r,s,x)K_r(x) - K_s(x) = E(r,s,x)K_r(x) - E(r,s,x)h_r(x+iy)$$
$$+ h_s(x+iy) - K_s(x) + h_s(x+iy)m_{rs}(x,y) ,$$

and $|E(r,s,x)| = 1$. We then have

$$||E(r,s,x)K_r(x) - K_s(x)|| \leq ||K_r(x) - h_r(x+iy)||$$
$$+ ||K_s(x) - h_s(x+iy)||$$
$$+ ||h_s(x+iy)m_{rs}(x,y)||$$

From (4.22) we see that, as $y \downarrow 0$,

$$||K_r(x) - h_r(x+iy)|| \to 0$$

and

$$||K_s(x) - h_s(x+iy)|| \to 0 .$$

Also,

$$||h_s(x+iy)m_{rs}(x,y)|| \leq ||\{h_s(x+iy) - K_s(x)\}m_{rs}(x,y)|| + ||K_s(x)m_{rs}(x,y)|| .$$

By (4.18) we see that

$$||\{h_s(x+iy) - K_s(x)\}m_{rs}(x,y)|| \le \left(1 + e^{I(r,s)}\right)||h_s(x+iy) - K_s(x)||$$

$$\to 0 \quad \text{as} \quad y \downarrow 0 \text{ , since } \quad I(r,s) < \infty$$

Moreover $|K_s(x)| = \phi_s(x)$ a.e., and thus $|K_s(x)| = \phi(x)$ when $|x| \le s$. Hence by dominated convergence and (4.21),

$$||K_s(x)m_{rs}(x,y)|| \to ||\phi(x)\{e^{i\rho_{rs}(x)} - 1\}||$$

as $y \downarrow 0$. If we write $A, A', \ldots$ and so on for various constants, (4.9) shows

$$||\phi(x)\{e^{i\rho_{rs}(x)} - 1\}|| \le A\left\{\int_{r<|t|<s} \frac{\log\psi(t)/\phi(t)}{|t|^3}\,dt\right\}||x^2\phi(x)||$$

$$= A'\int_{r<|t|<s} \frac{\log\psi(t)/\phi(t)}{|t|^3}\,dt \ ,$$

since we may now suppose $x^2\phi(x) \in L_2$. Thus we may conclude

$$||E(r,s,x)K_r(x) - K_s(x)|| \le A'\int_{r<|t|<s} \frac{\log\psi(t)/\phi(t)}{|t|^3}\,dt \ .$$

But (4.13) shows

$$||G_r(x) - G_s(x)|| = ||E(r,s,x)K_r(x) - K_s(x)|| \ .$$

Thus, rather as before, we can deduce that $\{G_r(x)\}$ is a fundamental sequence, in L_2 this time. There exists an L_2 function $G(x)$ such that $G_s(x)$ tends to $G(x)$, as $s \to \infty$, in L_2 , and

$$\int_{-\infty}^{+\infty} |G_r(x) - G(x)|^2 dx \le \left\{A' \int_{r<|t|<s} \frac{\log\psi(t)/\phi(t)}{|t|^3}\, dt\right\}^2 + 2\int_{\zeta r<|t|<r} \{\phi(t)\}^2 dt$$

$$+ \int_r^\infty [\{\phi(t)\}^2 + \{\psi(t)\}^2] dt \ .$$

Now

$$\int_{\zeta r}^\infty \{\phi(t)\}^2 dt \le \frac{1}{\zeta^4 r^4} \int_{\zeta r}^\infty t^4 \{\phi(t)\}^2 dt = O\left(\frac{1}{r^4}\right) ,$$

so, by Lemma 3.2 (especially (3.4)), we find

$$\int_{\zeta r}^\infty \{\phi(t)\}^2 dt = O\left(\int_r^\infty \frac{\log[1/\phi(t)]}{t^3}\, dt\right) .$$

A similar result holds with $\psi(t)$ in place of $\phi(t)$, and we are led, much as in the earlier case, to the result

$$(4.29) \quad \int_{-\infty}^{+\infty} |G_r(x) - G(x)|^2 dx \quad \left\{A'' \int_{|t|>r} \frac{\log[1/\phi(t)]}{t^3}\, dt\right\}^2 .$$

But, by Parseval's theorem,

$$\int_{-\infty}^{+\infty} |G_r(x) - G(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |g_r(\theta) - g(\theta)|^2 d\theta .$$

Since $g(\theta) \equiv 0$ in $\theta \geq \beta(r)$, we can hence infer from (4.29) the result

$$\left\{ \int_{\beta(r)}^{\infty} |g(\theta)|^2 d\theta \right\} \leq A'' \int_{|t|>r} \frac{\log[1/\phi(t)]}{|t|^3} dt .$$

The remaining substitution of $b(r)$ for $\beta(r)$ then completes the proof of Theorem 2.

5. Proof of Theorem 3

Little need be said concerning this proof since an argument almost identical with the sufficiency part of the proof of Theorem 1 is all that is needed, after an appeal to Theorem 2. Set $\phi(\theta) = 1/(1 + 16\theta^2)M(4|\theta|)$. Then $\phi(\theta) \in L_1 \cap L_2$ and:

(a) if [3a] holds, $\theta^2\phi(\theta) \in L_1$;

(b) if [3b] holds, $\theta^2\phi(\theta) \in L_2$.

Suppose [3a] holds. Then Theorem 2 shows the existence of a continuous and bounded $g(x)$ in L_2 which is identically zero in $x \geq 1$, non-zero in some interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, and with a Fourier Transform $g^\dagger(\theta)$ such that $|g^\dagger(\theta)| \equiv \phi(\theta)$. As before, one can take $v(x) = |g(x)g(-x)|^2 \geq 0$ and deduce that $v^\dagger(\theta) = O(\phi(\theta/4))$, so that $\int_{-\infty}^{+\infty} M(|\theta|)|v^\dagger(\theta)|d\theta < \infty$. But $v(x) = O(|g(|x|)|^2)$, and, by Theorem 2,

$g(b(|x|)) = O(a(|x|))$, where

$$(5.1) \qquad a(|x|) = \int_{|x|}^{\infty} \frac{\log(1+16t^2)M(4|t|)}{t^3}\,dt$$

and

$$b(|x|) = \frac{2}{\pi}\int_{\lambda}^{|x|} \frac{\log(1+16t^2)M(4|t|)}{t^2}\,dt \le \rho + b_2(|x|) ,$$

as defined in the enunciation of Theorem 3. We also have, from (5.1), that

$$a(|x|) = a_2(|x|) + O\left(\frac{\log|x|}{x^2}\right) ,$$

where $a_2(|x|)$ is also defined in that enunciation. But, under [3a], $1/[M(4|t|)]^{\frac{1}{2}}$ belongs to L_2 . Thus we can deduce from Lemma 3.2 that for all large $|x|$,

$$\int_{|x|}^{\infty} \frac{\log M(4|t|)}{t^3}\,dt > \frac{\log|x|}{2x^2}$$

Thus $a(|x|) = O(a_2(|x|))$, and part of Theorem 3 is proved. It should be clear that the remaining part, when [3b] holds, follows from Theorem 2 in a very similar way.

We end with an example. Intuitively one might feel that if $M(x)$ were monotone and increasing sufficiently fast to make

$$\int_1^{\infty} \frac{|\log M(x)|}{x^2}\,dx = \infty ,$$

Then $[M(|x|)]^{-1}$ would necessarily belong to L_1, say. Condition [3a], for example, would be unnecessary. However this intuition is false and [3a] cannot be dropped. To see this, we shall construct an increasing sequence of reals $\{\xi_n\}$ as follows. Suppose $\xi_1 = 1$ and $\xi_2, \xi_3, \ldots, \xi_n$ have been defined. Define $M(x) = x$ in $\xi_n \le x < 2\xi_n$ and set $\xi_{n+1} = e^{2\xi_n}$. Then $\xi_{n+1} > \xi_n$ and $\xi_n \to \infty$ as $n \to \infty$, as desired. Also

$$\int_{\xi_n}^{\xi_{n+1}} \frac{dx}{M(x)} > \int_{\xi_n}^{2\xi_n} \frac{dx}{x} = \log 2$$

so that

$$\int_1^\infty \frac{dx}{M(x)} = \infty .$$

Further, define $M(x) = e^{2\xi_n}$ in $2\xi_n \le x < \xi_{n+1}$. It is clear that this ensures that $M(x)$ is non-decreasing. But

$$\int_{2\xi_n}^{\xi_{n+1}} \frac{|\log M(x)|}{x^2}\, dx = 2\xi_n \left\{ \frac{1}{2\xi_n} - \frac{1}{\xi_{n+1}} \right\} = 1 - 2\xi_n e^{-2\xi_n} \to 1 \text{ , as } n \to \infty .$$

Thus

$$\int_1^\infty \frac{|\log M(x)|}{x^2}\, dx = \infty$$

and the demonstration is apparent. Clearly a similar example will show the need for [3b].

References

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