*Supported by the U.S. Army Research Office, Durham, under Contract No. DAHC04-71-C-0042.

ROBUSTNESS OF CERTAIN TESTS OF CENSORING OF EXTREME SAMPLE VALUES II: SOME EXACT RESULTS FOR EXPONENTIAL POPULATIONS*

by

N. L. Johnson

Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 940
July 1974

ROBUSTNESS OF CERTAIN TESTS OF CENSORING OF EXTREME SAMPLE VALUES - II
SOME EXACT RESULTS FOR EXPONENTIAL POPULATIONS

by

N. L. Johnson

University of North Carolina at Chapel Hill

1. Introduction

In [1] we have given general formulae for evaluating the effect of an incorrect choice of population distribution on certain tests of sample censoring. (For definitions, etc. see [2][3].)

These formulae apply when the form of population distribution in known, but certain parameters $\underset{\sim}{\theta}$ have to be estimated. The formulae apply when possibly incorrect values $\underset{\sim}{\theta}^*$ are used for $\underset{\sim}{\theta}$.

When $\underset{\sim}{\theta}$ has to be estimated, there will usually be random variation in $\underset{\sim}{\theta}^*$. The overall effect of this can be evaluated by taking expected values of the formulae in [1] with respect to suitable distributions for $\underset{\sim}{\theta}^*$.

We shall suppose throughout the paper that there is sampling from a population in which the measured character has an absolutely continuous distribution.

2. General Discussion

As in [1] and earlier papers we suppose r values $X_1 \leq X_2 \leq \ldots \leq X_r$ are available and wish to test whether they are a complete random sample, or come from a complete random sample of size $(s_0 + r + s_r)$, from which the s_0 smallest and s_r largest values have been censored. We denote this last hypothesis by H_{s_0,s_r} ; $H_{0,0}$ corresponds to the case of no censoring.

The critical regions of the tests described in [1][2] are all defined in terms of the probability integral transforms

$$(1) \qquad Y_1 = \int_{-\infty}^{X_1} f(x|\underset{\sim}{\theta})dx \ ; \ Y_r = \int_{-\infty}^{X_r} f(x|\underset{\sim}{\theta})dx$$

of the minimum and maximum values X_1, X_r respectively. Here $f(x|\underset{\sim}{\theta})$ is the density function of X. If, in fact, an incorrect value of $\underset{\sim}{\theta}$ is used - $\underset{\sim}{\theta}^*$, say - then instead of calculating Y_1, Y_r we actually calculate

$$(2) \qquad Y_j^* = \int_{-\infty}^{X_j} f(x|\underset{\sim}{\theta}^*)dx \qquad (j = 1, r) \ .$$

Now provided $f(x|\underset{\sim}{\theta}) > 0$ for all x and $\underset{\sim}{\theta}$, (1) and (2) define Y_j^* as a unique function $h^*(Y_j)$ of Y_j. So when we use a critical region $w_{(k)}$ defined by

$$w_{(k)} \ (Y_1^*, Y_r^*) \geq C_\alpha^{(k)}$$

($C_\alpha^{(k)}$ is a constant which would give 100α% significance level if $H_{0,0}$ were valid *and* the correct $\underset{\sim}{\theta}$ were used) we are actually using a critical region $w_{(k)}^*$ defined by

$$(3) \qquad w_{(k)} \left(h^*(Y_1), h^*(Y_r)\right) \geq C_\alpha^{(k)}$$

in terms of the correct Y_1 and Y_r. The latter have joint density function (under H_{s_0,s_r})

$$(4) \qquad \frac{(r+s_0+s_r)!}{s_0!(r-2)!s_r!} y_1^{s_0}(y_r - y_1)^{r-2} y_r^{s_r} \qquad (0 \le y_1 \le y_r \le 1)$$

For a given $\underset{\sim}{\theta}^*$, the power $\beta_{(k)}(s_0, s_r|\underset{\sim}{\theta}^*)$ of the test with critical region $w^*_{(k)}$ is the integral of (4) over the region

$$(5) \qquad w_{(k)}\left(h^*(y_1), h^*(y_r)\right) \ge C_\alpha^{(k)} .$$

The unconditional, overall power $\beta_{(k)}(s_0, s_r)$ is the expected value, with respect to variation in $\underset{\sim}{\theta}^*$ of $\beta_{(k)}(s_0, s_r|\underset{\sim}{\theta}^*)$. In particular the actual significance level is

$$(6) \qquad \beta_{(k)}(0, 0) = r(r - 1)E\left[\int_{w^*_{(k)}}\int (y_r - y_1)^{r-2}\, dy_1\, dy_r\right]$$

(θ^*, of course appears implicitly in $w^*_{(k)}$.) This will not in general be equal to α, if $C_\alpha^{(k)}$ is calculated in accordance with the formulae in [2]. It will, in particular cases, be possible to choose $C_\alpha^{(k)}$ so that the overall significance level is α, but the necessary formulae will depend on the density function of X, and on that of $\underset{\sim}{\theta}^*$, while the values in ( ) do not.

As a concrete example we consider the case of normal variation.

In [1] it was shown that if $\underset{\sim}{\theta} = (\xi, \sigma)$ and

$$f(x|\xi, \sigma) = (\sigma\sqrt{2\pi})^{-1} \exp\left(-(x - \xi)^2(2\sigma^2)^{-1}\right)$$

then $\beta(s_0, s_r|\xi^*, \sigma^*)$ depends only on

$$\Delta^* = (\xi^* - \xi)/\sigma \quad \text{and} \quad \varepsilon^* = \sigma^*/\sigma .$$

If ξ^* and σ^{*2} are calculated from an independent random sample of size N, as the sample mean and variance respectively then

(i) Δ^* and ε^* are independent

(ii) Δ^* is distributed $N(0, N^{-1})$ which gives useful practical indicati...

(iii) ε^* is distributed $(N - 1)^{-\frac{1}{2}}\chi_{N-1}$.

This information is sufficient to enable us to calculate

$$\beta(s_0, s_r) = E[\beta(s_0, s_r|\xi^*, \sigma^*)] \tag{7}$$

the expectation being with respect to variation of ξ^* and σ^* .

Although this may be evaluated explicitly using numerical cubature applied to $\beta(s_0, s_r|\xi^*, \sigma^*)$ which is itself quite a complicated function, the calculations are quite heavy even with the assistance of an electronic computer. (It is hoped to provide some numerical tables in the third report in this series.) It is rather easier to use a two-entry table of values of $\beta(s_0, s_r|\xi^*, \sigma^*)$ and (very roughly) average with respect to the distributions of ξ^* and σ^* .

A further possible complication is that it may be necessary to estimate $\underset{\sim}{\theta}$ from the r values $X_1, \ldots, X_r$ actually available, so that $\underset{\sim}{\theta}^*$ and (Y_1, Y_r) are no longer mutually independent. We do not consider this situation in the present paper, except for a brief discussion in the final section.

3. Exponential Populations

The main aim of the present paper is to evaluate performance of the three tests described in [2] when applied to a population in which the (unorthodox) X's have an exponential distribution with expected value θ . We will suppose θ to be estimated as the arithmetic mean of a random sample (independent of the r values to be analysis) of size N , so that:

$$\text{(8)} \qquad \theta^{*}/\theta \quad \text{is distributed as} \quad (2N)^{-1}\,\chi^{2}_{2N} \,.$$

Under these conditions it is possible to get explicit formulae for properties of the tests which are relatively simple (as compared, for example, with the normal case).

This discussion will, it is hoped, help to make clearer the general description in Section 2.

The population density function is $\theta^{-1}\exp(-x/\theta)$ $(\theta, x > 0)$ corresponding to a $\frac{1}{2}\theta\chi^{2}_{2}$ distribution.

We have $Y = \int_0^X \theta^{-1}\exp(-x|\theta)dx = 1 - e^{-X/\theta}$ $(X > 0)$.

Hence

$$\text{(9)} \qquad Y^{*} = 1 - e^{-X/\theta^{*}} = 1 - (1 - Y)^{\theta/\theta^{*}} \,.$$

The critical regions to be used in testing for extreme censoring (1) form below (2) symmetrically or (3) in general, are (see [2]) respectively

(10.1) (Below) $$Y_1 \geq C_\alpha^{(1)}$$

(10.2) (Symmetric) $$Y_1(1 - Y_r) \geq C_\alpha^{(2)}$$

(10.3) (General) $$Y_1 + (1 - Y_r) \geq C_\alpha^{(3)}$$

If an incorrect value, θ^*, is used for θ, the regions actually used are, respectively

(11.1) $$1 - (1 - Y_1)^{\theta/\theta^*} \geq C_\alpha^{(1)}$$

(11.2) $$\{1 - (1 - Y_1)^{\theta/\theta^*}\} (1 - Y_r)^{\theta/\theta^*} \geq C_\alpha^{(2)}$$

(11.3) $$1 - (1 - Y_1)^{\theta/\theta^*} + (1 - Y_r)^{\theta/\theta^*} \geq C_\alpha^{(3)}$$

4. Censoring from Below

The inequality (11.1) can also be written

(11.1)' $$Y_1 \geq 1 - (1 - C_\alpha^{(1)})^{\theta^*/\theta} = 1 - (1 - C_\alpha^{(1)})^T$$

where we put

(12) $$T = \theta^*/\theta .$$

Since, given H_{s_0,s_r} the distribution of Y_1 is standard beta with parameters $s_0 + 1$, $r + s_r$ we have

(13) $$\beta_{(1)}(s_0, s_r|\theta^*) = \Pr[Y_1 \geq 1 - (1 - C_\alpha^{(1)})^T | s_0, s_r]$$

$$= I_{(1-C_\alpha^{(1)})^T}(r + s_r, s_0 + 1)$$

where $I_p(\alpha, \gamma) = \{B(\alpha, \gamma)\}^{-1} \int_0^p t^{\alpha-1} (1 - t)^{\gamma-1} dt$ is the incomplete beta function ratio.

In particular the conditional significance level is

(14) $$\beta_{(1)}(0,0|\theta^*) = I_{(1-C_\alpha^{(1)})^T}(r, 1) = (1 - C_\alpha^{(1)})^{rT} .$$

The unconditional significance level is

(15) $$\begin{aligned} E[\beta_{(1)}(0, 0|\theta^*)] &= E[(1 - C_\alpha^{(1)})^{rT}] \\ &= E\left[(1 - C_\alpha^{(1)})^{\frac{1}{2}\chi^2_{2N}/N} \right] \\ &= \{1 - rN^{-1} \log(1 - C_\alpha^{(1)})\}^{-N} . \end{aligned}$$

In order to make the significance level equal to α, we must choose $C_\alpha^{(1)}$ (= $C_\alpha^{(1)}(N)$, say) to satisfy

$$\{1 - rN^{-1} \log(1 - C_\alpha^{(1)})\}^{-N} = \alpha$$

i.e.

(16) $$C_\alpha^{(1)}(N) = 1 - \exp\{r^{-1}N(1 - \alpha^{-1/N})\} .$$

If the value $C_\alpha^{(1)} = 1 - \alpha^{1/r}$, which is appropriate when θ is known, is used then the overall significance level is actually

$$(17) \qquad (1 - N^{-1} \log \alpha)^{-N} .$$

As would be expected this tends to α as N tends to infinity.

The power with respect to $H_{s_0, 0}$ (i.e. actual censoring from below) is

$$\beta_{(1)} (s_0, 0) = E\left[I_{(1-C_\alpha^{(1)})T} (r, s_0 + 1) \right]$$

$$(18) \qquad = \frac{(r+s_0)!}{(r-1)!s_0!} \sum_{j=0}^{s_0} (-1)^j \binom{s_0}{j} (r+j)^{-1} E\left[(1-C_\alpha^{(1)})^{\frac{1}{2}(r+j)\chi_{2N}^2/N} \right]$$

$$= \frac{(r+s_0)!}{(r-1)!s_0!} \sum_{j=0}^{s_0} (-1)^j \binom{s_0}{j} (r+j)^{-1} \{1-(r+j)N^{-1}\log(1-C_\alpha^{(1)})\}^{-N}.$$

Inserting the value for $C_\alpha^{(1)}$ from (16) (which gives overall significance level equal to α) we obtain

$$(19) \quad \beta_{(1)}(s_0, 0) = \frac{(r+s_0)!}{(r-1)!s_0!} \sum_{j=0}^{s_0} (-1)^j \binom{s_0}{j} (r+j)^{-1} \{1 + (1+jr^{-1})(\alpha^{-1/N}-1)\}^{-N} .$$

Table 1 gives values of $\beta_{(1)} (s_0, 0)$ for $r = 5(5)25$, $N = 5,10,25,50$ and $s_0 = 2(2)8$ with $\alpha = 0.05$. These are powers when the actual significance level is 0.05 . Table 2 gives the actual significance levels and powers obtained if the value $C_\alpha^{(1)} = 1 - \alpha^{1/r}$ is used. In this case

$$(20)\quad \beta_{(1)}(s_0,0) = \frac{(r+s_0)!}{(r-1)!s_0!}\sum_{j=0}^{s_0}(-1)^j\binom{s_0}{j}(r+j)^{-1}\{1-(1+jr^{-1})N^{-1}\log\alpha\}^{-N} .$$

(Some values of $\beta_{(1)}(s_0,\ s_r)$ with $s_r \neq 0$ are also given.)

5. Symmetrical and General Censoring

Similar calculations for the tests with critical regions (11.2) and (11.3) are complicated by the fact that both Y_1 and Y_r appear in the inequalities. In these cases we use the fact that the joint density functions of $Z_1 = 1 - Y_1$ and $Z_r = 1 - Y_r$ under H_{s_0,s_r} is

$$(21)\quad \frac{(r+s_0+s_r)!}{s_0!(r-2)!s_r!}(1 - z_1)^{s_0}(z_1 - z_r)^{r-2} z_r^{s_r} \qquad (0 \le z_r \le z_1 \le 1)$$

and so the conditional power $\beta_{(k)}(s_0,\ s_r|\theta^*)$ is obtained by integrating (21) over the regions

$$(22.2)\quad \left(1 - z_1^{\theta/\theta^*}\right) z_r^{\theta/\theta^*} \ge C_\alpha^{(2)} \qquad \text{for} \quad k = 2 \quad \text{(symmetrical censoring)}$$

or

$$(22.3)\quad 1 - z_1^{\theta/\theta^*} + z_r^{\theta/\theta^*} \quad C_\alpha^{(3)} \qquad \text{for} \quad k = 3 \quad \text{(general censoring)}$$

The resultant value has in each case, to be averaged over the distribution θ^* (or equivalently of $T = \theta^*/\theta$).

Making the transformations $v_j = z_j^{\theta/\theta^*}$ $(j = 1,2)$ we have

$$(23)\quad \beta_{(k)}(s_0,s_r|\theta^*) = \frac{(r+s_0+s_r)!}{s_0!(r-2)!s_r!}\iint T^2(1-v_1^T)^{s_0}(v_1^T-v_r^T)^{r-2}v_1^{T-1}v_r^{(s_r+1)T-1}\,dv_1dv_r$$

The regions of integration are:

(24.2) For $k = 2$: $(1 - v_1)v_r \geq C_\alpha^{(2)}$; $0 \leq v_r \leq v_1 \leq 1$

(24.3) For $k = 3$: $1 - v_1 + v_r \geq C_\alpha^{(3)}$; $0 \leq v_r \leq v_1 \leq 1$.

The overall power is obtained by taking the expected value with respect to the distribution of T , which is $(2N)^{-1}\chi^2_{2N}$.

By performing the integration first with respect to T , and then with respect to v_r , we can reduce the triple integral to a finite sum of single integrals.

The integrand in (23) is

$$T^2\left\{\sum_{i=0}^{s_0}(-1)^i\binom{s_0}{i}v_1^{iT}\right\}\left\{\sum_{j=0}^{r-2}(-1)^j\binom{r-2}{j}(v_1^{r-2-j}v_2^j)^T\right\}v_1^T v_r^{(s_r+1)T}(v_1v_r)^{-1}$$

$$= (v_1v_r)^{-1}T^2\sum_{i=0}^{s_0}\sum_{j=0}^{r-2}(-1)^{i+j}\binom{s_0}{i}\binom{r-2}{j}v_1^{(r+i-j-1)T}v_r^{(s_r+j+1)T} .$$

The expected value of this with respect to variation in T is

$$(v_iv_r)^{-1}\sum_{i=0}^{s_0}\sum_{j=0}^{r-2}(-1)^{i+j}\binom{s_0}{i}\binom{r-2}{j}(2N)^{-2}\times$$

$$\times E\left[\chi^4_{2N}\exp\{(2N)^{-1}[(r+i-j-1)\log v_1 + (s_r+j+1)\log v_r]\chi^2_{2N}\}\right]$$

(25)

$$= (v_1v_r)^{-1}N^{-1}(N+1)\sum_{i=0}^{s_0}\sum_{j=0}^{r-2}(-1)^{i+j}\binom{s_0}{j}\binom{r-2}{j}\times$$

$$\times [1 - N^{-1}\{(r+i-j-1)\log v_i + (s_r+j+1)\log v_r\}]^{-(N+2)} .$$

(Using the relation $E[\chi_\nu^4 \exp(A\chi^2)] = \nu(\nu+2)(1-2A)^{-\frac{1}{2}\nu-2}$ for $A < \frac{1}{2}$.)

The indefinite integral of (25) with respect to v_r is

$$v_1^{-1} \sum_{i=0}^{s_0} \sum_{j=0}^{r-2} (-1)^{i+j} \binom{s_0}{j} \binom{r-2}{j} (s_r + j + 1)^{-1} \times$$

$$[1 - N^{-1} \{(r + i - j - 1) \log v_1 + (s_r + j + 1) \log v_r\}]^{-(N+1)} .$$

For $k = 2$, the range of integration for v_r is $(1 - v_1)^{-1} C_\alpha^{(2)} \le v_r \le v_1$ and for v_1 , $v_1(1 - v_1) \ge C_\alpha^{(2)}$ i.e. $V_1^- \le v_1 \le V_1^+$ where

(26) $$V_1^{\pm} = \frac{1}{2}\left[1 \pm \sqrt{1 - 4C_\alpha^{(2)}}\right] .$$

Hence

(27.2) $$\beta_{(2)}(s_0, s_r) = \frac{(r+s_0+s_r)!}{s_0!(r-2)!s_r!} \sum_{i=0}^{s_0} \sum_{j=0}^{r-2} (-1)^{i+j} \binom{s_0}{i} \binom{r-2}{j} (s_r+j+1)^{-1} \times$$

$$\times \Bigg[(r + s_r + i)^{-1} \Big\{ [1 - N^{-1}(r + s_r + i) \log V_1^+]^{-N}$$

$$- [1 - N^{-1} (r + s_r + i) \log V_1^-]^{-N} \Big\}$$

$$- \int_{V_1^-}^{V_1^+} v_1^{-1} \{1 - N^{-1}[(s_r+j+1) \log \left(\frac{C_\alpha^{(2)}}{1-v_1}\right) + (r+i-j-1) \log v_1]\}^{-(N+1)} dv_1 \Bigg] .$$

When $k = 3$, the range of integration for v_r is $v_1 - (1-C_\alpha^{(3)}) \le v_r \le v_1$, and the range of integration for v_1 is $1 - C_\alpha^{(3)} \le v_1 \le 1$.

Hence

$$(27.3)\quad \beta_{(3)}(s_0,s_r) = \frac{(r+s_0+s_r)!}{s_0!(r-2)!s_r!}\sum_{i=0}^{s_0}\sum_{j=0}^{r-2}(-1)^{i+j}\binom{s_0}{j}\binom{r-2}{j}(s_r + j + 1)^{-1} \times$$

$$\times \Bigg[(r + s_r + i)^{-1}\left\{1 - [1 - N^{-1}(r + s_r + i)\log(1 - C_\alpha^{(3)})]^{-N}\right\}$$

$$- \int_{1-C_\alpha^{(3)}}^{1} v_1^{-1}\left\{1-N^{-1}[(r+i-j-1)\log v_1+(s_r+j+1)\log(v_1-1+C_\alpha^{(3)})]\right\}^{-(N+1)} dv_1 \Bigg] .$$

6. Comments on Tables

Note that in all cases, for $s_0 + s_r$ constant, the power decreases as s_0 decreases. This is to be expected, since this test is designed to detect alternative hypotheses of the hype $H_{s_0,0}$. Table 1 shows that the use of $C_\alpha^{(1)} = 1 - \alpha^{1/r}$ as critical limit increases the significance level, and also the power. As might be expected, this effect decreases as N increases.

If $C_\alpha^{(1)}(N)$ is used so that the actual significance level is α , Table 2 shows that the power increases with N . This is not always the case in Table 1. Only for $s_0 + s_r$ sufficiently big, *and* s_0 not too small, does the power increase for N in Table 1.

As $N \to \infty$ the values in both Tables tend to the values for the case when the population distribution is known, (which do not depend on the actual distribution, so long as it is known and continuous).

I would like to express my gratitude to Mr. Kerry L. Lee for programming the computation of extensive tables of which these are a part.

7. No Prior Estimate of Parameter Available

In this case we may (assuming the null hypothesis $H_{0,0}$ to be valid) estimate θ by θ^*, the arithmetic mean of the r available X's. If $H_{0,0}$ is valid then θ^* is distributed as $(2r)^{-1}\theta\chi^2_{2r}$, but $X_1, X_2, \ldots, X_r$ are not independent of θ^*. In fact

$$\Pr[X_1(r\theta^*)^{-1} < K | H_{0,0}] = 1 - r(1 - K)^{r-1} + \binom{r}{2}(1 - 2K)^{r-1} \ldots$$

(the series terminates at the m-th term when $1 - mk < 0 \le 1 - (m - 1)K$).

In this case, the actual significance level for $k = 1$ (i.e. when testing for censoring from below) is

$$\begin{aligned} &\Pr[Y_1^* > C_\alpha^{(1)} | H_{0,0}] \\ (28) \qquad &= \Pr[X_1(r\theta^*) > -r^{-1}\log(1 - C_\alpha^{(1)})] \\ &= r\{1+r^{-1}\log(1-C_\alpha^{(1)})\}^{r-1} - \binom{r}{2}\{1+2r^{-1}\log(1-C^{(1)})\}^{r-1} + \ldots \end{aligned}$$

It is possible to choose $C_\alpha^{(1)}$ to make the actual significance level equal to α but we cannot give an explicit formula, as in (16).

REFERENCES

[1] Johnson, N.L. (1973) Robustness of certain tests of censoring of extreme sample values, *University of North Carolina Mimeo Series No. 866.*

[2] Johnson, N.L. (1970), A general purpose test of censoring of sample extreme values, (In *Essays in Probability and Statistics (S.N. Roy Memorial Volume)*), Chapel Hill, University of North Carolina Press, pp. 379 - 384.

[3] Johnson, N.L. (1971), Comparison of some tests of sample censoring of extreme values, *Austral. J. Statist.*, 13, pp. 1 - 6.

TABLE 1

Power w.r.t. H_{s_0,s_r} of (11.1) with $C_\alpha^{(1)} = 1 - \alpha^{\frac{1}{r}}$ (α = 0.05)

r=		5			
N=		5	10	25	50
s_0	s_r				
0	0	.0956	.0728	.0590	.0545
2	0	.3896	.3572	.3334	.3245
1	1	.1810	.1438	.1192	.1106
4	0	.6287	.6237	.6211	.6204
3	1	.4350	.4035	.3793	.3699
6	0	.7789	.7955	.8109	.8174
5	1	.6348	.6316	.6309	.6309
4	2	.4681	.4385	.4150	.4056
8	0	.8673	.8919	.9124	.9205
7	1	.7679	.7858	.8032	.8109
6	2	.6398	.6383	.6394	.6403

r=		10				15			
N=		5	10	25	50	5	10	25	50
s_0	s_r								
0	0	See r=5 values				See r=5 values			
2	0	.4285	.4003	.3796	.3719	.4433	.4167	.3973	.3901
1	1	.2218	.1853	.1605	.1517	.2379	.2019	.1774	.1686
4	0	.7005	.7050	.7101	.7124	.7271	.7345	.7418	.7449
3	1	.5321	.5140	.5007	.4957	.5695	.5564	.5472	.5439
6	0	.8546	.8754	.8925	.8993	.8800	.9006	.9169	.9231
5	1	.7516	.7638	.7755	.7805	.7924	.8082	.8221	.8278
4	2	.6121	.6050	.6008	.5994	.6672	.6677	.6697	.6708
8	0	.9311	.9514	.9657	.9707	.9497	.9668	.9779	.9816
7	1	.8748	.8979	.9165	.9238	.9073	.9287	.9448	.9507
6	2	.7900	.8082	.8248	.8318	.8399	.8606	.8782	.8852

(cont.)

TABLE 1 (cont.)

r=			20				25		
N=		5	10	25	50	5	10	25	50
s_0	s_r								
0	0	See r=5 values				See r=5 values			
2	0	.4511	.4253	.4067	.3997	.4559	.4307	.4124	.4056
1	1	.2465	.2109	.1865	.1778	.2519	.2165	.1922	.1836
4	0	.7409	.7496	.7579	.7614	.7493	.7588	.7676	.7713
3	1	.5891	.5786	.5716	.5690	.6013	.5923	.5865	.5845
6	0	.8925	.9127	.9282	.9340	.8999	.9197	.9347	.9402
5	1	.8128	.8298	.8444	.8503	.8251	.8426	.8574	.8633
4	2	.6957	.6998	.7045	.7067	.7132	.7192	.7254	.7282
8	0	.9582	.9734	.9829	.9860	.9630	.9770	.9856	.9883
7	1	.9224	.9423	.9567	.9618	.9310	.9498	.9630	.9677
6	2	.8639	.8849	.9019	.9085	.8778	.8986	.9150	.9213

TABLE 2

Power w.r.t. H_{s_0,s_r} of (11.1) with $C_\alpha^{(1)}=1-\exp\{-r^{-1}N(\alpha^{-1/N}-1)\}$ (=0.05)

r=			5		
N=		5	10	25	50
s_0	s_r				
2	0	.2408	.2736	.2973	.3059
1	1	.0964	.0993	.1008	.1013
4	0	.4396	.5188	.5764	.5975
3	1	.2678	.3075	.3370	.3480
6	0	.5965	.7027	.7746	.7996
5	1	.4368	.5200	.5826	.6062
4	2	.2874	.3329	.3677	.3810
8	0	.7099	,8216	.8885	.9096
7	1	.5751	.6855	.7634	.7913
6	2	.4346	.5210	.5881	.6139

(cont.)

TABLE 2 (cont.)

	r=	10				15			
	N=	5	10	25	50	5	10	25	50
s_0	s_r								
2	0	.2731	.3132	.3420	.3526	.2858	.3286	.3594	.3706
1	1	.1241	.1327	.1385	.1405	.1355	.1465	.1541	.1568
4	0	.5174	.6077	.6701	.6923	.5484	.6414	.7042	.7261
3	1	.3537	.4133	.4371	.4733	.3895	.4562	.5043	.5219
6	0	.7008	.8054	.8678	.8877	.7402	.8402	.8965	.9137
5	1	.5682	.6689	.7376	.7617	.6204	.7229	.7893	.8118
4	2	.4212	.4981	.5548	.5759	.4794	.5656	.6270	.6492
8	0	.8196	.9107	.9542	.9659	.8575	.9362	.9700	.9784
7	1	.7253	.8325	.8946	.9137	.7798	.8776	.9289	.9437
6	2	.6090	.7173	.7898	.8147	.6782	.7851	.8509	.8723

	r=	20				25			
	N=	5	10	25	50	5	10	25	50
s_0	s_r								
2	0	.2926	.3369	.3686	.3802	.2969	.3420	.3743	.3861
1	1	.1417	.1541	.1626	.1656	.1456	.1588	.1679	.1712
4	0	.5650	.6591	.7217	.7434	.5754	.6699	.7323	.7538
3	1	.4091	.4791	.5292	.5474	.4214	.4934	.5446	.5631
6	0	.7608	.8574	.9100	.9257	.7733	.8676	.9178	.9326
5	1	.6481	.7503	.8145	.8358	.6652	.7667	.8292	.8497
4	2	.5115	.6015	.6641	.6864	.5317	.6235	.6865	.7087
8	0	.8763	.9477	.9766	.9834	.8874	.9541	.9801	.9861
7	1	.8073	.8985	.9436	.9561	.8237	.9103	.9516	.9628
6	2	.7144	.8180	.8787	.8977	.7364	.8372	.8942	.9117