ZERO-ONE LAWS FOR EXTREME MEASURES

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ABSTRACT

The notion of an extreme measure is extended to certain linear spaces and useful characterizations of such measures are derived. If μ is an extreme probability measure on the Borel sets $\mathcal{B}^{\infty}$ of R^{∞}, it is shown that any subgroup in the completion of $\mathcal{B}^{\infty}$ with respect to μ has measure 0 or 1. This result leads, by means of an extreme-preserving mapping theorem, to an analogous result on the Borel σ-field of an arbitrary separable Hilbert space. A series expansion for weakly continuous stochastic processes makes it possible to obtain this 0-1 dichotomy for the space $C(T)$ of real-valued continuous functions defined on an arbitrary Lebesgue measurable set T, with the σ-field of cylinder sets. In each setting, these results include as special cases the previously known results for Gaussian measures.

1. INTRODUCTION. In 1970, Kallianpur [10] proved the following zero-one law:

THEOREM. Let X be a linear space of real-valued functions defined on a complete metric space T and let $\mathcal{B}(X)$ be the σ-field generated by the cylinder sets. If μ is any zero-mean Gaussian measure on $(X, \mathcal{B}(X))$ with a continuous covariance function, then $\mu(G) = 0$ or 1 for any $\mathcal{B}(X)$-measurable subgroup G of X.

At about the same time, Jamison and Orey [9] obtained the same zero-one result for completion measurable subgroups of the probability space $(R^\infty, \mathcal{B}^\infty, \mu)$, where μ is the product normalized Gaussian measure on the countable product of the real line R^∞ with Borel σ-field $\mathcal{B}^\infty$. They then applied this result to prove the Kallianpur theorem for completion measurable subgroups in the case where μ is any zero-mean Gaussian measure on the space $X = C(I)$ of continuous functions on the unit interval. Subsequently, Jain [8] extended the more general result of Kallianpur to completion measurable subgroups, and Cambanis and Masry [3] demonstrated that the restrictions on T and the covariance function could be dropped.

More recently, work has been done on extending these results to certain classes of non-Gaussian measures. Zinn [15] has exhibited a class of non-Gaussian measures for which the zero-one law is valid for completion measurable subgroups of R^∞. Dudley and Kanter [6] have proved that all measurable subspaces of a vector space have measure zero or one with respect to any stable measure.

We further examine this problem by considering yet another class of measures. Specifically we use the notion of an extreme measure, a term introduced by Skorokhod [13] in the context of Hilbert spaces. We prove the zero-one law for extreme probability measures on the space $(R^\infty, \mathcal{B}^\infty)$ [Theorem 3.2], and we then use this result, together with the technique of Jamison and Orey [9], to derive zero-one laws on other spaces as well. Finally, it is shown that in the contexts considered, Gaussian measures are extreme so that these results are in fact extensions of the results previously cited.

2. EXTREME MEASURES. Let X be a vector space over the field R of real numbers and let $\mathcal{B}$ be a σ-field of subsets of X such that for every $a \in X$, the transformation $\sigma_a: (X, \mathcal{B}) \to (X, \mathcal{B})$, defined by $\sigma_a(x) = x + a$, is measurable. A measure μ on X shall be assumed to mean, unless otherwise stated, a finite measure on $(X, \mathcal{B})$. For any measure μ on $(X, \mathcal{B})$, $\mathcal{B}_\mu$ is the σ-field obtained by forming the completion of $\mathcal{B}$ with respect to μ, and we shall also use μ to represent the complete measure on $(X, \mathcal{B}_\mu)$.

For two measures μ and ν on $(X, \mathcal{B})$ we write $\mu \ll \nu$ to mean μ is absolutely continuous with respect to ν and $\mu \perp \nu$ to mean μ is mutually singular with respect to ν. The equivalence relation "$\sim$" is defined on the set of all measures on $(X, \mathcal{B})$ by $\mu \sim \nu$ if and only if both $\mu \ll \nu$ and $\nu \ll \mu$. For any two subsets E and F of X we define

$$E - F = \{e - f : e \in E , f \in F\} .$$

In particular, we define the *difference set* of E, $D(E)$, by $D(E) = E - E$ and we write, for any $a \in X$, $E - a$ for the set $E - \{a\}$. For each measure

μ and element a of X, define a new measure μ_a by

(2.1) $$\mu_a(B) = \mu(B - a) \quad \text{for each } B \in \mathcal{B} .$$

The element a of X is said to be an *admissible translate* of μ if $\mu_a \ll \mu$. The set of all admissible translates of μ is denoted by M_μ. For any subset L of X, M_L denotes the set of measures μ on $(X, \mathcal{B})$ such that $L \subset M_\mu$.

<u>DEFINITION 2.1.</u> *For any subset* L *of* X*, a measure* μ *on* X *is* L*-extreme, or extreme with respect to* L*, if* $\mu \in M_L$ *and* μ *cannot be written as the sum of two non-zero mutually singular measures in* M_L*. We will say that* μ *is simply extreme if there is some subset* L *of* X *for which* μ *is* L*-extreme.*

It is easy to see that if L and L' are subsets of X with $L \subset L'$, then $M_{L'} \subset M_L$. Consequently, if μ is L-extreme and $\mu \in M_{L'}$, then μ is L'-extreme. In particular, then, if there is some subset L of X for which μ is L-extreme, then μ is extreme with respect to the set M_μ. Consequently, we could just as well have defined a measure to be extreme if it is M_μ-extreme in the sense of Definition 2.1. In applications, however, it may be that the existence of sufficient conditions for admissibility of a translate enables one to utilize a proper subset L of M_μ, even though an explicit characterization of all admissible translates is not known. In addition, it is sometimes convenient to choose a subset L of admissible translates satisfying some structural property not necessarily satisfied by the set M_μ.

<u>DEFINITION 2.2.</u> *Let* μ *be a measure on* $(X, \mathcal{B})$*. Then* $A \in \mathcal{B}_\mu$ *is* μ*-trivial or trivial if* $\mu(A) = 0$ *or* $\mu(A^c) = 0$.

For any two subsets E and F of X, the symmetric difference of E and F, $E \Delta F$, is defined by $E \Delta F = [(E \cap F^c) \cup (E^c \cap F)]$. Note that if $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$ for any measure μ.

DEFINITION 2.3. *Let* $L \subset X$. *Then* $A \in B_\mu$ *is* L-*invariant with respect to* μ, *or simply* L-*invariant, if* $\mu[A \Delta (A - a)] = 0$ *for each* $a \in L$.

LEMMA 2.1. *Let* $A \in B_\mu$. A *is* L-*invariant with respect to* μ *if and only if* A^c *is* L-*invariant with respect to* μ.

PROOF. Using the fact that

$$(E - a)^c = E^c - a \quad \text{for any} \quad E \subset X, \; a \in X, \tag{2.2}$$

it can be verified that $[A \Delta (A - a)] = [A^c \Delta (A^c - a)]$, from which the result follows.

LEMMA 2.2. *A measure* μ *on* (X, B) *is in* M_L *if and only if every* μ-*trivial set is* L-*invariant with respect to* μ.

PROOF. If $\mu \in M_L$, $a \in L$, and $\mu(A) = 0$, then $\mu(A - a) = \mu_a(A) = 0$. Hence $\mu[A \Delta (A - a)] \leq \mu(A) + \mu(A - a) = 0$, so A is L-invariant. If $\mu(A^c) = 0$, then the preceding argument shows that A^c is L-invariant and hence A is L-invariant by Lemma 2.1. Conversely, suppose $\mu(A) = 0$ and $a \in L$. A is invariant and so $\mu_a(A) = \mu(A - a) = \mu(A) = 0$. Thus $a \in M_\mu$ for every $a \in L$. Therefore, $\mu \in M_L$.

For $E \in \mathcal{B}_\mu$, define the measure μ^E on $(X, \mathcal{B})$ by $\mu^E(B) = \mu(E \cap B)$ for all $B \in \mathcal{B}$.

LEMMA 2.3. *Let* μ *be a measure in* M_L. *If* A *is* L-*invariant, then* $\mu^A \in M_L$.

PROOF. It is straightforward to verify that, for any $A, B \in \mathcal{B}$, $a \in X$,

$$(2.3) \qquad [(A \cap B) - a] \,\Delta\, [A \cap (B - a)] \subseteq [A \,\Delta\, (A - a)] .$$

In particular, then, if A is L-invariant and $a \in L$,

$$(2.4) \qquad \mu[(A \cap B) - a] = \mu[A \cap (B - a)] .$$

Let $B \in \mathcal{B}$ be such that $\mu^A(B) = 0$. Then $\mu[(A \cap B) - a] = 0$ for each $a \in L$ as $\mu \in M_L$. Thus by (2.4), $(\mu^A)_a(B) = \mu[A \cap (B - a)] = \mu[(A \cap B) - a] = 0$, demonstrating that $(\mu^A)_a \ll \mu^A$ for every $a \in L$; that is, $\mu^A \in M_L$.

THEOREM 2.1. *A measure* μ *is* L-*extreme if and only if the families of* L-*invariant sets and* μ-*trivial sets coincide.*

PROOF. Suppose μ is L-extreme. In particular, $\mu \in M_L$ so the L-invariance of every μ-trivial set follows from Lemma 2.2. To show that every L-invariant set is μ-trivial, let A be L-invariant; by Lemmas 2.1 and 2.3, μ^A and μ^{A^c} are both in M_L. Thus $\mu = \mu^A + \mu^{A^c}$ is a decomposition of μ into two mutually singular measures in M_L. But μ is extreme so that either μ^A or μ^{A^c} is identically 0; that is, A is μ-trivial.

Conversely, suppose μ is not L-extreme. By Lemma 2.2, the two families of sets cannot coincide if $\mu \notin M_L$; it therefore suffices to consider the case $\mu \in M_L$. Hence there exist non-zero mutually singular measures μ_1 and μ_2 in M_L such that $\mu = \mu_1 + \mu_2$. Then there is a set $A \in B_\mu$ such that $\mu_1 = \mu^A$, $\mu_2 = \mu^{A^c}$. A is not μ-trivial as μ_1 and μ_2 are both non-zero measures. Since $\mu^A(A^c) = 0$ and $\mu^A \in M_L$, it follows (using (2.2)) that, for every $a \in L$,

$$\mu[A \cap (A - a)^c] = \mu^A(A^c - a) = (\mu^A)_a(A^c) = 0 \ .$$

Likewise,

$$\mu[A^c \cap (A - a)] = \mu^{A^c}(A - a) = (\mu^{A^c})_a(A) = 0 \ ,$$

and hence $\mu[A \Delta (A - a)] = 0$. Thus A is an L-invariant set which is not μ-trivial. This completes the proof.

Since equivalent measures have the same set of admissible translates, a corollary of the preceding theorem is that measures equivalent to an extreme measure are themselves extreme.

3. A ZERO-ONE LAW FOR R^∞ .

Let R^∞ be the set of all sequences $x = (x_1, x_2, \ldots)$ of real numbers. We will use the characterization of extreme measures given by Theorem 2.1 to show that measurable subgroups have measure 0 or 1 with respect to any extreme probability measure. This result includes the result for Gaussian measures appearing in Jamison and Orey [9]. In addition, the result will be useful in deriving similar zero-one laws in other types of linear spaces.

The general setting throughout this section is as follows. Let $\mathcal{B}^\infty$ be the σ-field of Borel sets in R^∞. Let ε_k be the element of R^∞ which has a value of 1 in the k^{th} coordinate and 0 elsewhere. For every n, let R_n be the subspace of R^∞ spanned by $\{\varepsilon_1, \dots, \varepsilon_n\}$ and let R^n be the subspace spanned by $\{\varepsilon_k: k > n\}$. Let $\mathcal{B}_n$, $\mathcal{B}^n$ be the Borel σ-fields induced by the subspace topologies on R_n and R^n, respectively.

LEMMA 3.1. [13, p. 562] *If* ν *is a probability measure on* R_n *and* $\nu_a \ll \nu$ *for all* $a \in R_n$, *then* $\nu \sim m_n$, *where* m_n *is Lebesgue measure on* $(R_n, \mathcal{B}_n)$.

Note that R^∞ can be identified with $R_n \times R^n$ in a natural manner. Moreover, $\mathcal{B}^\infty = \mathcal{B}_n \times \mathcal{B}^n$ as $\mathcal{B}^\infty$ is generated by finite-dimensional rectangles. Thus we can consider $(R^\infty, \mathcal{B}^\infty)$ as the product of the measurable spaces $(R_n, \mathcal{B}_n)$ and $(R^n, \mathcal{B}^n)$ and we will make this identification for the remainder of the section.

For each $n \geq 1$ and $y \in R^n$ and each subset E of R^∞, we define the R^n-section of E determined by y by

$$E_n^y = \{x \in R_n: (x,y) \in E\} .$$

Similarly, for $x \in R_n$, we denote by E_x^n the R_n-section of E determined by x. The following result is a modification of a result stated only in the context of Hilbert spaces in Skorokhod [13].

THEOREM 3.1. *Let* $n \geq 1$ *and suppose* μ *is a probability measure on* $(R^\infty, \mathcal{B}^\infty)$ *such that* $R_n \subset M_\mu$. *Let* λ *and* ρ *be the projections of* μ *onto* $(R_n, \mathcal{B}_n)$

and $(R^n, \mathcal{B}^n)$, *respectively. Then* $\lambda \sim m_n$ *and* $\mu \ll \lambda \times \rho$.

PROOF. λ is defined by

$$\lambda(B_n) = \mu(B_n \times R^n) \quad \text{for} \quad B_n \in \mathcal{B}_n .$$

There is a family of probability measures $\{\mu(x,\circ): x \in R_n\}$ defined on $(R^n, \mathcal{B}^n)$ such that $\mu(x, B^n)$ is $\mathcal{B}_n$-measurable in x for each $B^n \in \mathcal{B}^n$ and such that

$$\mu(B_n \times B^n) = \int_{B_n} \mu(x, B^n)\lambda(dx) \tag{3.1}$$

for all $B_n \in \mathcal{B}_n$, $B^n \in \mathcal{B}^n$ (Loève [11, p. 36]). From (3.1) we obtain the representation

$$\mu(B) = \int_{R_n} \mu(x, B^n_x)\lambda(dx) \quad , \quad B \in \mathcal{B}^\infty = \mathcal{B}_n \times \mathcal{B}^n . \tag{3.2}$$

Consider now the measures μ_a on $(R^\infty, \mathcal{B}^\infty)$ for $a \in R_n$. For any $B_n \times B^n \in \mathcal{B}_n \times \mathcal{B}^n$,

$$\mu_a(B_n \times B^n) = \int_{B_n} \mu(x - a, B^n)\lambda_a(dx) . \tag{3.3}$$

By hypothesis $\mu_a \ll \mu$ and $\mathcal{B}^n$ is separable. Hence, by Theorem 4 of Skorokhod [13], the following two conditions are satisfied for each $a \in R_n$:

$$\lambda_a \ll \lambda \tag{3.4}$$

$$\mu(x - a, \circ) \ll \mu(x,\circ) \text{ a.s. } [\lambda_a(dx)] . \tag{3.5}$$

Condition (3.4), together with Lemma 3.1, yields $\lambda \sim m_n$. Let $f(x) = \frac{d\lambda}{dm_n}(x)$. Then, for $B^n \in \mathcal{B}^n$,

(3.6) $$\rho(B^n) = \mu(R_n \times B^n) = \int_{R_n} \mu(x , B^n)\lambda(dx) .$$

Let $\tilde{\mu}$ denote the measure $\lambda \times \rho$ on $(R_n , \mathcal{B}_n) \times (R^n , \mathcal{B}^n)$. Then, for $E \in \mathcal{B}^\infty$,

(3.7) $$\tilde{\mu}(E) = \int_{R_n} \rho(E_a^n) f(a) m_n(da) .$$

Using (3.6) in (3.7),

(3.8) $$\tilde{\mu}(E) = \int_{R_n} \int_{R_n} \mu(x , E_a^n)\lambda(dx)\lambda(da) .$$

Note also that, by (3.2),

$$\mu_{-x}(E) = \int_{R_n} \mu(a , E_{a-x}^n) f(a) m_n(da) \quad , \quad \text{for } x \in R_n .$$

By the invariance of m_n , we then have

(3.9) $$\mu_{-x}(E) = \int_{R_n} \mu(x + a , E_a^n) f(x + a) m_n(da) .$$

From (3.8) and (3.9) we may deduce that $\mu \ll \tilde{\mu}$ as follows. Suppose $E \in \mathcal{B}^\infty$ and $\tilde{\mu}(E) = 0$. By (3.8), there is a λ-null set N such that for $a \notin N$,

(3.10) $$\int_{R_n} \mu(x , E_a^n)\lambda(dx) = 0 .$$

Temporarily fix $a \notin N$. From (3.10) it follows that $\mu(x , E_a^n) = 0$ a.s. $[\lambda(dx)]$.

Thus, using (3.4) and (3.5),

$$(3.11) \qquad \mu(x + a\ ,\ E_a^n) = 0 \text{ a.s. } [\lambda_{-a}(dx)] \ .$$

Hence

$$(3.12) \qquad \int_{R_n} \mu(x + a\ ,\ E_a^n)\lambda_{-a}(dx) = 0 \quad , \quad a \notin N \ .$$

Since N is m_n-null,

$$\begin{aligned} 0 &= \int_{R_n}\int_{R_n} \mu(x + a\ ,\ E_a^n)\lambda_{-a}(dx)m_n(da) \\ &= \int_{R_n}\int_{R_n} \mu(x\ ,\ E_a^n)f(x)m_n(dx)m_n(da) \\ &= \int_{R_n}\int_{R_n} \mu(x + a\ ,\ E_a^n)f(x + a)m_n(da)m_n(dx) \\ &= \int_{R_n} \mu_{-x}(E)m_n(dx) \ , \end{aligned}$$

where the last equality follows from (3.9). Thus $\mu_{-x}(E) = 0$ a.e. $m_n(dx)$. But $\mu_x \ll \mu$ for each $x \in R_n$ so that $\mu_x \sim \mu$ for all $x \in R_n$. Therefore, $\mu(E) = 0$ and so $\mu \ll \tilde{\mu} = \lambda \times \rho$.

__LEMMA 3.2.__ *Let* E *be a Lebesgue measurable subset of* R_n *and let* m_n *denote Lebesgue measure on* R_n . *If* $m_n(E) > 0$ *, then there exists an* n-*dimensional interval, centered at the origin, contained in* $D(E)$.

The proof of this result can be obtained by a modification of the proof for the special case $n = 1$ appearing in Halmos [7, p. 68].

Let L_0 be the linear span of the set $\{\varepsilon_k: k = 1,2,\dots\}$ so that $L_0 = \cup_{n=1}^{\infty} R_n$.

THEOREM 3.2. *Let* μ *be an* L_0*-extreme probability measure on* $(R^\infty , \mathcal{B}^\infty)$ *and let* $G \in \mathcal{B}_\mu^\infty$ *be a group. Then* G *is* μ*-trivial.*

PROOF. Suppose that $L_0 \subset G$. Then, for every $a \in L_0$, $x \in G$ if and only if $x + a \in G$. Thus $G = G - a$ for every $a \in L_0$ so that $G \Delta (G - a) = \phi$. Hence G is L_0-invariant with respect to μ , and consequently, by Theorem 2.1, G is μ-trivial; that is, $\mu(G) = 0$ or 1 .

Suppose now that $L_0 \not\subset G$. Choose $n \geq 1$ such that $G_n^0 = \{x \in R_n: (x,0) \in G\} \neq R_n$. G_n^0 is a subgroup of R_n . Since $G \in \mathcal{B}_\mu^\infty$, G can be written as the disjoint union of a $\mathcal{B}^\infty$-measurable set H and a subset of a $\mathcal{B}^\infty$-measurable set of μ-measure 0 . As before, let m_n be Lebesgue measure on R_n and suppose that for some $y \in R^n$, $m_n(H_n^y) > 0$. By Lemma 3.2, there exists an n-dimensional interval I_0 in R_n , centered at the origin, such that $I_0 \subset D(H_n^y) \subset D(G_n^y) \subset G_n^0$. But this implies that $G_n^0 = R_n$, since if $x \in R_n$, we may choose N sufficiently large so that $x/N \in I_0 \subset G_n^0$ and hence $x = N \circ (x/N) \in G_n^0$, as G_n^0 is a group. This contradicts the assumption that $G_n^0 \neq R_n$. Hence $m_n(H_n^y) = 0$ for every $y \in R^n$.

Since $R_n \subset L_0$ and μ is L_0-extreme, it follows that $R_n \subset M_\mu$. By Theorem 3.1, $\mu << \lambda \times \rho$, where λ and ρ are the projections of μ onto $(R_n , \mathcal{B}_n)$ and $(R^n , \mathcal{B}^n)$, respectively. Furthermore, $\lambda \sim m_n$. Since $m_n(H_n^y) = 0$, it follows that $\lambda(H_n^y) = 0$ for all $y \in R^n$. But this implies that $(\lambda \times \rho)(H) = 0$. Hence $\mu(H) = 0$, and consequently $\mu(G) = 0$.

We remark that Zinn [15, Corollary 1.3] has proved the same result for measures μ satisfying the conditions

(i) $\lambda_n \ll m_n$ for all n, where λ_n is the projection of μ onto $(R_n, \mathcal{B}_n)$

(ii) μ is a product measure.

Theorem 3.1 shows that (i) is implied by our assumption that $L_0 \subset M_\mu$, but Condition (ii) is much more restrictive.

We next establish the zero-one theorem of Jamison and Orey [9] for the Gaussian measures on $(R^\infty, \mathcal{B}^\infty)$ as a special case of Theorem 3.2.

THEOREM 3.3. *Let* μ *be a Gaussian product measure on* $(R^\infty, \mathcal{B}^\infty)$. *Then* μ *is* L_0*-extreme and all* $\mathcal{B}_\mu^\infty$*-measurable subgroups are* μ*-trivial.*

PROOF. Define projection maps P_k on R^∞ by

(3.13) $$P_k(x) = x_k \quad , \quad x = (x_n) \in R^\infty ,$$

with covariance function $K(j,k) = \mathrm{Cov}(P_j, P_k)$. It is well-known (Parzen [12]) that $M_\mu = H(K)$, where $H(K)$ is the reproducing kernel Hilbert space determined by K. Furthermore, using the characterization of $H(K)$ given by Parzen [12], it can be shown that $\varepsilon_n \in H(K)$ for every n and hence $L_0 \subset H(K) = M_\mu$. Thus $\mu \in \mathcal{M}_{L_0}$ and it follows from Lemma 2.2 that every μ-trivial set is L_0-invariant.

Conversely, let A be any L_0-invariant set. As $\mathcal{B}^n = \sigma\{P_{n+1}, P_{n+2}, \ldots\}$ and $\{P_k : k = 1,2,\ldots\}$ is a sequence of independent random variables, it follows from the zero-one law for the tail σ-field generated by a sequence of independent random variables that the σ-field $\Gamma = \mathcal{B}_\mu^\infty \cap \bigcap_{n=1}^{\infty} \mathcal{B}_\mu^n$ consists of μ-trivial sets. We will show that the invariant set A is μ-trivial by showing that $A \in \Gamma$.

By the L_0-invariance of A, there is a μ-null set N_a for each $a \in L_0$ such that

$$(3.14) \qquad \chi_A(x) = \chi_A(x + a) \quad \text{for} \quad x \notin N_a \ .$$

For fixed n, let ϕ be the density of the normal distribution in R_n with mean 0 and covariance matrix the identity matrix I_n. For $x \in R^\infty$, define

$$g(x) = \int_{R_n} \chi_A(x + y)\phi(y)m_n(dy) \ .$$

Since

$$\int_{R^\infty} |g(x) - \chi_A(x)|\mu(dx) \le \int_{R_n}\int_{R^\infty} |\chi_A(x + y) - \chi_A(x)|\mu(dx)\phi(y)m_n(dy)$$

and $R_n \subset L_0$, it follows from (3.14) that $g(x) = \chi_A(x)$ a.s. $[\mu(dx)]$. Hence there is a μ-null set N such that

$$(3.15) \qquad g(x) = \chi_A(x) \quad \text{for} \quad x \notin N \ .$$

The set $M_a = N \cup N_a \cup (N - a)$ is also μ-null for each $a \in L_0$ and

$$(3.16) \qquad g(x) = \chi_A(x) = \chi_A(x + a) = g(x + a) \quad \text{for} \quad x \notin M_a \ .$$

Let $\{a_m\}$ be a dense subset of R_n and let $M = \bigcup_{m=1}^{\infty} M_{a_m}$. For given $a \in R_n$, let $\{a_{m_k} : k = 1,2,\ldots\}$ be a subsequence such that $\lim_{k\to\infty} a_{m_k} = a$. It is straightforward to show that for $x \in R^\infty$, $g(x + a)$ is a continuous function of $a \in R_n$. This fact, together with (3.16), implies that for $x \notin M$,

$$g(x + a) = \lim_{k\to\infty} g(x + a_{m_k}) = g(x) .$$

Thus $g(x + a)$ is a constant function of $a \in R_n$ almost surely; that is,

$$g(x) = g\big(P^n(x)\big) \text{ a.s. } [\mu(dx)] , \tag{3.17}$$

where $P^n(x) = (0,\ldots,0 , x_{n+1} , x_{n+2} , \ldots)$. Since $g\big(P^n(x)\big)$ is $\mathcal{B}^n$-measurable, g is $\mathcal{B}^n_\mu$-measurable by (3.17). That χ_A is $\mathcal{B}^n_\mu$-measurable for each $n = 1,2,\ldots$ now follows from (3.15) so that $A \in \mathcal{B}^\infty_\mu \cap \bigcap_{n=1}^\infty \mathcal{B}^n_\mu = \Gamma$. Thus A is μ-trivial. Therefore μ is L_0-extreme and consequently, by Theorem 3.2, every $\mathcal{B}^\infty_\mu$-measurable subgroup is μ-trivial.

4. THE TRANSFORMATION THEOREM.

The following theorem provides a means of transition from the result in R^∞ given by Theorem 3.2 to analogous results in other spaces.

THEOREM 4.1. *Let* X *be a real linear space,* $\mathcal{B}$ *a translation invariant* σ-*field of subsets of* X *and* L *a subset of* X . *Let* μ *be a measure on* $(X,\mathcal{B})$ *and let* Y *be a* $\mathcal{B}$-*measurable subgroup of* X *containing* L *such that* $\mu(Y^c) = 0$. *Let* π *be a one-to-one linear map from* X *into* R^∞ *such that* $\pi(L) = L_0$ *and the restriction* π' *of* π *to* Y *is a* $\mathcal{B}(Y)$ - $\mathcal{B}^\infty$ *bimeasurable map from* Y *onto* $\pi(Y)$, *where* $\mathcal{B}(Y) = \mathcal{B} \cap Y$. *If* $\nu = \mu\pi^{-1}$ *denotes the induced measure on* $(R^\infty , \mathcal{B}^\infty)$, *it follows that* μ *is* L-*extreme if and only if* ν *is* L_0-*extreme.*

PROOF. If $B \in \mathcal{B}$, then $\pi(B \cap Y) = \pi'(B \cap Y) \in \mathcal{B}^\infty$ and so

$$\mu(B) = \mu(B \cap Y) = \nu\big(\pi(B \cap Y)\big) \quad \text{for all} \quad B \in \mathcal{B} . \tag{4.1}$$

In particular,

$$\mu(B) = \nu\big(\pi(B)\big) \quad \text{for all} \quad B \in \mathcal{B}(Y) .$$

It follows that the μ-trivial subsets of Y and the ν-trivial subsets of R^∞ correspond under π' , and hence π' is $\mathcal{B}_\mu(Y)$ - $\mathcal{B}_\nu^\infty$ bimeasurable.

Under the given hypotheses, π establishes a one-to-one correspondence between elements a of L and elements $a' = \pi(a)$ of L_0 . Also note that for $B' \in \mathcal{B}^\infty$,

$$\pi^{-1}(B' - a') = \pi^{-1}(B') - a \qquad \text{for} \quad a \in L \ , \ a' = \pi(a) \in L_0 \ .$$

Using the identity

$$(B \cap Y) - a = (B - a) \cap Y \ , \ B \in \mathcal{B} \ , \ a \in L \ , \tag{4.2}$$

it is easy to show that $\mu \in \mathcal{M}_L$ if and only if $\nu \in \mathcal{M}_{L_0}$.

Suppose then that $\nu \in \mathcal{M}_{L_0}$ but ν is not L_0-extreme. By Lemma 2.2 and Theorem 2.1, there is some set $B' \in \mathcal{B}_\nu^\infty$ which is L_0-invariant but not ν-trivial. Then $\pi^{-1}(B') \in \mathcal{B}_\mu$ is not μ-trivial. But for any $a \in L$, $a' = \pi(a) \in L_0$,

$$\pi^{-1}(B') \,\Delta\, \big(\pi^{-1}(B') - a\big) = \pi^{-1}\big(B' \,\Delta\, (B' - a')\big) .$$

Hence

$$\mu[\pi^{-1}(B') \,\Delta\, (\pi^{-1}(B') - a)] = \nu(B' \,\Delta\, (B' - a')) = 0 \,.$$

As $\pi^{-1}(B')$ is L-invariant but not μ-trivial, μ is not L-extreme.

On the other hand, suppose $\mu \in M_L$ but μ is not L-extreme. There is a set $B \in B_\mu$ which is L-invariant but not μ-trivial. By (4.1), $\pi(B \cap Y)$ is not ν-trivial. For any $a' \in L_0$, $a = \pi^{-1}(a') \in L$ and thus by (4.2) and the properties of π,

$$\pi(B \cap Y) \,\Delta\, (\pi(B \cap Y) - a') = \pi\left[(B \,\Delta\, (B - a)) \cap Y\right] .$$

This identity, together with (4.1), yields

$$\nu\left[\pi(B \cap Y) \,\Delta\, (\pi(B \cap Y) - a')\right] = \mu[B \,\Delta\, (B - a)] = 0 \,.$$

Thus $\pi(B \cap Y)$ is L_0-invariant but not ν-trivial, so ν is not L_0-extreme.

COROLLARY 4.1. *Let* μ *be a probability measure on* (X,B) *satisfying the hypotheses of the preceding theorem for some subset* L *of* X, *a subgroup* Y *of* X, *and a map* π *from* X *into* R^∞. *If* μ *is* L*-extreme, then every* B_μ*-measurable subgroup of* X *has measure* 0 *or* 1.

PROOF. If $G \in B_\mu$ is a group, then $\pi(G \cap Y)$ is a B^∞-measurable subgroup. From Theorem 4.1, ν is an L_0-extreme probability measure on (R^∞, B^∞). Thus $\nu(\pi(G \cap Y))$ is 0 or 1 by Theorem 3.2. That G is μ-trivial follows from (4.1).

It should be noted that the subgroup Y of Theorem 4.1 and Corollary 4.1 can be taken to be all of X when it is convenient to do so.

5. HILBERT SPACES. We first apply Theorem 4.1 to the case where X is a separable Hilbert space.

THEOREM 5.1. *Let* H *be a real separable Hilbert space with* $\mathcal{B}(H)$ *the* σ-*field of Borel sets of* H . *Let* $\{e_k\}$ *be a complete orthonormal set in* H *and let* L *be the linear hull of* $\{e_k\}$. *If* μ *is an* L-*extreme probability measure on* $(H , \mathcal{B}(H))$ *and* G *is any* $\mathcal{B}_\mu(H)$-*measurable subgroup, then* G *is* μ-*trivial.*

PROOF. Let $<\circ,\circ>$ and $||\circ||$ denote the inner product and norm, respectively, in H . Then any $x \in H$ can be written in the form

$$x = \sum_{k=1}^{\infty} <x , e_k> e_k ,$$

where the convergence is with respect to the Hilbert space norm and

$$||x||^2 = \sum_{k=1}^{\infty} <x , e_k>^2 < \infty .$$

Furthermore, if (x_k) is any point in R^∞ such that $\sum_{k=1}^{\infty} x_k^2 < \infty$, then the sequence $y_n = \sum_{k=1}^{n} x_k e_k$ converges in H to some element $x = \sum_{k=1}^{\infty} x_k e_k$ in H and $x_k = <x , e_k>$. Thus the mapping π from H into R^∞ defined by

$$\pi(x) = (<x , e_k>) \tag{5.1}$$

is a one-to-one linear transformation from H onto

$$\pi(H) = \{(x_k) \in R^\infty: \sum_{k=1}^{\infty} x_k^2 < \infty\} .$$

For each k , $\pi(e_k) = \varepsilon_k$ so that $\pi(L) = L_0$. $\mathcal{B}(H)$ is the smallest σ-field for which the functions Γ_n, defined by $\Gamma_n(x) = \langle x , e_n \rangle$, are $\mathcal{B}(H) - \mathcal{B}(R)$ measurable (Ahmad [1, p. 100]) and $\mathcal{B}^\infty$ is the smallest σ-field in R^∞ for which the projections P_n defined on R^∞ by $P_n\big((x_k)\big) = x_n$ are $\mathcal{B}^\infty - \mathcal{B}(R)$ measurable. Thus π is $\mathcal{B}(H) - \mathcal{B}^\infty$ measurable. Moreover, $\pi(H) \in \mathcal{B}^\infty$. If $B \in \mathcal{B}(H)$ is of the form

$$B = \{x \in H: \langle x , e_n \rangle \le t\} \tag{5.2}$$

then

$$\pi(B) = \{(x_k) \in R^\infty: x_n \le t\} \cap \pi(H) \in \mathcal{B}^\infty .$$

Since π is one-to-one, the set of all subsets B **of** H for which $\pi(B) \in \mathcal{B}^\infty$ is a σ-field containing all sets of the form (5.2). Thus $\pi(B) \in \mathcal{B}^\infty$ for any $B \in \mathcal{B}(H)$ so that π is bimeasurable. Thus, by Corollary 4.1, every $\mathcal{B}_\mu(H)$-measurable subgroup of H is μ-trivial.

A probability measure μ on $\big(H , \mathcal{B}(H)\big)$ is said to be *Gaussian* if every continuous linear functional defined on H is a Gaussian random variable on $\big(H , \mathcal{B}(H) , \mu\big)$. Suppose μ is Gaussian. The mean element of μ is defined to be the unique element m in H satisfying

$$\int_H \langle x,y \rangle \mu(dx) = \langle m,y \rangle , \text{ for every } y \in H . \tag{5.3}$$

The covariance operator S of μ is the bounded linear operator defined on H satisfying

$$\int_H \langle x - m , y \rangle \langle x - m , z \rangle \mu(dx) = \langle Sy , z \rangle \text{ for all } y,z \in H . \tag{5.4}$$

If Γ_x denotes the Gaussian random variable defined on $(H , \mathcal{B}(H) , \mu)$ by $\Gamma_x(y) = \langle x,y \rangle$, then $\langle x,m \rangle$ is the expectation of Γ_x and $\langle Sx , y \rangle$ is the covariance of Γ_x and Γ_y for all $x,y \in H$.

COROLLARY 5.1. *Let* μ *be Gaussian on* $(H , \mathcal{B}(H))$. *Then* μ *is extreme and hence all* $\mathcal{B}_\mu(H)$- *measurable subgroups of* H *are* μ-*trivial.*

PROOF. Let $\{e_k\}$ be a complete orthonormal set of eigenvectors of S with corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$. Let L be the linear span of the collection $\{e_k\}$ and define the map π as in (5.1). The random variables Γ_k defined by $\Gamma_k(x) = \langle x , e_k \rangle$ are all Gaussian with mean $\langle m , e_k \rangle$; that is,

$$\int_H \langle x , e_k \rangle \mu(dx) = \langle m , e_k \rangle .$$

Moreover, using (5.4),

$$\mathrm{Cov}(\Gamma_k , \Gamma_j) = \lambda_k \delta_{kj} \quad , \text{ where } \delta_{kj} = \begin{cases} 1 , & k = j \\ 0 , & k \neq j \end{cases} .$$

Thus $\{\Gamma_k : k = 1,2,\ldots\}$ is a family of independent Gaussian random variables. It follows that the set $\{P_k : k = 1,2,\ldots\}$ of projections introduced in the proof of Theorem 3.3 is a sequence of independent Gaussian random variables with respect to the induced measure $\nu = \mu\pi^{-1}$ on $(R^\infty , \mathcal{B}^\infty)$; that is, ν is product Gaussian on $(R^\infty , \mathcal{B}^\infty)$. By Theorem 3.3, ν is L_0-extreme on $(R^\infty , \mathcal{B}^\infty)$. It follows from Theorem 4.1 that μ is L-extreme and hence all measurable subgroups of H are μ-trivial by Corollary 4.1.

6. SPACES OF CONTINUOUS FUNCTIONS. Let $X = C(T)$, the set of all continuous functions defined on some Lebesgue measurable subset T of the real line. Let

$\mathcal{B}(X)$ be the σ-field generated by "cylinder" sets of the form $\{x \in X: [x(t_1), \ldots, x(t_n)] \in B_n\}$ for all $n \geq 1$, $t_i \in T$, $1 \leq i \leq n$, and all n-dimensional Borel sets B_n in R_n. Let μ be a probability measure on $(X, \mathcal{B}(X))$. By the *process* $\{x(t): t \in T\}$ we mean the stochastic process $\{\Gamma(t,x): t \in T\}$ defined on $(X, \mathcal{B}(X))$ by $\Gamma(t,x) = x(t)$. Note that $\mathcal{B}(X)$ is precisely the smallest σ-field making $\Gamma(t,x)$ a random variable on $(X, \mathcal{B}(X))$ for all $t \in T$. We first develop the series representation for the process $\{x(t): t \in T\}$ described by Cambanis and Masry [3]. The desired zero-one theorem will then follow by arguments very similar to those used in the Hilbert space case.

It follows from the assumption of path continuity that the process $\{x(t): t \in T\}$ is a $(T \times X, \mathcal{B}(T) \times \mathcal{B}(X))$-measurable process. Let $K(s,t)$ be the autocorrelation function of the process:

$$(6.1) \qquad K(s,t) = E[x(s) \circ x(t)] = \int_X x(s) \circ x(t)\mu(dx) .$$

We can choose a measure ν on $(T, \mathcal{B}(T))$ such that ν is absolutely continuous with respect to Lebesgue measure m, with derivative $\frac{d\nu}{dm}(t) \neq 0$ a.e. $[m]$ and such that

$$\int_T K(t,t)\nu(dt) < \infty .$$

Such measures always exist by means of a construction given in [3]. Henceforth, ν will always denote a measure with the properties given above. For such a measure ν, it follows that $x(t) \in L_2(T, \mathcal{B}(T), \nu) = L_2(\nu)$ a.s.$[\mu]$, since

$$E\left[\int_T |x(t)|^2 \nu(dt)\right] = \int_T K(t,t)\nu(dt) < \infty .$$

Moreover, by Schwarz's inequality and the fact that $K(t,t) \in L_1(T , \mathcal{B}(T) , \nu)$, $K(s,t) \in L_2(\nu \times \nu) = L_2(T \times T , \mathcal{B}(T) \times \mathcal{B}(T) , \nu \times \nu)$. Thus the integral operator K defined on $L_2(\nu)$ by

$$(6.2) \qquad (Kf)(t) = \int_T K(s,t)f(s)\nu(dx)$$

is a Hilbert-Schmitt operator. K is self-adjoint and completely continuous. (See [2] for details.)

Let $H(x,T)$ denote the subspace of $L_2(\mu) = L_2(X , \mathcal{B}(X) , \mu)$ spanned in the mean square sense by $\{x(t)\}$. The process $\{x(t): t \in T\}$ is said to be weakly continuous if for all $\xi \in H(x,T)$, $\lim_{s\to t} E[x(s) \circ \xi] = E[x(t) \circ \xi]$.

Since K is self-adjoint and completely continuous, it follows (Akhiezer and Glazman [2, p. 127]) that there is an orthonormal set of eigenfunctions $\{f_k\}$ of the operator K corresponding to nonzero eigenvalues $\{\lambda_k\}$ - not necessarily distinct - and such that $\{f_k\}$ is complete in the range of K . The orthogonal complement of the range of K is the null space of K , since for any $h \in L_2(\nu)$,

$$(Kh , f_k) = (h , Kf_k) = (h , \lambda_k f_k) = \lambda_k (h , f_k) ,$$

where $(\circ,\circ)$ denotes the inner product in $L_2(\nu)$. If the process $\{x(t): t \in T\}$ is weakly continuous, $K(\circ,t)$ is continuous for each $t \in T$. From this it follows that the range of K consists of continuous functions; in particular, the eigenfunctions f_k are all members of $X = C(T)$.

The following result, while not appearing in quite this form, is essentially contained in Cambanis and Masry [3].

THEOREM 6.1. *Let* $\{x(t): t \in T\}$ *be a second order weakly continuous process and define random variables* ξ_k *by*

$$\xi_k(x) = \int_T x(t) f_k(t) \nu(dt) \ . \tag{6.3}$$

Then each ξ_k *is in* $L_2(\mu)$ *and, in fact,* $H(x,T) = H(\xi)$ *, the subspace of* $L_2(\mu)$ *spanned by* $\{\xi_k\}$ *. Moreover,* $x(t)$ *has the representation*

$$x(t) = \sum_k f_k(t) \xi_k(x) \tag{6.4}$$

where the convergence is in $L_2(\mu)$ *and in* $L_2(\nu)$ a.s. $[\mu]$.

THEOREM 6.2. *Let* $\{x(t): t \in T\}$ *be a weakly continuous second order process defined on* $X = C(T)$ *and let* L *be the linear span of* $\{f_k\}$ *. If* μ *is* L*-extreme and* G *is any* $B_\mu(X)$*-measurable subgroup, then* G *is* μ*-trivial.*

PROOF. We use the representation developed in Theorem 6.1 to define a mapping satisfying the conditions of Theorem 4.1. Given the representation (6.4) of Theorem 6.1, define a mapping π from $(X , \mathcal{B}(X))$ into $(R^\infty , \mathcal{B}^\infty)$ by

$$\pi(x) = (\xi_k(x))_k \ . \tag{6.5}$$

π is one-to-one, as $\xi_k(x_1) = \xi_k(x_2)$ for all k implies $x_1(t) = x_2(t)$ a.s. $[\nu]$, which implies that $x_1 = x_2$ since the paths $x(t)$ are continuous.

Let Y be the set of all x in X such that (6.4) converges in $L_2(\nu)$. By Theorem 6.1, $\mu(Y) = 1$. Y is a group containing L and π is a linear map. Since $\mathcal{B}(X)$ is the smallest σ-field making each $x(t)$ measurable, the representation (6.4) implies that $\mathcal{B}(X)$ is the smallest σ-field making each ξ_k measurable. Thus $\mathcal{B}(X)$ is generated by sets of the form

$$B = \{x: [\xi_{k_1}(x), \ldots, \xi_{k_n}(x)] \in B_n\}, \tag{6.6}$$

where B_n is an n-dimensional Borel set. Thus π is $\mathcal{B}(X) - \mathcal{B}^\infty$ measurable. Moreover, $\pi(Y) \in \mathcal{B}^\infty$ as $(x_k) \in \pi(Y)$ if and only if $\sum_k x_k f_k(t)$ converges in $L_2(\nu)$, and the latter is true if and only if $\sum_k x_k^2 < \infty$. In fact, $\pi' = \pi_{|Y}$ is bimeasurable, since if B is of the form (6.6) and $B \subseteq Y$,

$$\pi(B) = \{(x_k): [x_{k_1}, \ldots, x_{k_n}] \in B_n\} \cap \pi(Y) \in \mathcal{B}^\infty .$$

Since $\xi_k(f_j) = (f_j , f_k) = \delta_{jk}$, $\pi(f_j) = \varepsilon_j$, and thus $\pi(L) = L_0$. The conclusion of the theorem now follows from Theorem 4.1.

We say that μ is a Gaussian measure on $(X , \mathcal{B}(X))$ if the process $\{x(t): t \in T\}$ defined on $(X , \mathcal{B}(X))$ is Gaussian with respect to μ.

COROLLARY 6.1. *If μ is a Gaussian measure on $(X , \mathcal{B}(X))$ then μ is L-extreme and thus all $B_\mu^\infty(X)$-measurable subgroups of G are μ-trivial.*

PROOF. Define π as in (6.5). By Theorem 6.1, the random variables ξ_k, as defined by (6.3), are in $H(x,T)$. Thus each ξ_k is Gaussian. Since $\mathrm{Cov}(\xi_k , \xi_j) = \lambda_k \delta_{kj}$, $\{\xi_k\}$ is a sequence of independent Gaussian random variables. Just as in the proof of Corollary 5.1, it follows that the induced measure $\nu = \mu\pi^{-1}$ is product-Gaussian on (R^∞, B^∞). By Theorem 3.3, ν is L_0-extreme on (R^∞, B^∞) and thus μ is L-extreme by Theorem 4.1. Hence all measurable subgroups of X are μ-trivial by Corollary 4.1.

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