SOME NOTES ON MULTIPLE-COMPARISON PROCEDURES BASED ON RANK SCORES IN THE MULTISAMPLE MULTIVARIATE LOCATION PROBLEM

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Some Notes on Multiple-Comparison Procedures

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SUMMARY. Gabriel and Sen (1968) considered the problem of simultaneous inference based on rank scores in the multisample multivariate location problem. The family of hypotheses consists of all component hypotheses on equalities in the location parameters of subsets of variates across subsets of populations. However, their testing family* is not strictly monotone* and thus, not consonant*. In this paper two alternative procedures based on rank scores and S.N. Roy's Union-Intersection principle are discussed. The first (Section 2) is based on the maximum of all pairwise Hotelling's T^2-type statistics and the second (Section 3) depends on the maximum range across variates. The resulting Simultaneous Test Procedures (STP's*) are more resolvent* than the procedure given by Gabriel and Sen (1968). In Section 4, the problem of many-one comparisons (i.e. comparisons of several treatments with a control) is discussed. Critical points for the implementation of the new multiple comparisons procedures in large samples, obtained by simulation, are given in the appendix.

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*See Gabriel (1969) for a definition

I. Introduction

Consider $k\geq 2$ independent samples of sizes $n_1,\ldots n_k$ from CDF's $F_1,\ldots,F_k$ respectively. Let $y_{i\gamma}^{(j)}$ be the γ^{th} observation on the j^{th} variate in the i^{th} sample. Let $R_{i\gamma}^{(j)}, R_{ii',\gamma}^{(j)}$, be the ranks of $y_{i\gamma}^{(j)}$ among the ordered $\{y_{i1}^{(j)},\ldots,y_{in_i}^{(j)}\}$ and $\{y_{i1}^{(j)},\ldots,y_{in_i}^{(j)}, y_{i'1}^{(j)},\ldots,y_{i'n_{i'}}^{(j)}\}$ respectively. Further define corresponding rank scores $E_{i\gamma}^{(j)}$ and $E_{ii',\gamma}^{(j)}$, $j=1,\ldots,p$ based on p score generating functions as in Gabriel and Sen (1968). Let $T_{ii'}^{(j)}$ denote the mean score for the i^{th} sample on the j^{th} variate in the pooled ranking of the $(i,i')^{th}$ pair of samples. Put $U_{ii'}^{(j)}=T_{ii'}^{(j)}-T_{i'i}^{(j)}$, $N=\Sigma n_i$ and $V_N=(v_{Njj'})_{j,j'=1,\ldots,p}$ where $U_{ii}^{(j)}=0$ for all $j=1,\ldots,p$; $i=1,\ldots,k$ and

$$v_{Njj'} = \frac{1}{N}\sum_{i=1}^{k}\{\sum_{\gamma=1}^{n_i}E_{i\gamma}^{(j)}E_{i\gamma}^{(j')}-[\sum_{\gamma=1}^{n_i}E_{i\gamma}^{(j)}]\,[\sum_{\gamma=1}^{n_i}E_{i\gamma}^{(j')}]/n_i\} \tag{1.1}$$

We use G_e and S_a as generic notations for a group of -e- out of the k populations and a set of -a- out of the p variables. Let $H_{0,G_e}^{S_a}$ be the hypothesis that specifies the equality of the location vectors of the variables in S_a for all samples in G_e. Let Ω denote the family of all $H_{0,G_e}^{S_a}$ and put $N(G_e) = \sum_{i\varepsilon G_e} n_i$. Let $v_N^{jj'(a)}$ denote the $(j,j')^{th}$ element in the inverse of the principle minor in V_N corresponding to the variables in S_a.

Based on the Chatterjee and Sen permutational (conditional) approach to the multivariate multisample location problem (see Puri and Sen (1971), Ch.5) the statistic

$$L_{G_e}^{S_a} = \sum_{ii'\epsilon G_e} n_i n_{i'} \sum_{jj'\epsilon S_a} U_{ii'}^{(j)} U_{ii'}^{(j')} v_N^{jj'(a)}/2N \tag{1.2}$$

is proposed by Gabriel and Sen (1968) as an appropriate conditionally distribution free statistic for the hypothesis $H_{0,G_e}^{S_a}$. These authors consider the family Ω and prove that $\mathcal{L} = \left\{(H_{0,G_e}^{S_a}, L_{G_e}^{S_a}) : H_{0,G_e}^{S_a} \varepsilon \Omega\right\}$ is, under the permutational (conditional) probability law, P_N, say, a joint* monotone increasing testing family. Under some regularity conditions they show that, under $H_{0,G_e}^{S_a}$, the asymptotic distribution of $NL_{G_e}^{S_a}/N(G_e)$ is a central chi-square with a(e-1) d.f.

The large sample STP of experimentwise level α proposed by these authors is: reject $H_{0,G_e}^{S_a}$ iff

$$L_{G_e}^{S_a} > \chi_{p(k-1)}[1-\alpha] \tag{1.3}$$

where $\chi_\nu[\alpha]$ is the α th quantile of the central chi-square distribution with ν d.f. This STP is clearly coherent*, however, it is not consonant because $\mathcal{L}$ is not strictly monotone. In the next two sections we propose two STP's which are more resolvent than this one. In particular, the procedure to be discussed in Section 3 is a consonant one being based on a strictly monotone testing family.

2. Alternative Procedures - I. Let $\underset{\sim}{T}_{ii'} = (T_{ii'}^{(1)},\ldots,T_{ii'}^{(p)})'$, $\underset{\sim}{U}_{ii'} = \underset{\sim}{T}_{ii'} - \underset{\sim}{T}_{i'i}$, $1 \leq i < i' \leq k$. Under some regularity conditions, assumed hereafter, the asymptotic common distribution of the $\underset{\sim}{T}_{ii'}$'s is normal. (See Theorem 3.1 in Gabriel and Sen (1968) and Ch. 6 in Puri and Sen (1971)). One of the implications of these conditions is that consistent estimates of the asymptotic dispersion parameters of the $T_{ii'}$'s are given by

$$\underset{\sim}{\mathrm{COV}}(\underset{\sim}{T}_{ii'},\underset{\sim}{T}_{jj'}) = \begin{cases} \dfrac{n_{i'}}{n_i n_{ii'}} \underset{\sim}{V}_N & i=j \ , \ i'=j'. \\ \dfrac{n_{i'} n_{j'}}{n_i n_{ii'} n_{ij'}} \underset{\sim}{V}_N & i=j;\ i,i',j' \text{ different.} \\ \underset{\sim}{0}: p\times p & i,i',j,j' \text{ all different.} \end{cases} \tag{2.1}$$

where $n_{ii'} = n_i + n_{i'}$, $1 \leq i < i' \leq k$.

Let $b_{ii'}$, $1 \leq i < i' \leq k$, be real numbers. We define ξ_N by the equation

$$\Pr\{ \max_{1 \leq i < i' \leq k} (b_{ii'} \underset{\sim}{U}'_{ii'} \underset{\sim}{V}_N^{-1} \underset{\sim}{U}_{ii'}) < \xi_N \mid P_N \} = 1-\alpha \tag{2.2}$$

and introduce the decision rule: reject $H_{0,G_e}^{S_a}$ iff

$$T_{G_e}^{S_a} = \max_{i,i' \varepsilon G_e} (b_{ii'} \sum_{j,j' \varepsilon S_a} U_{ii'}^{(j)} U_{ii'}^{(j')} v_N^{jj'(a)}) > \xi_N \tag{2.3}$$

Lemma 2.1. (i) The family $\mathcal{T} = \left\{ (H_{0,G_e}^{S_a}, T_{G_e}^{S_a}) : H_{0,G_e}^{S_a} \varepsilon \Omega \right\}$ is a joint monotone testing family under P_N.

(ii) On letting $b_{ii'} = n_i n_{i'}/2N$, the new procedure is strictly more parsimonious and strictly more resolvent than the one considered by Gabriel and Sen(1968).

Proof. (i) Follows from considerations as in Gabriel and Sen (1968).

(ii) By Theorems 4 and 5 in Gabriel(1969) it is sufficient to prove that $\mathcal{T}$ is strictly narrower* than $\mathcal{L}$. This is easily verified since for any G_e which contains only a pair of samples and any S_a, we have $L_{G_e}^{S_a}=T_{G_e}^{S_a}$, and for any G_e with more than a pair of samples, in particular for $G_e=G_k$, we have $L_{G_e}^{S_a}>T_{G_e}^{S_a}$ with probability 1.

Hereafter we refer to this STP as the T^2_{max} approach. This STP was first introduced by Roy and Bose(1953) under normal theory. Approximations to the critical points were studied by Siotani(1960).

Let $\underset{\sim}{x}_i$:p×1, i=1,...,k be k independent random vectors each following the distributional law $\mathcal{N}(\underset{\sim}{0},\underset{\sim}{\Sigma})$ and define ξ by the equation

$$\Pr\{\max_{1\leq i<i'\leq k}[(\underset{\sim}{x}_i-\underset{\sim}{x}_{i'})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}_i-\underset{\sim}{x}_{i'})]<\xi\}=1-\alpha \tag{2.4}$$

Lemma 2.2. (i) When sample sizes are equal ($n_1=n_2=\ldots=n_k=n$, say) and we let $b_{ii'}=n$, $1\leq i<i'\leq k$, we get: $\xi_N\to\xi$ in probability.

(ii) If sample sizes are different, on letting $b_{ii'}=n_in_{i'}/n_{ii'}$, $1\leq i<i'\leq k$ we get: $\text{plim}\,\xi_N<\chi_p[(1-\alpha)^{1/k^*}]$, where $k^*=k(k-1)/2$.

Proof. (i) Follows from (2.1), the invariance of ξ to any choice of a p.d. $\underset{\sim}{\Sigma}$, and our assumption on the asymptotic normality of the $\underset{\sim}{T}_{ii'}$'s.

(ii) Follows from considerations as in (i) and Corollary 4, Khatri(1967).

Remark 2.1. Gabriel and Sen's procedure gives different error rates for different pairwise comparisons in the case of non-equal sample sizes. The T^2_{max} procedure of (ii) in Lemma 2.2 distributes equally the error rates among the various pairwise contrasts.

Remark 2.2. Alternative procedures in the case of unequal sample sizes can be based on straightforward multivariate generalizations of the results in Hochberg(1974). Also, using the procedure given in (ii) of Lemma 2.2, alternative approximations of $\operatorname{plim}\xi_N$ may be used, for example, one may use Siotani(1960)'s modified second order Bonferroni approximation.

Remark 2.3. Analogeous results in terms of simultaneous interval estimation can be obtained. Such procedures utilize the 'Sliding Principle' in estimation based on rank statistics. (See Puri and Sen(1971), Ch. 6).

Some estimates of $\xi/2$ generated by simulation are given in the appendix.

3. Alternative Procedures - II. Define ψ_N by the equation

$$\Pr\{\max_{1\leq i<i'\leq k}\max_{1\leq j\leq p}(b_{ii'}^{1/2}|U_{ii'}^{(j)}|/v_{Njj})<\psi_N|P_N\}=1-\alpha \tag{3.1}$$

and introduce the decision rule: reject $H_{0,G_e}^{S_a}$ iff

$$R_{G_e}^{S_a}=\max_{i,i'\varepsilon G_e}\max_{j\varepsilon S_a}(b_{ii'}^{1/2}|U_{ii'}^{(j)}|/v_{Njj})>\psi_N \tag{3.2}$$

Lemma 3.1. (i) The family $\mathcal{R}=\left\{(H_{0,G_e}^{S_a},R_{G_e}^{S_a}):\ H_{0,G_e}^{S_a}\varepsilon\Omega\right\}$ is a joint strictly monotone testing family under P_N.

(ii) The STP based on $R_{0,G_e}^{S_a}$ is strictly more parsimonious and strictly more resolvent than the T_{max}^2 procedure.

Proof. (i) Follows from considerations as in Gabriel and Sen(1968).

(ii) This is proved using arguments as in the proof of (ii) in Lemma 2.1.

Put $\underset{\sim}{V}_{N,D}=\operatorname{Diag}(v_{N11}^{-1/2},\ldots,v_{Npp}^{-1/2})$, $\underset{\sim}{C}_N=\underset{\sim}{V}_{N,D}\underset{\sim}{V}_N\underset{\sim}{V}_{N,D}$ and $\underset{\sim}{C}=\lim\underset{\sim}{C}_N$. Let

$\underset{\sim}{z}_i=(z_{i1},\ldots,z_{ip})'$, $i=1,\ldots,k$ be k independent random vectors each following the distributional law $\mathcal{N}(\underset{\sim}{0},\underset{\sim}{C})$ and define $M = \max_{1\leq i<i'\leq k}\ \max_{1\leq j\leq p} (|z_{ij}-z_{i'j}|)$.

Lemma 3.2. (i) In the case of equal sample sizes (n), on letting $b_{ii'}=n$, $1\leq i<i'\leq k$, R^{Sa}_{Ge} is asymptotically distributed as M.

(ii) In general, on letting $b_{ii'}=2n_i n_{i'}/n_{ii'}$, $1\leq i<i'\leq k$, an upper bound on $\operatorname{plim}\psi_N$ is given by $q_k[(1-\alpha)^{1/p}]$ where $q_k[\alpha]$ is the α^{th} quantile of the range of k i.i.d. unit normal variables.

Proof. (i) Follows from our assumption on the asymptotic normality of the $T_{ii'}$'s and from (2.1).

(ii) Follows from Corollary 4, Khatri(1967).

In the case of equal sample sizes, the problem of obtaining the asymptotic critical values amounts to finding quantiles in the distribution of M. The distribution of M has been explicitly derived, first, in the bivariate case by Hartely (1950) and later in the general multivariate setup by Mardia (1964). The distribution of M depends on the correlations (the offdiagonal elements in $\underset{\sim}{C}$) and thus tabulation of its quantiles for $p>2$ is rejected from economical considerations. Even in the bivariate case there are some problems involved in obtaining 'precise' upper quantiles of M by numerical integration (see the CDF of M in Hartely (1950)). As a consequence, some upper quantiles of M were obtained by simulation for selected values of $|\rho|$ and k when $p=2$. For $p>2$, one may either use the upper bound given in (ii) of Lemma 3.2 or a modified second order Bonferroni approximation based on Siotani (1960) and our tables for the bivariate case. Further discussion on the critical values to use with the STP of this section is given in the appendix.

Remark 3.1. In analogy with Remark 2.2 we note that the results of Hochberg (1974) may be used to produce alternative procedures similar to those considered here for the unbalanced case.

Remark 3.2. It is clear that for simultaneous interval estimation (based on the 'sliding principle') of all contrasts among the location parameters on any of the individual responses (see Puri and Sen (1971), Ch. 6) the shortest intervals are obtained when using the rank score statistics of this section.

4. Comparisons of several treatments with a control. Suppose that the sample from F_1 represents measurments of a control group which is to be compared with k-1 treatments represented by the samples from $F_2,\ldots,F_k$. All notation introduced above is retained. The hypothesis $H^{S_a}_{0;i_1,\ldots,i_d}$, $2\leq i_1<i_2<\ldots<i_d\leq k$, $d<k$, specifies the equality of the location sub-vectors corresponding to variables in S_a across samples from $F_{i_1},\ldots,F_{i_d}$. Let Ω_c denote the family of all hypotheses of that form. We now introduce two sets of statistics:

$$T^{S_a}_{i_1,\ldots,i_d} = \max_{i\varepsilon\{i_1,\ldots,i_d\}} \Big(b_{i1} \sum_{j,j'\varepsilon S_a} U^{(j)}_{i1} U^{(j')}_{i1} v^{jj'}_N(a)\Big)$$

$$R^{S_a}_{i_1,\ldots,i_d} \max_{i\varepsilon\{i_1,\ldots,i_d\}} \max_{1\leq j\leq p} \big(b^{1/2}_{i1} |U^{(j)}_{i1}|/v_{Njj}\big)$$

In complete analogy with the definitions of ξ_N, ψ_N and the corresponding STP's based on Ω and the statistics $T^{S_a}_{G_e}$, $R^{S_a}_{G_e}$ (Sections 2 and 3, respectively) we define ξ^c_N, ψ^c_N and corresponding STP's based on Ω_c and the statistics $T^{S_a}_{i_1,\ldots,i_d}$ and $R^{S_a}_{i_1,\ldots,i_d}$. Next we introduce some quantities which will be used to define the large sample approximations of ξ^c_N and ψ^c_N. Let $\Phi_\rho(\underset{\sim}{t})$ denote the CDF of a multivariate standardized (zero means and unit variances) normal vector with correlation matrix ρ:p×p at the point $\underset{\sim}{t}=(t_1,\ldots,t_p)'$. Define $D_T(\underset{\sim}{t}_1,\underset{\sim}{t}_2)=\sum_{i=1}^{p}(t_{1i}-t_{2i})^2$; $D_R(\underset{\sim}{t}_1,\underset{\sim}{t}_2)=\max_{1\le i\le p}\{|t_{1i}-t_{2i}|\}$ and for any $\underset{\sim}{u}\in\mathcal{R}^p$

$$\lambda_{T,\underset{\sim}{u}}(s)=\int_{D_T(\underset{\sim}{u},\underset{\sim}{t})\le s} d\Phi_I(\underset{\sim}{t}) \quad ; \quad \lambda_{R,\underset{\sim}{u}}(s)=\int_{D_R(\underset{\sim}{u},\underset{\sim}{t})\le s} d\Phi_\rho(\underset{\sim}{t})$$

Suppose that $\underset{\sim}{V}_N\to\underset{\sim}{V}$ w.p.1., the corresponding correlation matrix of which is ρ. Define the quantities ξ_c and ψ_c by the equations

$$\int_{\underset{\sim}{u}\in\mathcal{R}^p}\lambda_{T,\underset{\sim}{u}}^{k-1}(\xi_c)d\Phi_I(\underset{\sim}{u})=1-\alpha$$

$$\int_{\underset{\sim}{u}\in\mathcal{R}^p}\lambda_{R,\underset{\sim}{u}}^{k-1}(\xi_c)d\Phi_\rho(\underset{\sim}{u})=1-\alpha$$

The following lemma (the proof of which goes along the same line used earlier and thus omitted) summarizes the properties of the many-one STP's considered here.

Lemma 4.1. (i) Both families $\mathcal{T}_c=\{(H_{0;i_1,\ldots,i_d}^{Sa},\ T_{i_1,\ldots,i_d}^{Sa}):H_{0;i_1,\ldots,i_d}^{Sa}\in\Omega_c\}$ and $\mathcal{R}_c=\{(H_{0;i_1,\ldots,i_d}^{Sa},\ R_{i_1,\ldots,i_d}^{Sa}):H_{0;i_1,\ldots,i_d}^{Sa}\in\Omega_c\}$ are joint monotone testing families.(Under P_N).

(ii) When sample sizes are equal (n) and we let $b_{i1}=n$, we get $\text{plim}\,\xi_N^c=\xi_c$; $\text{plim}\,\psi_N^c=\psi_c$.

(iii) In general, on letting $b_{i1}=\frac{n_i n_1}{n_{i1}}$ we get

$$\text{plim}\,\xi_N^c<2\chi_p^2[(1-\alpha)^{1/(k-1)}]$$

$$\text{plim}\,\psi_N^c<\sqrt{2}d_{k-1}[(1-\alpha)^{1/p}]$$

,where $d_k[\alpha]$ is the α^{th} quantile of the maximum absolute value among k standertized normal variables with common correlation 0.5.

Appendix

(In cooperation with Rodriguez German, Biostatistics, UNC)

I. Let $\underset{\sim}{x}_i : p\times 1$, $i=1,\ldots,k$ be k independent random vectors each following the distributional law $\mathcal{N}(0\,\underset{\sim}{\Sigma})$ where $\underset{\sim}{\Sigma}$ is p.d. Define the statistic $S = \max_{1\leq i<j\leq k} [(\underset{\sim}{x}_i - \underset{\sim}{x}_j)' \underset{\sim}{\Sigma}^{-1} (\underset{\sim}{x}_i - \underset{\sim}{x}_j)]$.

The quantity ξ_α of section 2 is the $(1-\alpha)^{th}$ quantile of S. The distribution of S is invariant to different choices of $\underset{\sim}{\Sigma}$, we may take $\underset{\sim}{\Sigma}=\underset{\sim}{I}$. Thus, the distribution of S has only two parameters, namely, k and p. Table 1 gives estimates of some $\xi_\alpha/2$ $(\hat{\xi}/2)$ obtained by simulating 10^4 independent values of S for each of several values of k and p. Also in Table 1 we provide the upper bounds $\chi_p[(1-\alpha)^{2/[k(k-1)]}]$ $(\bar{\xi}/2)$ given by Lemma 2.2.

Table 1.

p	k \ α	.20 $\hat{\xi}/2$	.20 $\bar{\xi}/2$	.10 $\hat{\xi}/2$	.10 $\bar{\xi}/2$	.05 $\hat{\xi}/2$	.05 $\bar{\xi}/2$	.01 $\hat{\xi}/2$	.01 $\bar{\xi}/2$
2	3	4.88	5.27	6.40	6.73	7.76	8.15	11.07	11.43
	4	6.09	6.62	7.72	8.10	9.37	9.53	12.81	12.75
	5	7.02	7.62	8.75	9.12	10.22	10.55	13.42	13.82
	6	7.68	8.43	9.25	9.92	10.71	11.37	13.97	14.62
	7	8.27	9.10	9.88	10.60	11.41	12.02	15.10	15.28
	8	8.87	9.67	10.56	11.17	12.07	12.61	15.66	15.86
3	3	6.68	7.01	8.37	8.64	10.03	10.19	13.79	13.73
	4	8.03	8.51	9.73	10.14	11.45	11.69	14.84	15.14
	5	8.97	9.62	10.68	11.24	12.27	12.79	16.05	16.27
	6	9.81	10.49	11.52	12.11	13.18	13.66	16.53	17.11
	7	10.44	11.22	12.20	12.84	13.94	14.37	17.63	17.82
	8	11.02	11.84	12.96	13.45	14.53	14.98	18.04	18.42
4	3	8.32	8.61	10.08	10.38	11.78	12.05	15.79	15.80
	4	9.72	10.24	11.54	11.99	13.22	13.65	17.09	17.29
	5	10.79	11.43	12.70	13.17	14.48	14.81	18.31	18.47
	6	11.70	12.37	13.63	14.09	15.47	15.73	19.19	19.35
	7	12.40	13.15	14.23	14.86	15.89	16.49	19.82	20.09
	8	12.95	13.81	14.87	15.51	16.62	17.12	20.17	20.72

II. Here in Table 2 we tabulate estimates (obtained by simulation) of some upper quantiles of M (see section 3) in the bivariate case. Let ρ be the offdiagonal element in the correlation matrix $\underset{\sim}{C}:2\times2$. The estimates ere obtained by simulating 10^4 independent values of M for each of several values of ρ and k. (Note that a standardized bivariate normal pair (X,Y) with correlation ρ is easily constructed from two independent unit normal variables U,V by the transformation $X=U;\ Y=\rho U+(1-\rho^2)^{1/2} V$). Several issues here deserve some attention.

A. It is easily verified that the distribution of M in the bivariate case is only a function of $|\rho|$ (and k).

B. We conjecture that M is stochastically monotonically decreasing with $|\rho|$ (i.e. quantiles in the distribution of M are monotonically decreasing with $|\rho|$ for any given k). This monotonicity is a special case of a general conjectured monotonicity in multivariate normal probabilities of convex symmetric sets (see Šidák (1973)).

C. Our simulation does not categorically support this conjecture for low values of $|\rho|$. However, it seems to us that this should be attributed to the extremely low decreasing in the upper quantiles of M as a function of $|\rho|$ over such low values of $|\rho|$, and the errors in our estimates. Thus, Table 1 is not completely consistent with our conjecture and, in a few cases reveals some inconsistency with the upper bounds obtained when $\rho=0$. If our conjecture is proved, clearly, such inconsistencies should be removed by fitting "smooth" monotonic functions to these quantiles such as $\alpha+\beta|\rho|^{\gamma}$ or $\alpha\log[(|\rho|-\beta)/\gamma]$. Alternatively we tried smoothing the percentiles for any given $|\rho|$ as a function of k using the above monotonic "smooth" functions. However, the monotonicity in $|\rho|$ was not obtained by these fittings and thus finally, only the crude estimates for $|\rho|=.2,.4,.6,.8,.9$ are given in Table 2 together with the upper bounds obtained when $\rho=0$. It is clear from the table that for $|\rho| \leq .6$, the upper bounds are very sharp due to the low dependence of the high quantiles of M on $|\rho|$ in this range. Actually, by comparing the entries for $|\rho|=.9$ with those of $|\rho|=1.0$ (i.e. the quantiles of the range in the univariate case) we see that it is only in the very large values of $|\rho|$ that considerable drops in quantiles take place when moving to higher values of $|\rho|$.

Table 2.

k	|ρ|	α = .20	.10	.05	.01
2	.00	2.29	2.76	3.16	3.97
	.20	2.30	2.74	3.14	3.99
	.40	2.27	2.72	3.11	3.90
	.60	2.17	2.66	3.07	3.87
	.80	2.13	2.61	3.05	3.84
	.90	2.08	2.59	3.00	3.83
3	.00	2.87	3.30	3.67	4.42
	.20	2.86	3.28	3.64	4.34
	.40	2.85	3.29	3.67	4.34
	.60	2.81	3.23	3.61	4.35
	.80	2.72	3.17	3.57	4.26
	.90	2.61	3.05	3.44	4.21
4	.00	3.21	3.62	3.98	4.69
	.20	3.20	3.62	3.95	4.68
	.40	3.16	3.61	3.97	4.65
	.60	3.12	3.54	3.92	4.66
	.80	3.06	3.49	3.89	4.58
	.90	3.03	3.46	3.84	4.55
5	.00	3.45	3.84	4.13	4.89
	.20	3.44	3.82	4.17	4.87
	.40	3.41	3.83	4.17	4.87
	.60	3.41	3.83	4.19	4.92
	.80	3.29	3.71	4.04	4.77
	.90	3.25	3.69	4.03	4.73
6	.00	3.63	4.02	4.36	5.03
	.20	3.61	4.00	4.35	4.97
	.40	3.60	4.00	4.34	5.03
	.60	3.57	3.96	4.31	4.95
	.80	3.53	3.93	4.26	4.87
	.90	3.43	3.86	4.22	4.89
7	.00	3.78	4.16	4.49	5.15
	.20	3.77	4.14	4.44	5.08
	.40	3.75	4.13	4.46	5.15
	.60	3.72	4.10	4.43	5.10
	.80	3.65	4.04	4.40	5.09
	.90	3.60	4.01	4.36	5.03
8	.00	3.90	4.28	4.60	5.26
	.20	3.91	4.28	4.60	5.29
	.40	3.87	4.26	4.56	5.23
	.60	3.87	4.25	4.57	5.17
	.80	3.80	4.20	4.52	5.14
	.90	3.73	4.13	4.49	5.14
9	.00	4.01	4.27	4.69	5.34
	.20	3.99	4.34	4.66	5.28
	.40	3.99	4.36	4.69	5.36
	.60	3.95	4.31	4.63	5.27
	.80	3.90	4.29	4.61	5.28
	.90	3.84	4.25	4.60	5.24
10	.00	4.10	4.46	4.78	5.42
	.20	4.08	4.44	4.76	5.42
	.40	4.07	4.44	4.75	5.43
	.60	4.05	4.42	4.73	5.38
	.80	3.99	4.38	4.69	5.39
	.90	3.94	4.33	4.64	5.32

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