IS THE SINGLE-SAMPLE TEST FOR THE NEGATIVE EXPONENTIAL DISTRIBUTION ROBUST?

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Summary

The robustness of the power function of the standard one-sample parametric test for the mean of the negative exponential distribution is examined. The main form of departure from the exponential assumption is a mixture of negative exponential components although an alternative Gamma distribution is also examined. It is found that the test is sensitive to these departures although the effect of mixtures with short-tails is less dramatic than those with long-tails. An alternative test for large sample sizes is given and shown not only to be robust but also asymptotically efficient.

1. Introduction

The usual parametric test that the mean μ of a negative exponential (N.E.) distribution is equal to some prescribed value μ_0 is based upon the fact that $\bar{x}$ the sample mean then has a χ^2 distribution. In fact with probability density function given by

$$f(x;\ \mu) = \frac{1}{\mu} e^{-x/\mu} \quad ; \quad x,\ \mu\ > 0 \tag{1}$$

the test against the one-sided alternative $\mu > \mu_0$ relies upon the critical region for sample size n given by

$$\bar{x}\ \geq\ \frac{\mu_0}{2n}\quad \chi^2_{\alpha,2n} \tag{2}$$

where $\chi^2_{\alpha,2n}$ is the upper 100α %point of the χ^2 distribution with 2n degrees of freedom. The critical region for the two-sided alternative $\mu \neq \mu_0$ may be similarly specified either using a central acceptance region with equal rejection probabilities in each tail or adjusting the tail probabilities to achieve an unbiased test.

However one can seldom be sure of the precise form of the underlying probability distribution and there are many instances with small or moderate sample sizes where we would be unlikely to detect the difference between the sample generated from the assumed form or some similar alternative. For the one-sample problem described above Zelen and Dannemiller [6] have investigated the robustness of the test when the probability distribution actually generating the data is of the Weibull form. In this note we examine in detail the power function of the test when the deviation from the N.E. assumption is in the form of a mixture of two N.E. components.

$$g(x;\ p,\ \theta_1,\ \theta_2) = \frac{p}{\theta_1}\ e^{-x/\theta_1} + \frac{q}{\theta_2}\ e^{-x/\theta_2}\ ;\ x,\ \theta_1,\ \theta_2 > 0;\ 0 \leq p \leq 1;$$

$$p + q = 1 \tag{3}$$

and the parameter of interest, the mean, takes the form $\mu = p\theta_1 + q\theta_2$.

In addition we examine the robustness of the test when the departure from the N.E. assumption is in the form of a Gamma distribution with shape parameter γ close to unity,

$$h(x;\ \gamma,\ \theta) = \frac{x^{\gamma-1}\ e^{-x/\theta}}{\theta^{\gamma}\ \Gamma(\gamma)} \quad , \quad x,\ \gamma,\ \theta\ > 0 \tag{4}$$

where the parameter of interest is $\mu = \gamma\theta$.

In the case of the mixture of N.E. components, since we are concerned with the situation when the investigator is unknowingly dealing with a mixture, we have concentrated our attention on situations where a large proportion of one N.E. component is mixed with a small proportion of the other ($p \doteq 1$ or $q \doteq 1$). Without loss of generality we assume $\theta_2 > \theta_1$ leading to "long-tailed contamination" when $p \doteq 1$ since the small proportion of the second component has a larger scale parameter and conversely "short-tailed contamination" when $q \doteq 1$. It is in situations like this with p close to zero or unity that it is particularly difficult to detect the presence of the mixture. In fact, with regard to the numerical results presented, it is generally true that much larger sample sizes than those considered, would be needed in order to detect the presence of the second N.E. component.

It is perhaps appropriate at this point to make a general comment about the accuracy of numerical results. We have relied upon numerical integration, sometimes of complicated functions, using an iterative form of Simpson's Rule with successively smaller step lengths until the procedure converged. This was found to be quite efficient for our purpose and checks indicate that results have an accuracy greater than the three decimal places presented in the tables of results.

2. Results for a mixture of two N.E. components

For a mixture of two N.E. components the probability density $\xi(\bar{x})$ of the sample mean $\bar{x}$ is given in the appendix. A complete derivation is given in Holt [3]. $\xi(\bar{x})$ is a function of n, p, θ_1 and θ_2 and our approach has been to use numerical integration techniques on it for given values of the parameters to obtain points on the power function. For the calculations it was assumed without loss of generality that the mean of the distribution under the null hypothesis was $\mu_0 = 1$. This is possible since if we consider the power function in the more general case of $\mu = \mu^*$ when the underlying distribution is a mixture as in (3) and the critical region is set as if for a N.E. distribution as in (2) then the power of the test is precisely as if the mixture was $g(x) = p\ f(x; \frac{\theta_1}{\mu^*}) + q\ f(x; \frac{\theta_2}{\mu^*})$ and the critical region was set to test $\mu = \mu_0 = 1$.

We now turn our attention to the numerical results of which only a sample are presented for reasons of space. We will present results only for tests which have a nominal size of $\alpha = 0.05$. Other values of α have been examined with a similar pattern of numerical values resulting. As we stated before we have concentrated our effort on mixtures which are particularly difficult to detect from the assumed single N.E. form. In fact we have taken the set of values for p : {0.99, 0.95, 0.90, 0.1, 0.05, 0.01} the first three values leading to "long-tailed contamination" and the remainder to "short-tailed contamination" under $\theta_2 > \theta_1$. We have examined the set of values θ_2/θ_1 : {3, 5, 10} for the ratio of scale parameters of the two components.

Tables 1 and 2 present points on the power functions of the usual one-sided test with alternative $\mu > \mu_0$ based upon the N.E. distribution and upon the various mixture alternatives we have examined. Table 1 deals with samples of size $n = 5$ and Table 2 with those of size $n = 20$. Points of interest are as follows:

(i) There is distortion of the power function both at the nominal size level and at other points on the power function. For the cases we consider actual sizes occur as large as 0.137 with $n = 5$ and 0.185 with $n = 20$ $\{p = 0.9, \theta_2/\theta_1 = 10\}$ rather than the nominal size of $\alpha = 0.05$.

(ii) The overall distortion in the power function is to lessen power causing estimated sample sizes for a particular test performance to be too small. Of course locally near the null hypothesis power tends to be increased since this is the effect on the actual size of the test but for larger values of the mean μ the actual power function is uniformly lower than that based upon the N.E. distribution.

(iii) As one might expect "short-tailed contamination" ($p \doteq 0$) is comparatively less of a problem, the larger distortions to the actual size of the test and the remainder of the power function occurring in the cases of "long-tailed contamination". Nevertheless distortion of the size and power function does occur and it is perhaps surprising to note that whether contamination is short or long-tailed (p close to zero or unity) the effect on the actual size is always to inflate it.

(iv) Corresponding results for the one-sided case when the alternative is $\mu < \mu_0$ are similar in a qualitative sense to those for $\mu > \mu_0$, but a larger percentage change in the mean of the distribution is needed before the loss in power shows up.

Tables 3 and 4 present results for the two-sided test corresponding to sample sizes of 5 and 20 respectively where the acceptance region is specified with equal probabilities in each tail (central interval). Here we must examine the bias of the test as well as its actual size and the effect on the remainder of the power function.

Points of interest are as follows:

(i) The standard N.E. test is slightly biased, although a comparison of Tables 3 and 4 suggests that the bias rapidly disappears. In the presence of a mixture of N.E. components there is evidence that the range of μ in which the power function is below 0.05 is enlarged and also shifted in an upwards direction. For example for $n = 5$ the mixture $p = 0.9$ $\theta_2/\theta_1 = 10$ has power function below the actual size in the range of $\mu(1.0, 1.59)$ with a minimum value of 0.227 (93.0% of 0.244). Thus while the drop in power is relatively slight it does persist over a wide range of μ. Again as n increases this range shortens and the loss in power below the actual size of the test becomes less. Table 5 presents for each mixture the range of μ in which the power function is below the actual size and the lowest level reached as a percentage of the actual size.

(ii) It would appear that in other respects the two-sided test is considerably less robust than the one-sided version. The actual size of the test reaches such extremes as 0.244 when $n = 5$ and 0.383 when $n = 20$ ($p = 0.9$, $\theta_2/\theta_1 = 10$). For a nominal test size of 0.05 such results are dramatic and even less extreme mixtures exhibit large distortions to the nominal test size.

(iii) As with the one-sided test the overall effect for more extreme values of μ is to reduce the power of the test. Of course with such large effects on the power functions in the neighbourhood of the null hypothesis it takes relatively large values of μ in some cases before the overall reduction in power is apparent. However the use of the power function in estimating required sample sizes is frequently from points on the power function in the range of power = 0.9 or 0.95 so that type I and type II errors are similar in magnitude. By this stage the power function is almost always less than that for the N.E. distribution for the mixtures we have considered.

Tables 6 and 7 correspond to Tables 3 and 4 except that following Ramachandran [4] the tail areas of the test have been adjusted to yield an unbiased test of size $\alpha = 0.05$ for the case of the N.E. distribution (which previously showed bias below $\mu = 1$). For the less extreme mixtures the effect appears to be to reduce the bias. However for the more extreme mixtures the range of values in which the test is biased is increased and the percentage of the actual size to which the power function falls at its minimum is lower too. Thus for $p = 0.9$ $\theta_2/\theta_1 = 10$ the range of bias is now (1.00 - 1.8) and the lowest value of the power function is 87.9% of the actual size. In other respects the results are similar to those in Tables 3 and 4 and the main points are still appropriate.

3. Results for the Gamma distribution

When the actual distribution is of the Gamma form (4) then $\bar{x}$ the sample mean also follows this same form $\bar{x} \sim h(x; n\gamma, \theta/n)$ and it is straightforward using numerical techniques to examine the power function of the test. Table 8 presents points on the power functions derived from a variety of Gamma distributions $\gamma = \{0.5, 0.8, 0.9, 0.95, 0.99, 1.01, 1.05, 1.1, 1.2, 1.5\}$ for samples of size 5 and 20 for the one-sided test ($\mu > \mu_0$) and Table 9 presents the corresponding results for the two-sided test.

The main points are as follows:

(i) Unlike the mixture situation the actual size of the test can be inflated or reduced depending on whether γ is less than or greater than one.

(ii) The value of γ also determines the overall effect on the <u>extremes</u> of the power function. Thus for $\gamma > 1$ the tendency is for the actual power to be greater than that indicated under the N.E. assumption whereas for $\gamma < 1$ the reverse is true, (but this does not actually show up in the two-sided results for $n = 5$ for the range we have presented).

(iii) Distortion to the nominal size of the test can be great. In the one sided case $\mu > \mu_0$, for example, actual sizes range from 0.024 ($\gamma = 1.5$) to 0.112 ($\gamma = 0.5$) for $n = 20$.

(iv) As in the mixture case the effect on the two-sided test seems to be worse than that for the corresponding one-sided procedure. For example the test sizes corresponding to those in point (iii) above are 0.016 and 0.167 respectively for the same sample size.

(v) Whilst the test procedure in the two-sided case is biased the evidence is that this remains virtually unchanged over the range of γ which we have considered. For samples of size 5, the range of μ for which the power function is below the actual test size is from 0.87 - 1.00 for all Gamma distributions considered. Whilst the percentage which fall below the actual test size varies from case to case it is never less than 89.6% ($\gamma = 1.5$). The N.E. case is biased over the same range with a minimum percentage of 95.2% of the actual size. For samples of size 20 the range has shortened to 0.96 - 1.00 and the lowest point of any of the power functions is 97.6% of the actual size ($\gamma = 1.5$). In the case of the Gamma distribution the procedure of Ramachandran will remove the bias of the test not only in the N.E. case but also to a very large extent in the test based upon the Gamma distribution, leaving the rest of the conclusions for the biased test practically as they stand.

4. The two-sample test

The test corresponding to (2) for the equality of means based upon samples from two N.E. distributions is based upon $\overline{x}/\overline{y}$ the ratio of the sample means. This ratio has an F distribution with 2m and 2n degrees of freedom on the null hypothesis where m and n are the sizes of the two independent samples. Gehan and Thomas (1969) have examined the robustness of the procedure against the Weibull alternative.

When the two samples are actually drawn from mixtures of N.E. components the ratio of sample means has a very complex distribution. In the case when both samples came from the same mixture (a special case of the null hypothesis) the distribution of the ratio has been derived by Holt [3] and is given in the Appendix. Again using numerical techniques we are able to examine the actual size of the test in this special restricted case of the null hypothesis.

Table 10 presents the actual sizes of the nominal $\alpha = 0.05$ test for various mixtures of N.E. components and sample sizes. Whilst we have not examined the whole of the power function nor samples from different mixtures having the same mean the results are not encouraging in terms of the robustness of the nominal test size. It seems reasonable to conclude that the power function will also be distorted at least in the immediate neighbourhood of the null hypothesis. In view of the complexity of the calculations involved and the discouraging results obtained we have not pursued this study. Our experience would suggest that a better method of attack for a fuller investigation would be using computer simulation techniques but our feeling is that the few results we have obtained are enough to indicate that the two-sample test is no more robust than its one-sample counterpart.

5. Alternative test statistics

In view of the sensitivity of the N.E. tests to even slight departures from the basic assumptions one is led to look for alternative approaches. Gehan and Thomas [2] investigated the performance of certain non-parametric alternatives for Weibull departures and found them to be clear improvements. It seems likely that suitable non-parametric competitors would also be satisfactory for the two alternative distributional forms which we consider, and for small samples there seems little else to recommend. However for large samples we can take advantage of an optimal parametric test statistic which remains the same function of the data for samples from the single exponential, a mixture of K N.E. components or even a mixture of K Gamma distributions. It may be argued that robust large sample tests are of little value here since as the sample increases the form of the underlying distribution will become clear. However as we have pointed out, mixtures are usually difficult to detect even for large samples unless the components are quite dissimilar and also in substantial proportions. For this reason we feel that a discussion of large sample sizes is far from being valueless. We consider here the theory for the one-sample problem but that for the two-sample case is similar.

The basis of an optimal parametric test for all of distributions mentioned above is the fact that $\bar{x}$ the sample mean is the maximum likelihood estimate of the mean of the distribution in all cases. A proof of this is given in Fryer and Holt (1970). Thus the Wald [5] large-sample statistic for the hypothesis $\mu - \mu_0 = 0$ takes the form

$$t = \frac{n(\bar{x} - \mu_0)^2}{\hat{\sigma}^2}$$

and t has an asymptotic χ^2 distribution with one degree of freedom on H_0. Here $\hat{\sigma}^2$ is the maximum likelihood estimator of the variance of the distribution and in general depends upon the form of the distribution and the value of any parameters involved. However the replacement of $\hat{\sigma}^2$ by s^2 the sample variance gives a new statistic t_1 which is asymptotically equivalent to t. Clearly appropriate one-tailed tests can be based upon $t_1^{1/2}$ if it is properly signed.

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APPENDIX

1. Distribution of $y = \bar{x}$ the sample mean of a mixture of two negative exponential components.

From Holt (1969) the probability density function is given by

$$\xi(y) = \frac{n^n y^{n-1}}{\Gamma(n)} e^{-ny/\theta_1} \sum_{r=0}^{n} \binom{n}{r} p_1^{n-r} p_2^{r} K(r,n,ny\phi)$$

where $p_1 = p/\theta_1$; $p_2 = q/\theta_2$; $\phi = \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)$ and $K(r, n, ny\phi)$

is Kummer's confluent hypogeometric function

$$K(r, n, ny\phi) = \sum_{i=0}^{\infty} \frac{(r)_\ell}{(n)_\ell} \frac{(ny\phi)^p}{\ell!}$$

$(r)_\ell = r(r+1) \ldots (r+\ell-1)$ for example.

2. Distribution of $u = \bar{x}/\bar{y}$ the ratio of sample means for two independent samples of size m and n from the same mixture of two negative exponential components, also from Holt (1969).

$$\zeta(u) = \frac{m^m n^n}{\Gamma(m)\Gamma(n)} \sum_{r=0}^{m} \sum_{s=0}^{n} \binom{m}{r}\binom{n}{s} p_1^{m+n-r-s} p_2^{r+s} K_1(r, s, u, \theta_1, \theta_2)$$

where

$$K_1 = \sum_{t=0}^{\infty} \phi^t \left\{\frac{\theta_1}{mu+n}\right\}^{m+n+t} \Gamma(m+n+t) K_2(t, u, r, s)$$

and

$$K_2 = \sum_{\ell=0}^{t} \frac{u^{\ell+m-1} m^\ell n^{t-\ell} (r)_\ell (s)_{t-\ell}}{(m)_\ell (n)_{t-\ell} \ell! (t-\ell)!}$$

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Table 1. Points on the Power Function of the One-Sided Test for $\mu = 1.0$ for Mixtures of N.E. Components

Sample size of 5. Nominal size of test $\alpha = 0.05$

Parameter Values		Mean of Distribution															
P	θ_2/θ_1	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
Negative Exponential		050	123	219	324	426	518	597	665	722	768	807	838	864	885	903	918
0.99	3	053	124	218	320	420	511	591	659	715	762	801	833	860	882	900	915
	5	056	122	212	311	409	499	579	647	705	753	793	826	853	876	894	910
	10	056	111	190	281	375	464	544	614	674	725	768	803	833	858	879	896
0.95	3	064	130	215	310	403	491	569	636	694	742	782	816	844	868	887	903
	5	079	132	200	281	364	445	520	588	647	698	742	779	811	838	861	880
	10	101	132	171	222	277	339	403	466	525	580	630	674	714	748	779	805
0.90	3	074	138	217	304	392	474	549	616	673	722	763	798	828	853	874	891
	5	100	152	210	275	344	413	480	542	599	650	695	734	768	798	824	846
	10	137	175	211	245	281	320	361	404	447	491	533	574	612	647	680	710
0.01	3	051	124	220	324	425	517	596	664	720	767	805	837	863	884	902	916
	5	051	124	220	325	426	517	596	663	720	766	804	836	862	883	901	916
	10	051	124	221	325	426	517	596	663	719	766	804	835	861	883	900	915
0.05	3	053	127	223	325	425	514	593	659	715	761	799	831	857	879	897	912
	5	054	129	224	327	425	514	591	657	712	758	796	827	853	875	893	908
	10	056	130	226	328	425	513	590	655	710	755	793	824	850	872	890	905
0.10	3	056	131	226	326	423	511	588	653	708	754	792	824	850	872	891	906
	5	059	135	229	329	424	510	584	648	702	747	785	816	843	865	883	899
	10	061	138	232	331	425	509	582	645	698	742	779	810	836	858	877	892

Decimal points omitted: entries should be divided by 1000.

Table 2. Points on the Power Function of the One-Sided Test for $\mu = 1.0$ for Mixtures of N.E. Components. Sample size of 20. Nominal size of test $\alpha = 0.05$.

Parameter Values		Mean of Distribution					
P	θ_2/θ_1	1.0	1.2	1.4	1.6	1.8	2.0
Negative Exponential		050	223	478	701	846	926
0.99	3	056	225	472	693	839	921
	5	066	223	457	675	825	913
	10	085	205	406	618	781	883
0.95	3	073	233	458	667	816	905
	5	107	241	422	607	758	861
	10	164	260	365	482	605	720
0.90	3	085	243	452	649	795	888
	5	128	264	423	580	715	818
	10	185	295	403	504	592	673
0.01	3	051	224	478	700	845	925
	5	051	225	478	699	844	924
	10	051	225	478	699	844	924
0.05	3	053	227	477	696	840	921
	5	055	229	477	694	837	918
	10	056	231	477	692	835	916
0.10	3	057	231	476	691	834	916
	5	060	235	476	686	828	911
	10	063	239	476	683	823	906

Decimal points omitted: entries should be divided by 1000.

Table 3. Points on the Power Function of the Two-Sided Test for $\mu = 1.0$ for Mixtures of N.E. Components

Sample size of 5. Nominal Test Size $\alpha = 0.05$ (Central Interval)

Parameter Values		Mean of Distribution																	
P	θ_2/θ_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Negative Exponential		629	383	228	138	087	060	048	050	063	085	116	153	194	239	285	331	377	421
0.99	3	633	389	234	143	092	064	053	054	067	088	118	154	194	237	282	327	372	416
	5	644	402	245	153	101	072	060	060	071	090	118	151	189	231	274	318	362	405
	10	678	440	279	181	124	091	075	070	076	090	111	139	172	209	248	289	330	371
0.95	3	647	412	255	161	108	080	068	070	081	100	127	159	195	234	275	317	359	400
	5	682	462	303	203	145	114	100	098	104	117	136	160	188	219	253	288	325	361
	10	759	590	437	324	249	202	174	159	152	151	156	164	176	191	210	231	254	279
0.90	3	655	432	275	179	123	094	083	084	095	114	138	167	200	236	273	312	350	389
	5	694	504	352	249	186	151	135	132	137	148	163	183	205	230	257	285	314	344
	10	741	637	527	430	354	301	266	244	232	227	227	231	237	245	255	266	279	292
0.01	3	629	384	229	140	088	061	049	051	064	086	117	154	195	240	285	331	377	421
	5	628	384	230	140	089	061	050	052	064	087	118	154	196	240	286	332	377	421
	10	628	384	230	141	090	062	051	052	065	088	118	155	196	241	286	332	377	421
0.05	3	628	388	235	145	093	065	053	055	068	091	121	158	199	242	287	333	377	420
	5	628	390	238	149	097	068	057	058	071	094	124	161	201	245	289	334	378	421
	10	627	391	241	152	100	072	060	061	074	097	127	164	204	247	292	336	380	422
0.10	3	628	393	242	152	099	071	059	060	073	096	126	163	203	246	290	334	377	420
	5	627	397	249	160	107	077	065	067	080	102	133	169	209	251	294	337	380	421
	10	625	399	254	166	113	084	072	073	087	109	139	175	214	256	299	341	383	423

Decimal points omitted: entries should be divided by 1000.

Table 4. Points on the Power Function the Two-Sided Test for μ = 1.0 for Mixtures of N.E. Components. Sample size of 20. Nominal size of test α = 0.05 (central interval)

Parameter Values		Mean of Distribution															
P	θ_2/θ_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
Negative Exponential		1000	983	841	561	301	141	066	050	080	150	251	369	490	602	699	777
0.99	3	999	978	838	566	310	149	074	057	086	154	252	366	484	595	690	769
	5	997	967	835	580	330	167	090	072	097	158	248	356	469	577	673	753
	10	981	943	843	636	401	231	140	109	118	159	228	317	417	519	615	700
0.95	3	998	965	824	577	336	176	097	080	108	171	259	362	469	573	665	743
	5	989	933	804	610	403	247	162	136	153	199	265	345	432	522	605	682
	10	956	876	775	669	556	443	351	293	270	275	298	334	378	428	483	540
0.90	3	997	956	812	581	355	198	118	100	126	186	268	363	463	559	646	725
	5	987	918	780	605	432	296	214	184	195	235	293	361	434	508	579	646
	10	956	861	742	628	535	464	414	383	368	369	383	408	440	476	514	553
0.01	3	1000	982	840	561	302	142	068	051	081	151	252	370	490	602	698	776
	5	1000	982	840	561	303	143	068	052	081	151	252	370	490	601	697	775
	10	1000	982	840	561	303	144	069	052	082	152	252	370	490	602	698	776
0.05	3	1000	981	837	561	307	148	072	055	085	155	255	371	489	599	693	771
	5	1000	980	835	561	309	150	075	058	088	158	257	372	489	596	691	768
	10	1000	979	833	561	310	153	077	060	090	160	259	373	489	597	690	767
0.10	3	1000	979	832	562	312	154	078	061	091	160	259	372	488	595	688	765
	5	1000	977	828	561	316	160	083	066	096	166	263	375	488	592	684	760
	10	1000	975	824	560	319	165	089	071	102	177	267	377	488	591	681	755

Decimal points omitted: entries should be divided by 1000.

Table 5. Lowest Value of Power Function as a Percentage of the Actual Size of the Test and Range of Values for which Test is Biased Under a Mixture of N.E. Components

Parameters		Sample Size			
		5		20	
P	θ_2/θ_1	Range	Minimum % of size	Range	Minimum % of size
Negative	Exponential	0.87-1.00	95.2	0.96-1.00	98.8
0.99	3	0.88-1.00	95.8	0.96-1.00	99.0
	5	0.89-1.00	97.7	0.98-1.00	99.6
	10	0.99-1.00	100.0	1.00-1.05	99.1
0.95	3	0.88-1.00	97.0	0.96-1.00	99.3
	5	0.95-1.00	99.7	1.00-1.01	100.0
	10	1.00-1.35	95.0	1.00-1.28	91.9
0.90	3	0.87-1.00	97.4	0.97-1.00	99.6
	5	0.97-1.00	99.9	1.00-1.04	99.7
	10	1.00-1.59	93.0	1.00-1.30	95.7
0.01	3	0.87-1.00	95.3	0.96-1.00	98.8
	5	0.87-1.00	95.2	0.96-1.00	98.8
	10	0.87-1.00	95.4	0.97-1.00	98.8
0.05	3	0.87-1.00	95.5	0.96-1.00	99.6
	5	0.87-1.00	96.0	0.96-1.00	99.0
	10	0.87-1.00	96.1	0.96-1.00	99.0
0.10	3	0.87-1.00	95.9	0.96-1.00	98.8
	5	0.88-1.00	96.4	0.97-1.00	99.1
	10	0.88-1.00	96.7	0.97-1.00	99.2

Table 6. Points on the Power Function of the Two-Sided Test for μ = 1.0 for Mixtures of N.E. Components - Tail Areas Adjusted to Make N.E. Test Unbiased. Sample Size of 5. Nominal Test Size α = 0.05.

Parameter Values		Mean of Distribution																	
P	θ_2/θ_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Negative Exponential		697	449	278	174	111	075	056	050	055	070	093	124	159	198	240	284	327	370
0.99	3	700	455	285	179	116	080	060	055	060	074	096	125	160	198	239	281	323	366
	5	709	468	297	190	126	088	069	062	065	078	098	125	157	193	232	273	314	355
	10	739	507	333	221	151	110	086	075	074	081	096	117	144	175	210	247	285	324
0.95	3	709	477	307	199	133	096	076	071	075	088	108	134	164	199	236	274	313	353
	5	736	525	357	244	174	133	111	101	102	109	123	141	164	191	221	252	286	320
	10	794	644	493	373	288	231	195	172	160	155	155	159	167	179	193	210	229	251
0.90	3	713	494	328	218	150	111	091	085	090	102	121	145	173	204	237	273	309	345
	5	738	561	406	292	218	172	148	137	136	142	154	169	187	208	231	256	283	310
	10	763	671	570	475	395	335	292	264	246	236	232	232	235	240	248	257	267	278
0.01	3	696	450	280	175	113	076	057	051	056	071	095	125	160	199	241	284	328	370
	5	696	450	280	176	113	077	058	052	057	072	095	125	161	200	242	285	328	371
	10	696	450	281	177	114	077	058	052	058	072	096	126	161	200	242	285	328	371
0.05	3	695	453	285	181	118	081	061	055	060	076	099	129	164	203	244	286	329	371
	5	694	454	288	185	122	084	065	058	064	079	102	132	167	205	246	288	331	372
	10	692	454	290	188	125	088	068	062	067	082	105	135	170	208	249	291	332	374
0.10	3	694	457	292	188	125	087	067	061	066	081	104	134	169	207	247	289	330	371
	5	691	459	298	196	132	094	074	067	073	088	111	141	175	213	252	293	334	374
	10	689	460	302	202	139	101	081	074	080	095	118	147	181	219	259	298	338	377

Decimal points omitted: entries should be divided by 1000.

Table 7. Points on the Power Function of the Two-Sided Test for μ = 1.0

for Mixtures of N.E. Components - Tail Areas Adjusted to Make N.E. Test Unbiased

Sample Size of 20. Nominal Test Size α = 0.05

Parameter Values		Mean of Distribution																	
P	θ_2/θ_1	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
Negative Exponential		1000	986	861	592	328	157	074	050	071	133	227	342	461	574	674	756	821	871
0.99	3	1000	982	856	596	336	166	081	057	078	138	229	340	456	567	666	748	814	864
	5	997	971	852	609	356	184	098	072	090	144	228	331	442	550	648	731	799	852
	10	982	947	856	660	427	249	150	111	114	149	211	295	392	493	590	677	750	811
0.95	3	998	970	841	605	362	192	105	080	100	156	238	338	443	547	641	722	789	842
	5	991	939	818	631	426	264	171	138	148	188	250	326	411	498	582	660	729	787
	10	959	882	784	680	570	458	363	301	272	273	292	325	367	415	468	523	580	635
0.90	3	998	961	829	606	380	215	126	099	119	172	249	341	439	535	623	702	768	822
	5	988	925	794	623	451	312	223	186	191	226	280	345	416	489	560	627	689	744
	10	959	869	752	639	545	473	421	387	370	368	379	402	432	466	503	541	579	616
0.01	3	1000	986	860	592	329	159	075	051	072	134	228	342	461	574	673	755	820	870
	5	1000	986	859	592	329	159	075	051	073	135	229	342	461	574	672	754	819	869
	10	1000	985	859	592	329	160	076	051	073	135	229	343	461	574	672	754	819	868
0.05	3	1000	984	856	591	333	164	079	055	077	139	232	344	461	572	669	750	815	865
	5	1000	984	854	591	334	167	082	057	079	141	234	345	461	571	667	748	812	862
	10	1000	983	852	590	336	169	085	060	082	144	236	347	461	570	666	746	810	859
0.10	3	1000	983	851	591	337	170	085	060	082	144	236	346	460	569	664	744	809	859
	5	1000	981	847	590	341	176	091	066	088	150	241	349	461	567	661	739	803	853
	10	1000	980	844	588	344	181	096	071	093	155	245	352	462	566	658	735	798	848

Decimal points omitted: entries should be divided by 1000.

Table 8. Points on the Power Function of the One-Sided Test for μ = 1.0 for Gamma Distributions with Sample Sizes of 5 and 20. Nominal Size of Test α = 0.05.

Parameters		Mean of Distribution										
γ	Sample size n	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	3.0
Negative Exponential	5	050	123	219	324	426	518	597	665	722	768	807
1.01	5	049	122	219	324	426	518	596	666	723	770	808
1.05	5	047	119	216	323	427	521	603	672	729	776	814
1.10	5	043	114	212	321	428	524	608	678	736	784	822
1.20	5	038	107	206	318	429	530	617	690	749	797	836
1.50	5	025	087	187	309	433	546	642	721	783	832	869
0.99	5	051	124	220	324	425	517	596	664	720	767	805
0.95	5	054	128	223	326	425	514	592	658	714	760	798
0.90	5	058	132	227	327	423	510	586	651	706	751	789
0.80	5	066	142	234	330	420	502	574	636	688	733	770
0.50	5	103	178	257	334	406	470	527	577	620	659	692
Negative Exponential	20	050	223	478	701	846	926	966	984			
1.01	20	049	222	478	702	848	927	966	985			
1.05	20	046	219	479	707	853	931	969	986			
1.10	20	043	214	480	712	859	936	972	988			
1.20	20	037	206	481	723	871	945	978	991			
1.50	20	024	184	485	751	900	964	988	996			
0.99	20	051	224	478	700	845	925	965	984			
0.95	20	054	228	477	695	839	920	962	982			
0.90	20	058	233	476	689	831	914	957	979			
0.80	20	068	243	474	675	815	899	946	972			
0.50	20	112	277	463	625	748	834	891	929			

Decimal points omitted: entries should be divided by 1000.

Table 9. Points on the Power Function of the Two-Sided Test for $\mu = 1.0$ for Gamma Distributions with Sample Sizes of 5 and 20. Nominal Size of Test $\alpha = 0.05$.

Parameters		Mean of Distribution													
γ	n	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
Negative Exponential	5	383	228	138	087	060	048	050	063	085	116	153	194	239	285
1.01	5	381	226	137	086	058	047	049	062	084	115	152	193	238	284
1.05	5	377	220	131	081	054	043	045	057	079	110	147	189	235	282
1.10	5	371	213	124	075	049	038	040	052	074	105	142	185	231	279
1.20	5	361	199	111	064	040	030	032	043	065	095	133	176	223	274
1.50	5	334	164	081	041	023	015	016	026	045	073	110	155	205	259
0.99	5	384	229	140	089	061	050	051	064	087	117	154	195	240	285
0.95	5	389	236	146	094	066	054	056	069	092	122	159	200	243	288
0.90	5	395	245	155	102	073	061	063	076	099	129	165	205	248	291
0.80	5	408	264	174	121	091	078	080	093	116	145	180	218	258	299
0.50	5	459	339	259	209	180	169	170	181	200	221	250	278	308	338
Negative Exponential	20	983	841	561	301	141	066	050	080	150	251	369	490	602	699
1.01	20	983	842	561	300	140	065	049	078	148	250	369	490	603	700
1.05	20	985	846	562	296	134	061	045	074	144	246	367	491	606	704
1.10	20	986	851	562	290	128	055	040	068	138	242	365	492	610	710
1.20	20	989	860	562	280	116	046	032	059	128	234	362	494	617	720
1.50	20	995	884	563	252	088	028	016	039	104	212	352	500	636	748
0.99	20	982	840	561	303	143	068	051	081	151	252	370	490	601	697
0.95	20	980	836	561	307	149	073	056	086	156	255	371	489	598	693
0.90	20	978	830	561	314	156	080	063	093	163	260	373	488	594	686
0.80	20	971	820	562	327	174	097	080	110	178	271	378	485	586	673
0.50	20	938	776	565	379	251	183	167	193	247	320	400	481	558	627

Decimal points omitted: entries should be divided by 1000.

Table 10. Actual Size of Nominal 5% Level Two-Sample Test for Two Independent Samples of Sizes m and n From the Same Mixture of Negative Exponential Components and Various Values of θ_2/θ_1.

n	m	5			10			15			20		
	θ_2/θ_1 \ p	.90	.95	.99	.90	.95	.99	.90	.95	.99	.90	.95	.99
5	3	061	062	053	075	065	053	077	066	054	076	065	053
	5	100	080	057	107	085	058	108	087	059	107	086	059
	10	156	115	066	158	121	068	155	120	069	150	117	068
10	3	072	062	053	078	067	054	080	068	055	081	069	054
	5	106	082	057	120	094	061	122	097	062	122	098	062
	10	182	132	069	195	150	077	191	152	079	187	151	079
15	3	071	062	052	078	066	054	081	069	055	083	071	056
	5	106	082	057	123	095	061	128	101	063	130	104	065
	10	192	137	070	212	162	079	210	169	084	207	169	086
20	3	070	061	052	079	067	054	082	069	054			
	5	106	081	057	125	096	061	131	102	063			
	10	198	139	070	217	170	082	222	179	087			

Decimal points omitted: entries should be divided by 1000.