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SOME APPROXIMATIONS IN SELECTION THEORY

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SUMMARY

Simple approximations are given for the following: (1) the probability of correctly selecting the largest mean from K normal populations with common unknown variance using a subset selection approach (2) the same probabilities for selecting the largest or smallest normal variance using either the subset selection or indifference zone approaches. One class of approximations involves computing tail probabilities of the beta distribution, while another (less efficient) class may be computed by hand. The bounds are evaluated and shown to compare quite nicely with known exact results.

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1. INTRODUCTION

Dudewicz (1969), Dudewicz and Zaino (1971), and Ramberg (1972) considered the problem of approximating the probability of correctly selecting the largest mean when one is considering K normal populations with common known variance using Bechhofer's (1954) indifference zone approach; they then used their inequalities to derive explicit approximations (depending on the inverse of the standard normal c.d.f.) to find the sample size necessary to guarantee a probability requirement outside the indifference zone (see also McDonald (1971)). In this paper, explicit upper bounds on the probability of a correct selection are obtained for selecting a subset containing the largest or smallest normal mean when the common variance is unknown using Gupta's (1965) rule, and for the probability of correctly selecting the largest or smallest normal variance using either subset selection or the indifference zone. These bounds are not only easy to obtain, but some of them may be computed by hand. Using them, approximations are given for the various constants necessary in each problem, and, when compared to explicit tabulated results, are shown to behave quite favorably. The method of proof is simple and relies on two facts;

(1.1) If X is a r.v. and $r > 0$,

$$P\{X > a\} \quad e^{-ra} E(e^{rX}) .$$

(1.2) If $X_1, X_2, \ldots$ are exchangeable r.v.'s (i.e., the joint distribution of any m of them depends only on m), then for measurable sets $A_1, \ldots, A_n$,

$$P\{X_i \in A_i, i=1, \ldots, n\} \geq \prod_{i=1}^{n} P(X_i \in A_i) \geq 1 - \sum_{i=1}^{n} P(X_i \notin A_i) .$$

(1.2) is given in Dykstra, Hewett, and Thompson (1973). Finally, recall that the beta density with (α,β) degrees of freedom is

$$\big(B(\alpha,\beta)\big)^{-1} x^{\alpha-1}(1-x)^{\beta-1} \quad 0 < x < 1 ,$$

and that if V_1, V_2 are independent standard Gamma random variables with shape parameters r_1 and r_2 (see Section 3), then $V_1/(V_1 + V_2)$ has a beta density with (r_1, r_2) degrees of freedom.

2. THE BOUNDS FOR THE NORMAL MEANS PROBLEM

Suppose $\Pi_1, \ldots ,\Pi_k$ are normal populations with means $\mu_1, \ldots ,\mu_k$ and a common unknown variance σ^2. If $\mu_{[1]} \leq \ldots \leq \mu_{[k]}$ denotes the true (unknown) ordering, the goal is to find a rule R_1 for which the probability of a correct selection $\big(P(CS)\big)$ is at least as large as a specified constant P^* for all parameter configurations. If one takes a sample of size n from each population and forms the sample means $\bar{X}_1, \ldots ,\bar{X}_k$, and if S_r^2 is an independent estimate of σ^2 such that rS_r^2 has a chi-squared distribution with r degrees of freedom, Gupta (1965) proposed to eliminate Π_i if

$$\bar{X}_i \leq \max_{1\leq j\leq k} \bar{X}_j - cS_r n^{-1/2} . \tag{2.1}$$

Thus, one wants to find c, given k,r,P^*, such that $P(CS) \geq P^*$, whatever the parameter configuration $\mu_1, \ldots ,\mu_k$.

Define

$$(2.2)\qquad \begin{aligned} A(P^*) &= 2(P^*)^{1/k-1} - 1 \\ B(\alpha,\beta,\gamma) &= \Pr\{\text{Beta r.v. with } (\alpha,\beta) \text{ d.f.} \le \gamma\} \\ F(\alpha,\beta,\gamma) &= \Pr\{F \text{ r.v. with } (\alpha,\beta) \text{ d.f.} \le \gamma\}\ . \end{aligned}$$

__Lemma 2.1.__ Using the rule given by (2.1),

$$(2.3)\qquad P\{CS\} \ge \left\{1 - \tfrac{1}{2}\left(1 - F(1,r,c^2/2)\right)\right\}^{k-1}$$

$$(2.4)\qquad \ge \left\{1 - 2^{-3/2}\, c(1 + c^2/2r)^{-(r+1)/2}\, (1 + 1/r)^{(r+1)/2}\right\}^{k-1}$$

$$\ge \left\{1 - \left(2^{-1/2}(1 + c^2/4r)\right)^{-r/2}\right\}^{k-1}\ .$$

Hence, the value of c for which $\inf P(CS) = P^*$ satisfies

$$(2.5)\qquad c^2 \le 2rB^{-1}\left(\tfrac{1}{2},\tfrac{r}{2},A(P^*)\right)/\left(1 - B^{-1}(\tfrac{1}{2},\tfrac{r}{2},A(P^*)\right)$$

$$(2.6)\qquad \le c_0^2$$

$$\le 4r\{[2^{1/2}(1 - (P^*)^{1/k-1})]^{-2/r} - 1\}\ ,$$

where c_0 is the value for which (2.4) equals P^*.

__Proof:__ Letting IS denote an incorrect selection, one finds by using (1.2),

$$P\{IS\} \le 1 - \left[\ 1 - \tfrac{1}{2}\, P\left\{{}^{V_1}/_{V_2} \ge {}^{c^2}/_{2r}\right\}\ \right]^{k-1}\ ,$$

where V_1 and V_2 are independent chi-square r.v.'s with 1 and r d.f. respectively. Thus,

$$P\{IS\} \le 1 - \left[\ 1 - \tfrac{1}{2}\left(1 - F(1,r,c^2/2)\right)\ \right]^{k-1}\ .$$

Now, by using (1.1),

$$(2.7)\ \frac{1}{2}\,P\{V_1/V_2 \geq c^2/2r\} = \frac{1}{2}\int_0^\infty P\{bV_1 \geq c^2xb/2r\}dP\{V_2 \leq x\}$$
$$\leq \frac{1}{2}\,(1 - 2b)^{-1/2}\int_0^\infty e^{-xc^2b/2r}\,dP\{V_2 \leq x\}$$
$$= \frac{1}{2}\,(1 - 2b)^{-1/2}\,(1 + c^2b/r)^{-r/2}\ .$$

The last term achieves its minimum at

$$b = (c^2 - 2)/[2(c^2/r)(1 + r)]\ ,$$

which yields the two bounds on $P\{CS\}$.

Note that the value c_0 is easily computed using a method of bisection if one has extensive log and exponential tables or a reasonably sophisticated calculator. Note also that these approximations hold for the dual problem of selecting the smallest mean.

Remark 2.1. The case where $n = 1$ and s_r^2 is formed by a small amount of independent replication (and hence r is small) is certainly non-trivial. For example, in any large scale experiment with interactions where one wants the largest cell mean, taking more than one observation from each population may be infeasible economically; thus, the case where k is large (≥ 50) but r is small (≤ 60) is of interest. No tables are available for this problem, but the approximations (see Tables 1a and 1b) seem to perform quite well.

3. THE BOUNDS FOR THE GAMMA SCALE PARAMETER PROBLEM

Suppose that $\Pi_1, \ldots, \Pi_k$ are Gamma populations with densities $f_i(x) = (\Gamma(r)\beta_i^r)^{-1} x^{r-1} e^{-x/\beta_i}$, $x > 0$ $(i = 1, \ldots, k)$, where the shape parameter r is known (if $\beta_i = 1$, the density is called a standard Gamma). The goal is to find subset selection and indifference zone rules for selecting either the largest or smallest scale parameter β_i. The best set of tables for selecting the largest β_i are due to Gupta (1963), while Gupta and Sobel (1962) have tables for selecting the smallest β_i. It will be assumed that $X_1, \ldots, X_k$ follow the densities $f_1, \ldots, f_k$ (if samples of size n were available from each population, then $n\overline{X}_1, \ldots, n\overline{X}_k$ would still have Gamma densities with shape parameter nr, so it suffices to consider $n = 1$). The problem of selecting the largest β_i will be considered first. Let $\beta_{[1]} \le \beta_{[2]} \le \ldots \le \beta_{[k]}$ denote the correct (unknown) ranking of the parameters. The indifference zone formulation assumes $\beta_{[i]}/\beta_{[k]} \le \delta < 1$ and selects the population Π_i which gives rise to the largest of $\overline{X}_1, \ldots, \overline{X}_k$, where n is the first integer so that nr is as large as r_0, where $P\{CS \mid r = r_0, \beta_{[1]} = \ldots = \beta_{[k-1]} = \delta\beta_{[k]}\} = P^*$. Thus, this problem may be solved assuming $n = 1$. The subset selection approach assumes that $\delta \le 1$ and (assuming $n = 1$ with r known) selects Π_i if

$$(3.1) \qquad X_i \ge c \max_{1\le j\le k} X_j \quad (0 < c < 1) .$$

Since $c = 1$ in (3.1) corresponds to the indifference zone rule, one can get answers to both problems by using (3.1) with $c \le 1$, $\delta \le 1$ (but not $c = 1$ and $\delta = 1$ simultaneously). Let

$$A_2(P^*) = 1 - (P^*)^{1/k-1} \tag{3.2}$$
$$A_3(P^*) = \left(A_2(P^*)\right)^{-1/r} .$$

Lemma 3.1. Using the rule (3.1),

$$P\{CS\} \geq \left[1 - Pr\left\{B\left(r,r,\frac{c\delta}{1+c\delta}\right)\right\} \right]^{k-1} \tag{3.3}$$

$$\geq \left[1 - \{(1 + c\delta)^2/4c\delta\}^{-r} \right]^{k-1} ,$$

so that in the subset selection case,

$$c\delta \geq B^{-1}\left(r,r,A_2(P^*)\right)/\left\{1 - B^{-1}(r,r,A_2(P^*))\right\} \tag{3.5}$$

$$\geq 2\ A_2(P^*) - 1 - \left[\left(2\ A_2(P^*) - 1\right)^2 - 1\right]^{1/2} , \tag{3.6}$$

and in the indifference zone formulation $(c = 1)$, the smallest r such that $P(CS \mid \delta) \geq P^*$ when $\beta_{[i]}/\beta_{[k]} \leq \delta$ satisfies

$$r \leq r_1 \leq r_2 \text{ , where} \tag{3.7}$$

(3.8) r_1 = first number r such that $B\left(r,r,\delta/(1 + \delta)\right) \leq A_2(P^*)$

(3.9) r_2 = first number r such that $r \geq -\log A_2(P^*)/\log\ (1+\delta)^2/4\delta$.

Proof: Again, the proof is a simple application of (1.1) and (1.2). From (1.2), if V_1 and V_2 are independent with densities $\left(\Gamma(r)\right)^{-1}x^{r-1}e^{-x}$, then

$$(3.10)\qquad P\{CS\} \geq [1 - P\{V_2 \leq c\delta V_1\}]^{k-1}$$

$$= [1 - B(r,r,\frac{c\delta}{1+c\delta})]^{k-1} .$$

This gives (3.3). From (1.1),

$$(3.11)\quad P\{V_2 \leq c\delta V_k\} \leq \int_0^\infty e^{-x/a}(1 - c\delta/a)^{-r} dP(V_2 \leq x) \quad \text{for} \quad c\delta < a$$

$$= (1 - c\delta/a)^{-r}(1 + 1/a)^{-r} .$$

A simple argument shows that for $a > c\delta$,

$$(3.12)\qquad (1 + c\delta)^2/4c\delta \geq (1 - c\delta/a)(1 + 1/a) ,$$

which gives (3.4). (3.5) and (3.6) are now immediate. (3.7), (3.8) and (3.9) now follow from (3.3) and (3.4) with $c = 1$.

Remark 3.1. The approximations (3.5) and (3.7), while better than (3.6) and (3.8), do require the use of a computer routine to calculate the Beta distribution, while (3.6) and (3.8) may be done by hand. Note also that the results permit the use of a mixture of the subset selection and indifference zone approaches.

The corresponding problem of selecting the smallest β_i would use the rule that selects π_i if

$$(3.13)\qquad X_i \leq c_1 \min_{1\leq j\leq k} X_j \quad (1 \leq c_1)$$

with $\beta_{[i]}/\beta_{[1]} \geq \delta_1 \geq 1$ (but $c_1 = 1$ and $\delta_1 = 1$ may not happen simultaneously). For exact results, these rules are different and cannot be solved by solving the largest parameter case; however, the approximations are symmetric in that in Lemma 3.1, if one sets $\delta = 1/\delta_1$ and $c = 1/c_1$, one gets appropriate bounds.

Of course, the problem of selecting normal variances is covered by these results.

<u>Remark 3.2.</u> The simple method given here also can be used to yield approximations for the indifference zone problem of selecting the t largest (or smallest) β_i when $\beta_{[k-t]}/\beta_{[k-t+1]} \leq \delta < 1$ (or $\beta_{[t+1]}/\beta_{[t]} \geq \delta_1 > 1$). Since tables are not generally available for this problem if $t \geq 2$ (see Carroll, Gupta, and Huang (1974) however), the approximations so derived would be of interest in practice.

4. TABLES AND EVALUATION OF THE BOUNDS

The results of Sections 2 and 3 gave bounds using two methods; Method I uses (1.2) and requires evaluation of Beta probabilities, while Method II uses (1.1) and gives bounds which may be computed by hand. The Method I bounds are uniformly better than the Method II bounds. Letting Problem 1 be that of Section 2, Problem 2 that of selecting the largest or smallest gamma scale using subset selection, and Problem 3 that of selecting the largest or smallest gamma scale using the indifference zone, Tables 1 through 3 lead to the following conclusions:

(4.1) Method I bounds are very good for all three problems, and get better as as either r or P^* increase. It appears however, that as k

increases, the ratio of the Method I bounds to the exact results also increases slightly. Thus, if k_0 is the largest k covered by existing exact tables, and if the ratio of the bound to the exact results is a , one might, for $k > k_0$, divide the Method I bound by a

(4.2) Method II bounds perform uniformly well in Problem 1 and quite well for Problem 2 if $r \geq 10$. For Problem 3, they seem to be somewhat conservative. For Problem 1, the ratio of Method I bounds to the exact results decreases as r, k, or P^* increase. For Problem 2, this ratio decreases as r or P^* increase, but increases as k increases if one wants to select the largest gamma scale parameter. In this latter case, one should use the method suggested in (4.1).

(4.3) Conclusions about the bounds from Problem 3 are difficult to make, since this author can derive exact results from existing tables only for $\delta \leq .50$. It may be that the Method II bounds actually get worse as δ , or P^* increase, so that the suggestion in (4.1) might be applied. The Method II bounds seem to get worse as k increases if in Problem 3 one wants the largest parameter, but better otherwise.

(4.4) Recall that the Method II bounds (2.6), (3.6), and (3.9) may be computed by hand. Their good performance in these problems will be quite helpful.

Tables 1a and 1b compare the bounds given by (2.5) and (2.6) respectively with the tables of Gupta and Sobel (1957) for $r = 16(4)24$, 36, 40, 60, 120, 3[illegible] $k = 2$, 10, 18, 30, 50 and $P^* = .90$, .95, .99 . The numbers in parentheses are the ratio of these bounds to the exact results and are most informative.

Tables 2a and 2b are concerned with the bounds (3.5) and (3.6) respectively for the problem of selecting the largest gamma scale parameter. These bounds are compared with the exact results of Gupta (1963) for $\nu = 2r = 4(2)10, 10(10)50$, $k = 4(2)10$ and $P^* = .90, .95, .99$. The performance of the bounds of this paper for selecting the smallest gamma scale parameter, while not included here, turns out to be even better.

Tables 3a and 3b give the performance of the bounds derived from (3.8) and (3.9) for the indifference zone problem of selecting the largest gamma scale parameter (again, these bounds will perform slightly better if one wishes to select the smallest scale parameter). The exact results were derived from Gupta (1963) and Gupta and Sobel (1962); direct comparisons are fairly difficult since the range of δ values one can obtain using the above tables does not include anything for which $\delta > .50$, which is the most interesting case. Thus, the bound given in (3.8) seems very practical for use where $\delta > .5$ and any k, and use of the suggestion given in (4.1) should improve the performance of (3.9).

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TABLE 1a.

Values of the Bound given by (2.5)[a,b] .

r \ k	2	10	18	30	50
16	1.89 (1.00)	3.54 (1.10)	3.98 (1.13)	4.34 (1.15)	4.70 (1.18)
	2.46 (1.00)	4.04 (1.07)	4.47 (1.09)	4.83 (1.11)	5.18 (1.13)
	3.65 (1.00)	5.14 (1.04)	5.56 (1.06)	5.92 (1.07)	6.28 (1.09)
20	1.87 (1.00)	3.47 (1.10)	3.88 (1.12)	4.22 (1.14)	4.55 (1.16)
	2.43 (1.00)	3.94 (1.07)	4.34 (1.09)	4.67 (1.10)	4.99 (1.12)
	3.57 (1.00)	4.95 (1.03)	5.34 (1.05)	5.66 (1.06)	5.98 (1.07)
24	1.86 (1.00)	3.42 (1.09)	3.82 (1.11)	4.14 (1.13)	4.46 (1.15)
	2.42 (1.00)	3.87 (1.06)	4.25 (1.08)	4.57 (1.10)	4.87 (1.11)
	3.52 (1.00)	4.84 (1.03)	5.20 (1.04)	5.50 (1.05)	5.79 (1.06)
36	1.84 (1.00)	3.35 (1.09)	3.72 (1.10)	4.02 (1.11)	4.31 (1.13)
	2.38 (1.00)	3.77 (1.05)	4.12 (1.07)	4.41 (1.08)	4.68 (1.09)
	3.44 (1.00)	4.65 (1.02)	4.98 (1.03)	5.24 (1.04)	5.50 (1.05)
40	1.84 (1.00)	3.33 (1.08)	3.70 (1.10)	4.00 (1.11)	4.28 (1.13)
	2.38 (1.00)	3.75 (1.05)	4.10 (1.07)	4.38 (1.08)	4.65 (1.09)
	3.42 (1.00)	4.61 (1.02)	4.94 (1.03)	5.19 (1.04)	5.45 (1.05)
60	1.83 (1.00)	3.29 (1.08)	3.64 (1.09)	3.93 (1.11)	4.19 (1.12)
	2.36 (1.00)	3.69 (1.05)	4.02 (1.06)	4.29 (1.07)	4.54 (1.08)
	3.38 (1.00)	4.51 (1.02)	4.81 (1.02)	5.06 (1.03)	5.28 (1.04)
120	1.81 (1.00)	3.24 (1.07)	3.59 (1.09)	3.86 (1.11)	4.11 (1.11)
	2.34 (1.00)	3.63 (1.04)	3.95 (1.05)	4.20 (1.06)	4.44 (1.07)
	3.33 (1.00)	4.42 (1.02)	4.70 (1.02)	4.92 (1.03)	5.14 (1.03)
360	1.80 (1.00)	3.22 (1.07)	3.55 (1.08)	3.82 (1.09)	4.05 (1.10)
	2.32 (1.00)	3.60 (1.04)	3.91 (1.05)	4.14 (1.06)	4.38 (1.07)
	3.31 (1.00)	4.36 (1.02)	4.63 (1.02)	4.84 (1.02)	5.03 (1.03)

[a] For each r , the first row is for $P^* = .90$, the second for $P^* = .95$, and the third for $P^* = .99$

[b] The numbers in parentheses are the ratios of the bound in (2.5) with exact results given in Gupta and Sobel (1957).

TABLE 1b.

Values of the Bound given by (2.6)[a,b] .

r \ k	2	10	18	30	50
16	3.74 (1.97)	5.48 (1.70)	5.96 (1.70)	6.36 (1.69)	6.75 (1.69)
	4.34 (1.75)	6.01 (1.58)	6.50 (1.58)	6.90 (1.57)	7.30 (1.57)
	5.59 (1.53)	7.24 (1.47)	7.73 (1.47)	8.13 (1.47)	8.55 (1.47)
20	3.69 (1.97)	5.30 (1.68)	5.74 (1.66)	6.08 (1.64)	6.45 (1.64)
	4.25 (1.74)	5.79 (1.57)	6.21 (1.56)	6.57 (1.56)	6.92 (1.55)
	5.41 (1.51)	6.83 (1.44)	7.30 (1.43)	7.64 (1.43)	8.01 (1.43)
24	3.65 (1.96)	5.19 (1.66)	5.59 (1.63)	5.92 (1.62)	6.25 (1.61)
	4.20 (1.73)	5.65 (1.55)	6.05 (1.54)	6.37 (1.53)	6.68 (1.52)
	5.28 (1.50)	6.65 (1.42)	7.03 (1.41)	7.35 (1.40)	7.66 (1.40)
36	3.58 (1.93)	5.01 (1.63)	5.38 (1.60)	5.67 (1.57)	5.94 (1.56)
	4.09 (1.71)	5.41 (1.51)	5.76 (1.50)	6.05 (1.49)	6.32 (1.48)
	5.08 (1.47)	6.28 (1.38)	6.61 (1.37)	6.88 (1.36)	7.15 (1.36)
40	3.57 (1.93)	4.98 (1.62)	5.32 (1.59)	5.61 (1.57)	5.88 (1.55)
	4.07 (1.71)	5.38 (1.51)	5.70 (1.49)	5.99 (1.48)	6.25 (1.47)
	5.05 (1.47)	6.21 (1.38)	6.54 (1.37)	6.79 (1.36)	7.04 (1.36)
60	3.54 (1.93)	4.87 (1.60)	5.19 (1.56)	5.47 (1.54)	5.72 (1.52)
	4.01 (1.69)	5.23 (1.49)	5.56 (1.49)	5.81 (1.45)	6.05 (1.44)
	4.94 (1.46)	6.01 (1.36)	6.30 (1.34)	6.54 (1.32)	6.77 (1.32)
120	3.51 (1.92)	4.76 (1.58)	5.07 (1.54)	5.32 (1.52)	5.56 (1.50)
	3.96 (1.69)	5.12 (1.47)	5.41 (1.45)	5.63 (1.43)	5.87 (1.42)
	4.85 (1.45)	5.83 (1.34)	6.08 (1.32)	6.30 (1.32)	6.50 (1.31)
360	3.47 (1.91)	4.70 (1.57)	4.99 (1.53)	5.23 (1.50)	5.45 (1.48)
	3.92 (1.68)	5.03 (1.46)	5.30 (1.43)	5.52 (1.41)	5.74 (1.40)
	4.78 (1.44)	5.70 (1.33)	5.96 (1.32)	6.14 (1.30)	6.34 (1.29)
∞	3.46 (1.91)	4.67 (1.56)	4.96 (1.52)	5.18 (1.49)	5.39 (1.47)
	3.90 (1.67)	4.99 (1.45)	5.27 (1.43)	5.47 (1.40)	5.67 (1.39)
	4.74 (1.44)	5.65 (1.32)	5.88 (1.31)	6.06 (1.29)	6.25 (1.28)

[a]For each r , the first row is for $P^* = .90$, the second for $P^* = .95$, and the third for $P^* = .99$.

[b]The numbers in parentheses are the ratios of (2.6) with the exact results of Gupta and Sobel (1957).

TABLE 2a.

Values of the Bound given by (3.5)[a,b] .

ν \ k	4	6	8	10
4.00000	0.12558 (1.26609) 0.08388 (1.23980) 0.03513 (1.22411)	0.09372 (1.44039) 0.06307 (1.41118) 0.02682 (1.37937)	0.07818 (1.57333) 0.05269 (1.55640) 0.02272 (1.49637)	0.06806 (1.70445) 0.04623 (1.64387) 0.01967 (1.62714)
6.00000	0.17931 (1.17581) 0.14638 (1.14773) 0.07761 (1.12097)	0.05935 (1.27389) 0.11882 (1.24558) 0.06417 (1.21547)	0.13873 (1.35518) 0.10434 (1.32259) 0.05703 (1.26244)	0.12558 (1.41738) 0.09489 (1.38048) 0.05214 (1.32324)
8.00000	0.25298 (1.13051) 0.17931 (1.11499) 0.11821 (1.08283)	0.21148 (1.20581) 0.16728 (1.17764) 0.10137 (1.14431)	0.18897 (1.26475) 0.15024 (1.22472) 0.09198 (1.18510)	0.17398 (1.31053) 0.13873 (1.26868) 0.08561 (1.21484)
[illegible]0.00000	0.29743 (1.10613) 0.23933 (1.09052) 0.15413 (1.06405)	0.25452 (1.16691) 0.20719 (1.14387) 0.13557 (1.10644)	0.23114 (1.20706) 0.18897 (1.18008) 0.12497 (1.13631)	0.21507 (1.24611) 0.17667 (1.21128) 0.11760 (1.15647)
[illegible]0.00000	0.43367 (1.06071) 0.37588 (1.04820) 0.28200 (1.03190)	0.39083 (1.09510) 0.34163 (1.07426) 0.25992 (1.05032)	0.36670 (1.11535) 0.32172 (1.09413) 0.24612 (1.06452)	0.34959 (1.13276) 0.30821 (1.10639) 0.23709 (1.07555)
[illegible]0.00000	0.50866 (1.04589) 0.45300 (1.03753) 0.36125 (1.09620)	0.46863 (1.06695) 0.42074 (1.05290) 0.33812 (1.03512)	0.44480 (1.08364) 0.40034 (1.06909) 0.32428 (1.04539)	0.42867 (1.09408) 0.38706 (1.07734) 0.31493 (1.05103)
[illegible]0.00000	0.55801 (1.03762) 0.50644 (1.02678) 0.41583 (1.01964)	0.51985 (1.05607) 0.47391 (1.04450) 0.39367 (1.02624)	0.49762 (1.06707) 0.45506 (1.05480) 0.38052 (1.03280)	0.48245 (1.07577) 0.44175 (1.06170) 0.37036 (1.03953)
[illegible]0.00000	0.59439 (1.03298) 0.54508 (1.02371) 0.45817 (1.01491)	0.55801 (1.04837) 0.51423 (1.03650) 0.43669 (1.02132)	0.53696 (1.05781) 0.49653 (1.02906) 0.42272 (1.02906)	0.52211 (1.06491) 0.41388 (1.03171) 0.41388 (1.03171)

[a] For each r , the first row is for P* = .90 , the second for P* = .95, and the third for P* = .99 .

[b] The numbers in parentheses are the ratios of (3.5) to exact results given by Gupta (1963). ν = 2r .

TABLE 2b.

Values of the Bound given by (3.6)[a,b] .

ν \ k	4	6	8	10
	0.05133 (3.09740)	0.03897 (3.46429)	0.03258 (3.77528)	0.02853 (4.06533)
4.00000	0.03486 (2.98339)	0.02663 (3.34252)	0.02232 (3.67324)	0.01959 (3.87908)
	0.01489 (2.88734)	0.01146 (3.22880)	0.00966 (3.52010)	0.00850 (3.76508)
	0.09816 (2.36356)	0.08031 (2.52780)	0.07057 (2.66409)	0.06416 (2.77438)
6.00000	0.07409 (2.26748)	0.06105 (2.42415)	0.05386 (2.56198)	0.04910 (2.66787)
	0.04047 (2.14948)	0.03370 (2.31441)	0.02991 (2.40744)	0.02737 (2.52061)
	0.14005 (2.04217)	0.11895 (2.14385)	0.10713 (2.23089)	0.09922 (2.29803)
8.00000	0.11143 (1.97425)	0.09534 (2.06640)	0.08622 (2.13399)	0.08007 (2.19796)
	0.06866 (1.86426)	0.05940 (1.95300)	0.05406 (2.01631)	0.05042 (2.06272)
	0.17648 (0.86421)	0.15335 (1.93671)	0.14021 (1.98991)	0.13130 (2.04107)
10.00000	0.14501 (1.79983)	0.12691 (1.86746)	0.11650 (1.91410)	0.10941 (1.95601)
	0.09604 (1.70757)	0.08499 (1.76498)	0.07851 (1.80859)	0.07405 (1.83655)
	0.30324 (1.51696)	0.27673 (1.54666)	0.26117 (1.56606)	0.25039 (1.58151)
0.00000	0.26690 (1.47621)	0.24500 (1.49796)	0.23201 (1.51718)	0.22296 (1.52944)
	0.20543 (1.41657)	0.19036 (1.43412)	0.18126 (1.44541)	0.17486 (1.45835)
	0.38087 (1.39681)	0.35420 (1.41163)	0.33835 (1.42455)	0.32729 (1.43299)
0.00000	0.34421 (1.36545)	0.32172 (1.37698)	0.30822 (1.38862)	0.29874 (1.39588)
	0.28018 (1.41337)	0.26402 (1.32567)	0.25415 (1.33385)	0.24715 (1.33928)
	0.43513 (1.33063)	0.40899 (1.34233)	0.39334 (1.34996)	0.38237 (1.35733)
0.00000	0.39914 (1.30281)	0.37683 (1.31360)	0.36335 (1.32105)	0.35384 (1.32548)
	0.33512 (1.26523)	0.31868 (1.26771)	0.30860 (1.27351)	0.30140 (1.27735)
	0.47607 (1.28971)	0.45063 (1.29819)	0.43532 (1.30477)	0.42456 (1.30960)
0.00000	0.44100 (1.26531)	0.41911 (1.27175)	0.40582 (1.27642)	0.39642 (1.28147)
	0.37785 (1.23066)	0.36146 (1.23388)	0.35136 (1.23804)	0.34414 (1.24076)

[a] For each r , the first row is for P* = .90 , the second for P* = .95 , and the third for P* = .99 .

[b] The numbers in parentheses are the ratios of (3.6) to exact results given by Gupta (1963) . $\nu = 2r$

TABLE 3a.

Values of $\nu = 2r$ given by (3.8)[a,b] .

δ \ k	4		6		8		10	
.25	8	(1.00)	10	(1.25)	12	(1.20)	12	(1.20)
	11	(1.10)	13	(1.08)	14	(1.16)	15	(1.07)
	17	(1.00)	20	(1.11)	21	(1.05)	23	(1.15)
.30	11	(1.00)	13	(1.08)	15	(1.25)	16	(1.33)
	14	(1.00)	17	(1.21)	18	(1.12)	20	(1.25)
	22	(1.00)	25	(1.04)	27	(1.03)	28	(1.07)
.35	13	(1.08)	17	(1.21)	19	(1.18)	21	(1.31)
	18	(1.12)	21	(1.05)	24	(1.20)	25	(1.13)
	29	(1.03)	32	(1.06)	35	(1.09)	36	(1.05)
.40	17	(1.06)	21	(1.16)	24	(1.20)	26	(1.18)
	23	(1.04)	27	(1.12)	30	(1.15)	33	(1.17)
	37	(1.02)	42	(1.05)	45	(1.07)	47	(1.06)
.45	22	(1.10)	28	(1.16)	31	(1.19)	34	(1.21)
	30	(1.07)	36	(1.12)	39	(1.14)	42	(1.16)
	48	(1.04)	54	-	58	-	61	-

[a] For each δ , the first row is for $P^* = .90$, the second for $P^* - .95$, the third for $P^* = .99$.

[b] The numbers in parentheses are the ratios of these results to exact results given by Gupta (1963).

TABLE 3b.

Values of $\nu = 2r$ given by (3.9)[a,b] .

δ \ k	4		6		8		10	
.25	16	(2.00)	18	(2.25)	19	(1.90)	20	(2.00)
	19	(1.90)	21	(1.75)	23	(1.91)	24	(1.71)
	26	(1.44)	28	(1.55)	30	(1.50)	31	(1.55)
.35	26	(2.16)	30	(2.14)	32	(2.00)	34	(2.12)
	31	(1.93)	35	(1.75)	38	(1.90)	40	(1.81)
	44	(1.57)	48	(1.60)	50	(1.56)	52	(1.52)
.40	34	(2.12)	39	(2.16)	42	(2.10)	44	(2.00)
	41	(1.86)	46	(1.91)	49	(1.88)	51	(1.82)
	57	(1.58)	62	(1.55)	65	(1.54)	67	(1.52)
.425	38	(2.11)	44	(2.20)	48	(2.18)	51	(2.12)
	46	(1.91)	52	(1.85)	56	(1.83)	59	(1.96)
	65	(1.54)	70	(1.52)	74	(1.54)	77	(1.54)
.45	44	(2.00)	50	(2.08)	55	(2.11)	58	(2.07)
	53	(1.89)	60	(1.87)	64	(1.88)	67	(1.86)
	74	(1.60)	-		-		-	
.50	58	(2.23)	66	(2.20)	72	(2.00)	76	(2.11)
	70	(1.91)	78	(1.85)	84	(1.83)	88	(1.96)
	-		-		-		-	

[a] For each δ , the first row is for $P^* = .90$, the second for $P^* = .95$, and the third for $P^* = .99$.

[b] The numbers in parentheses are the ratios of these results to exact results given in Gupta (1963).