

*This research was partially supported by the C.S.I.R.

AN ASYMPTOTIC DISTRIBUTION FOR THE
j-th esf OF THE CHARACTERISTIC ROOTS
OF THE NON-CENTRAL WISHART MATRIX

D.J. de Waal*

University of North Carolina
University of the Orange Free State

Institute of Statistics Mimeo Series No. 960

November, 1974

An Asymptotic Distribution for the j-th esf of the
characteristic roots of the non-central Wishart matrix

D.J. de Waal*

University of North Carolina
University of the Orange Free State

1. Introduction. Let $nS = nXX'$ for $X(p \times n)$ be distributed $W(\Sigma, n, \Omega)$ where $\Omega = \frac{1}{2} \Sigma^{-1} MM'$ and $M = E(X)$. The asymptotic distribution for $|S|$ has been considered by Fujikoshi (1968) if $\Omega = O(1)$ and by Sugiura and Nagao (1970) if $\Omega = O(n)$ w.r.t.n. Fujikoshi (1970) has derived the asymptotic distribution for $\text{tr} R^{-1}S$, R any real fixed $p \times p$ symmetric matrix, for both cases $\Omega = O(1)$ and $\Omega = O(n)$. Fujikoshi (1970) has also considered the asymptotic distribution of $|S|$ for $\Omega = O(\sqrt{n})$. An attempt is made here to derive an asymptotic distribution for the j-th esf of the characteristic roots of S , denoted by $\text{tr}_j S$. This will generalize the above mentioned papers in the sense that

$$\text{tr}_j S = \begin{cases} \text{tr} S & \text{for } j = 1 \\ |S| & \text{for } j = p . \end{cases}$$

The two theorems that will be proved, are the asymptotic distributions of $\text{tr}_j R^{-1}S$ to order terms $1/\sqrt{n}$ for $\Omega = O(1)$ and $\Omega = O(n) = n\theta$ say. Since the characteristic roots of S are invariant under an orthogonal transformation

$$S \rightarrow HSH' \quad H \in O(p) ,$$

* This research was partially supported by the C.S.I.R.

Ω can be considered as diagonal under the assumption $\Sigma = I_p$. The theorems hold, however, for general Σ but since R is arbitrary, Σ is taken as the identity in the proofs.

2. Some useful results.

Lemma 2.1 de Waal (1973)

For any Q ($p \times p$)

$$2.1 \quad \frac{\partial \text{tr}_j QF}{\partial F} = (-1)^{j-1} Q' \sum_{i=1}^j (-1)^{j-i} (F'Q')^{i-1} \text{tr}_{j-i} QF .$$

Lemma 2.2 Let $E_{rs}^* = \frac{\partial \Sigma}{\partial \sigma_{rs}^*}$ and $\partial = \left(\frac{1}{2}(1 + \delta_{rs}) \frac{\partial}{\partial \sigma_{rs}} \right) = \left(\frac{\partial}{\partial \sigma_{rs}^*} \right)$, δ_{rs}

being the Kronecker's delta, then

$$2.2 \quad \frac{\partial \text{tr}_j R^{-1} \Sigma}{\partial \sigma_{rs}^*} = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} \text{tr}(R^{-1} (\Sigma R^{-1})^{i-1} E_{rs}^*) \text{tr}_{j-i} R^{-1} \Sigma = \text{tr} \Lambda_j E_{rs}^*$$

where

$$2.3 \quad \Lambda_j = \frac{\partial \text{tr}_j R^{-1} \Sigma}{\partial \Sigma} = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} R^{-1} (\Sigma R^{-1})^{i-1} \text{tr}_{j-i} R^{-1} \Sigma .$$

Proof: Since

$$\begin{aligned} \frac{\partial \text{tr}_j R^{-1} \Sigma}{\partial \sigma_{rs}^*} &= \text{tr} \left(\frac{\partial \text{tr}_j R^{-1} \Sigma}{\partial \Sigma} \cdot \frac{\partial \Sigma}{\partial \sigma_{rs}^*} \right) \\ &= \text{tr}(\Lambda_j E_{rs}^*) . \end{aligned}$$

Corollary 2.1

$$2.4 \quad \frac{\partial \text{tr}_j R^{-1} \Sigma}{\partial \sigma_{rs}^*} \Big|_{\Sigma = \frac{1}{v} I} = v^{1-j} \text{tr} \Gamma_j E_{rs}^*$$

and

$$2.5 \quad \frac{\partial \text{tr}_j R^{-1} \Sigma}{\partial \sigma_{rs}^*} \Big|_{\Sigma = \frac{1}{v}(I+2\Theta)} = v^{1-j} \text{tr} \Gamma_j^* E_{rs}^*$$

where

$$2.6 \quad \Gamma_j = \Lambda_j \Big|_{\Sigma = I} = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} R^{-i} \text{tr}_{j-i} R^{-1}$$

and

$$\Gamma_j^* = \Lambda_j \Big|_{\Sigma = I+2\Theta} = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} R^{-1} ((I+2\Theta)R^{-1})^{i-1} \text{tr}_{j-i} R^{-1} (I+2\Theta)$$

Proof:

These results follow directly from Lemma 2.2.

Lemma 2.3 Let

$$\tau^2 = 2 \text{tr} \Gamma_j^2$$

and

$$\eta^2 = 2 \text{tr} \Gamma_j^{*2}$$

then

$$2.8 \quad e^{-t^2 \text{tr} \partial^2 / \tau^2} \text{tr} \Omega \partial \exp(v^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma = \frac{1}{v} I_p} \\ = \exp(v^{-1} \text{tr}_j R^{-1}) e^{-\frac{1}{2} t^2 \text{tr} \Omega \Gamma_j} + o(v^{-1}).$$

$$2.9 \quad e^{-t^2 \text{tr} \partial^2 / \tau^2} \text{tr} \partial^3 \exp(v^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma = \frac{1}{v} I_p} \\ = \exp(v^{-1} \text{tr}_j R^{-1}) e^{-\frac{1}{2} t^2 \text{tr} \Gamma_j^3} + o(v^{-1}).$$

$$2.10 \quad e^{-t^2 \text{tr} \partial^2 / \eta^2} \text{tr} \partial^3 (I+6\Theta) \exp(v^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma = \frac{1}{v} (I+2\Theta)} \\ = \exp(v^{-1} \text{tr}_j R^{-1} (I+2\Theta)) e^{-\frac{1}{2} t^2 \text{tr} \Gamma_j^{*3} (I+2\Theta)} + o(v^{-1}).$$

Proof:

From 2.2 it is clear that

$$\begin{aligned}
& \text{tr} \Omega \partial \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I}_p \\
& = \nu^{j-1} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \Sigma_{r,s} \omega_{rs} \text{tr} \Lambda_j E_{sr}^* \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I}_p \\
& = \nu^{j-1} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Omega \Lambda_j \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I}_p \\
& = \exp(\nu^{-1} \text{tr}_j R^{-1}) \text{tr} \Omega \Gamma_j .
\end{aligned}$$

Hence

$$\begin{aligned}
& \text{tr} \partial^2 \text{tr} \Omega \partial \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I}_p \\
& = \Sigma_{r,s} \frac{\partial^2}{\partial \sigma_{rs}^* \partial \sigma_{sr}^*} \{ \nu^{j-1} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Omega \Lambda_j \} \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I} \\
& = \Sigma_{r,s} \frac{\partial}{\partial \sigma_{rs}^*} \left\{ \nu^{2(j-1)} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Lambda_j E_{sr}^* \text{tr} \Omega \Lambda_j \right. \\
& \quad \left. + \nu^{j-1} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \frac{\partial \text{tr} \Omega \Lambda_j}{\partial \sigma_{sr}^*} \right\} \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I} \\
& = \Sigma_{r,s} \nu^{3(j-1)} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Lambda_j E_{rs}^* \text{tr} \Lambda_j E_{sr}^* \text{tr} \Omega \Lambda_j \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I} + o(\nu^{-1}) \\
& = \exp(\nu^{-1} \text{tr}_j R^{-1}) \text{tr} \Gamma_j^2 \text{tr} \Omega \Gamma_j + o(\nu^{-1}) \\
& \text{tr}^2 \partial^2 \text{tr} \Omega \partial \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I}_p \\
& = \Sigma_{m,n} \Sigma_{r,s} \frac{\partial^4}{\partial \sigma_{mn}^* \partial \sigma_{nm}^* \partial \sigma_{rs}^* \partial \sigma_{sr}^*} \{ \nu^{j-1} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Omega \Lambda_j \} \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I}_p \\
& = \Sigma_{m,n} \Sigma_{r,s} \nu^{5(j-1)} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Lambda_j E_{mn}^* \text{tr} \Lambda_j E_{nm}^* \\
& \quad \text{tr} \Lambda_j E_{rs}^* \text{tr} \Lambda_j E_{sr}^* \text{tr} \Omega \Lambda_j \Big|_{\Sigma} = \frac{1}{\nu} \mathbb{I} + o(\nu^{-1}) \\
& = \exp(\nu^{-1} \text{tr}_j R^{-1}) \text{tr}^2 \Gamma_j^2 \text{tr} \Omega \Gamma_j + o(\nu^{-1}) .
\end{aligned}$$

Continuing in the same manner it is quite clear that

$$\begin{aligned} \text{tr}^k \partial^2 \text{tr} \Omega \partial \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) |_{\Sigma} &= \frac{1}{\nu} I_p \\ &= \exp(\nu^{-1} \text{tr}_j R^{-1}) \text{tr}^k \Gamma_j^2 \text{tr} \Omega \Gamma_j + 0(\nu^{-1}) \end{aligned}$$

and hence 2.8 follows.

Consider

$$\begin{aligned} \text{tr} \partial^3 \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) |_{\Sigma} &= \frac{1}{\nu} I_p \\ &= \sum_{r,s,e} \frac{\partial^3}{\partial \sigma_{rs}^* \partial \sigma_{se}^* \partial \sigma_{er}^*} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) |_{\Sigma} = \frac{1}{\nu} I \\ &= \sum_{r,s,e} \nu^{3(j-1)} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Lambda_j E_{er}^* \text{tr} \Lambda_j E_{se}^* \text{tr} \Lambda_j E_{rs}^* |_{\Sigma} = \frac{1}{\nu} I + 0(\nu^{-1}) \\ &= \exp(\nu^{-1} \text{tr}_j R^{-1}) \text{tr} \Gamma_j^3 + 0(\nu^{-1}) . \end{aligned}$$

It is therefore quite obvious that 2.8 is true.

Let $B = (I + 6\Theta)$, then

$$\begin{aligned} \text{tr} \partial^3 B \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) |_{\Sigma} &= \frac{1}{\nu} (I + 2\Theta) \\ &= \sum_{r,s,e,k} b_{rs} \frac{\partial^3}{\partial \sigma_{se}^* \partial \sigma_{ek}^* \partial \sigma_{kr}^*} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) |_{\Sigma} = \frac{1}{\nu} (I + 2\Theta) \\ &= \sum_{r,s,e,k} b_{rs} \nu^{3(j-1)} \exp(\nu^{j-1} \text{tr}_j R^{-1} \Sigma) \text{tr} \Lambda_j E_{se}^* \text{tr} \Lambda_j E_{ek}^* \\ &\quad \text{tr} \Lambda_j E_{rs}^* |_{\Sigma} = \frac{1}{\nu} (I + 2\Theta) + 0(\nu^{-1}) \\ &= \text{tr} \Gamma_j^3 (I + 6\Theta) \exp(\text{tr}_j R^{-1} (I + 2\Theta)) + 0(\nu^{-1}) \end{aligned}$$

and hence 2.10 follows easily.

3. The asymptotic distributions of $\text{tr}_j R^{-1} S$

From the result by Anderson (1946) we have that if nS is distributed $W(I, n, \Omega)$, then for B p.d.s. and scalar λ

$$3.1 \quad E \text{etr}(\sqrt{n} \lambda BS) \\ = \left| I - \frac{2\lambda B}{\sqrt{n}} \right|^{-\frac{1}{2}n} \text{etr}(-\Omega) \text{etr} \left(\Omega \left(I - \frac{2\lambda B}{\sqrt{n}} \right)^{-1} \right).$$

Using the expansions

$$\left| I - \frac{2\lambda B}{\sqrt{n}} \right|^{-\frac{1}{2}n} = \text{etr} \left(\frac{1}{2} \sum_{i=1}^6 n^{1-i/2} (2\lambda B)^i / i + o(n^{-7/2}) \right)$$

and

$$\text{etr} \left(I - \frac{2\lambda B}{\sqrt{n}} \right)^{-1} \Omega = \text{etr} \left(\sum_{i=0}^6 n^{-i/2} (2\lambda B)^i \Omega + o(n^{-7/2}) \right),$$

3.1 can be written as

$$3.2 \quad E \text{etr}(\sqrt{n} \lambda BS) \\ = \text{etr}(\sqrt{n} \lambda B) \text{etr} \left(\frac{1}{2} \lambda^2 B^2 \right) \left(1 + \frac{4}{3\sqrt{n}} \lambda^3 \text{tr} B^3 + o(n^{-1}) \right) \\ \text{etr} \left(\frac{2\lambda}{\sqrt{n}} B \Omega + \frac{(2\lambda)^2}{n} B^2 \Omega + \frac{(2\lambda)^3}{n\sqrt{n}} B^3 \Omega + o(n^{-2}) \right).$$

Theorem 3.1 Let nS be distributed as $W(\Sigma, n, \Omega)$ and let $\Omega = 0(1)$, then the asymptotic distribution of

$$\gamma = \frac{\sqrt{n}}{\tau} (\text{tr}_j R^{-1} S - \text{tr}_j R^{-1} \Sigma)$$

is given by

$$3.3 \quad P(\gamma < z) = \Phi(z) - \frac{2}{\sqrt{n}\tau} (\text{tr} \Omega \Gamma_j \Phi'(z) + \frac{2}{3\tau^2} \text{tr} \Gamma_j^3 \Phi^{(3)}(z)) + o(n^{-1})$$

where

$$\tau^2 = 2 \text{tr} \Gamma_j^2$$

and

$$\Gamma_j = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} (\Sigma^{1/2} R^{-1} \Sigma^{1/2})^i \text{tr}_{j-i} R^{-1} \Sigma.$$

$\phi^{(r)}(z)$ denotes the r -th derivative of the standard normal distribution function $\phi(z)$.

Proof: Without loss of generality assume $\Sigma = I$.

Expanding

$$\exp\left(\frac{it\sqrt{n}}{\tau} \text{tr}_j R^{-1} S\right)$$

as a Taylor series at

$$\frac{it\sqrt{n}}{\tau} S = \frac{it\sqrt{n}}{\tau} I_p$$

i.e.

$$\begin{aligned} & \exp\left(\frac{it\sqrt{n}}{\tau} \text{tr}_j R^{-1} S\right) \\ &= \text{etr}\left(\frac{it\sqrt{n}}{\tau} (S-I) \partial\right) \exp(v^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma} = \frac{1}{v} I_p \end{aligned}$$

where

$$v = \frac{\tau}{it\sqrt{n}},$$

the characteristic function of

$$\gamma = \frac{\sqrt{n}}{\tau} (\text{tr}_j R^{-1} S - \text{tr}_j R^{-1})$$

can be written as

$$\begin{aligned} 3.4 \quad \phi_{\gamma}(t) &= E e^{it\gamma} \\ &= \exp\left(-\frac{it\sqrt{n}}{\tau} \text{tr}_j R^{-1}\right) E \exp\left(\frac{it\sqrt{n}}{\tau} \text{tr}_j R^{-1} S\right) \\ &= \exp\left(-\frac{it\sqrt{n}}{\tau} \text{tr}_j R^{-1}\right) E \text{etr}\left(\frac{it\sqrt{n}}{\tau} S \partial\right) \\ &= \text{etr}\left(-\frac{it\sqrt{n}}{\tau} \partial\right) \exp(v^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma} = \frac{1}{v} I_p \\ &= \exp(-v^{-1} \text{tr}_j R^{-1}) \text{etr}(-t^2 \partial^2 / \tau^2) \left(1 - \frac{2it}{\sqrt{n}\tau} (\text{tr} \partial \Omega \right. \\ &\quad \left. + \frac{2(it)^2}{3\tau^2} \text{tr} \partial^3) + O(n^{-1})\right) \exp(v^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma} = \frac{1}{v} I \end{aligned}$$

Hence using 2.8 and 2.9 $\phi_Y(t)$ becomes

$$3.5 \quad \phi_Y(t) = e^{-\frac{1}{2}t^2} \left(1 + \frac{2it}{\sqrt{n}\tau} (\text{tr} \Omega \Gamma_j + \frac{2(it)^2}{3\tau^2} \text{tr} \Gamma_j^3) + o(n^{-1}) \right).$$

Inverting 3.5 gives the theorem

Theorem 3.2 Let nS be distributed as $W(\Sigma, n, \Omega)$ and let $\Omega = O(n) = n\Theta$ say, then the asymptotic distribution of

$$\zeta = \frac{\sqrt{n}}{\sigma} (\text{tr}_j R^{-1} S - \text{tr}_j \Sigma^{1/2} R^{-1} \Sigma^{1/2} (I+2\Theta))$$

is given by

$$3.6 \quad P(\zeta < z) = \Phi(z) - \frac{4}{3\sqrt{n}\sigma^3} \text{tr} \Gamma_j^{*3} (I+6\Theta) \Phi^{(3)}(z) + o(n^{-1})$$

where

$$\sigma^2 = 2 \text{tr} \Gamma_j^{*2}$$

and

$$\Gamma_j^* = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} \Sigma^{1/2} R^{-1} \Sigma^{1/2} ((I+2\Theta) \Sigma^{1/2} R^{-1} \Sigma^{1/2})^{i-1} \text{tr}_{j-i} \Sigma^{1/2} R^{-1} \Sigma^{1/2} (I+2\Theta).$$

Proof: Again we assume $\Sigma = I$.

From 3.2 under the assumption $\Omega = n\Theta$, the characteristic function of ζ can be written as

$$\begin{aligned} \phi_\zeta(t) &= \exp(-v^{-1} \text{tr}_j R^{-1} (I+2\Theta)) \text{etr}(-t^2 \partial^2 / \sigma^2) \\ &\quad \left(1 - \frac{4(it)^3}{3\sqrt{n}\sigma^3} \text{tr} \partial^3 (I+6\Theta) + o(n^{-1}) \right) \\ &\quad \exp(v^{j-1} \text{tr}_j R^{-1} \Sigma) \Big|_{\Sigma = \frac{1}{v}(I+2\Theta)} \end{aligned}$$

where $v = \frac{\sigma}{it\sqrt{n}}$.

Using 2.10, the characteristic function becomes

$$3.7 \quad \phi_{\zeta}(t) = \text{etr}\left(-\frac{1}{2}t^2\right) \left(1 - \frac{4(it)^3}{3\sqrt{n}\sigma^3} \text{tr}\Gamma_j^{*3}(I+6\Theta) + o(n^{-1})\right).$$

Inverting 3.7 gives the theorem.

An attempt has been made to derive these two theorems to higher order terms and will be given in a later communication.

It is interesting to note that if $j = 1$, the results coincide with the results of Fujikoshi (1970).

References

- Anderson, T.W. (1946): The non-central Wishart distribution and certain problems of multivariate statistics. *Ann. Math. Statist.* 17, 409-431.
- de Waal, D.J. (1973): On the elementary symmetric functions of the Wishart and correlation matrices. *S. Afr. Statist. J.*, 7, 41-60.
- Fujikoshi, Y. (1968): Asymptotic expansion of the distribution of the generalized variance in the noncentral case. *J. Sci. Hiroshima Univ.* Ser. A-I, 32, 293-299.
- Fujikoshi, Y. (1970): Asymptotic expansions of tests statistics in multivariate analysis. *J. Sci. Hiroshima Univ.* Ser. A-I, 34, 73-144.
- Sugiura, N. and Nagao (1971): Asymptotic expansion of the distribution of the generalized variance for noncentral Wishart matrix, when $\Omega = O(n)$. *Ann. Inst. Statist. Math.* 23, 469-475.