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AN ASYMPTOTIC DISTRIBUTION FOR THE j-th esf OF THE CHARACTERISTIC ROOTS OF THE NON-CENTRAL WISHART MATRIX

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1. Introduction. Let $nS = nXX'$ for $X(p\times n)$ be distributed $W(\Sigma,n,\Omega)$ where $\Omega = \frac{1}{2}\Sigma^{-1}MM'$ and $M = E(X)$. The asymptotic distribution for $|S|$ has been considered by Fujikoshi (1968) if $\Omega = 0(1)$ and by Sugiura and Nagao (1970) if $\Omega= 0(n)$ w.r.t.n. Fujikoshi (1970) has derived the asymptotic distribution for $\mathrm{tr}\, R^{-1}S$, R any real fixed $p\times p$ symmetric matrix, for both cases $\Omega = 0(1)$ and $\Omega = 0(n)$. Fuyikoshi (1970) has also considered the asymptotic distribution of $|S|$ for $\Omega = 0(\sqrt{n})$. An attempt is made here to derive an asymptotic distribution for the j-th esf of the characteristic roots of S, denoted by $\mathrm{tr}_j S$. This will generalize the above mentioned papers in the sense that

$$\mathrm{tr}_j S = \begin{cases} \mathrm{tr} S & \text{for } j = 1 \\ |S| & \text{for } j = p\ . \end{cases}$$

The two theorems that will be proved, are the asymptotic distributions of $\mathrm{tr}_j R^{-1}S$ to order terms $1/\sqrt{n}$ for $\Omega = 0(1)$ and $\Omega = 0(n) = n\theta$ say. Since the characteristic roots of S are invariant under an orthogonal transformation

$$S \to HSH' \qquad H \in 0(p)\ ,$$

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Ω can be considered as diagonal under the assumption $\Sigma = I_p$. The theorems hold, however, for general Σ but since R is arbitrary, Σ is taken as the identity in the proofs.

2. Some useful results.

Lemma 2.1 de Waal (1973)

For any Q (p×p)

$$2.1 \qquad \frac{\partial \mathrm{tr}_j QF}{\partial F} = (-1)^{j-1} Q' \sum_{i=1}^{j} (-1)^{j-i} (F'Q')^{i-1} \mathrm{tr}_{j-i} QF \ .$$

Lemma 2.2 Let $E^*_{rs} = \dfrac{\partial \Sigma}{\partial \sigma^*_{rs}}$ and $\partial = (\frac{1}{2}(1+\delta_{rs})\dfrac{\partial}{\partial \sigma_{rs}}) = (\dfrac{\partial}{\partial \sigma^*_{rs}})$, δ_{rs} being the Kronecker's delta, then

$$2.2 \qquad \frac{\partial \mathrm{tr}_j R^{-1}\Sigma}{\partial \sigma^*_{rs}} = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} \mathrm{tr}(R^{-1}(\Sigma R^{-1})^{i-1} E^*_{rs}) \mathrm{tr}_{j-i} R^{-1}\Sigma = \mathrm{tr}\Lambda_j E^*_{rs}$$

where

$$2.3 \qquad \Lambda_j = \frac{\partial \mathrm{tr}_j R^{-1}\Sigma}{\partial \Sigma} = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} R^{-1} (\Sigma R^{-1})^{i-1} \mathrm{tr}_{j-i} R^{-1}\Sigma \ .$$

Proof: Since

$$\frac{\partial \mathrm{tr}_j R^{-1}\Sigma}{\partial \sigma^*_{rs}} = \mathrm{tr}\left(\frac{\partial \mathrm{tr}_j R^{-1}\Sigma}{\partial \Sigma} \circ \frac{\partial \Sigma}{\partial \sigma^*_{rs}}\right)$$

$$= \mathrm{tr}(\Lambda_j E^*_{rs}) \quad .$$

Corollary 2.1

$$2.4 \qquad \frac{\partial \mathrm{tr}_j R^{-1}\Sigma}{\partial \sigma^*_{rs}} \Big|_{\Sigma = \frac{1}{\nu} I} = \nu^{1-j} \mathrm{tr}\Gamma_j E^*_{rs}$$

and

2.5 $$\left.\frac{\partial \operatorname{tr}_j R^{-1}\Sigma}{\partial \sigma^*_{rs}}\right|_{\Sigma = \frac{1}{\nu}(I+2\Theta)} = \nu^{1-j}\operatorname{tr}\Gamma^*_j E^*_{rs}$$

where

2.6 $$\Gamma_j = \Lambda_j\big|_{\Sigma = I} = (-1)^{j-1}\sum_{i=1}^{j}(-1)^{j-i}R^{-i}\operatorname{tr}_{j-i}R^{-1}$$

and

$$\Gamma^*_j = \Lambda_j\big|_{\Sigma = I+2\Theta} = (-1)^{j-1}\sum_{i=1}^{j}(-1)^{j-i}R^{-1}((I+2\Theta)R^{-1})^{i-1}\operatorname{tr}_{j-i}R^{-1}(I+2\Theta)$$

Proof:

These results follow directly from Lemma 2.2.

Lemma 2.3 Let

$$\tau^2 = 2\operatorname{tr}\Gamma_j^2$$

and

$$\eta^2 = 2\operatorname{tr}\Gamma_j^{*2}$$

then

2.8 $$e^{-t^2\operatorname{tr}\partial^2/\tau^2}\operatorname{tr}\Omega\partial\,\exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\big|_{\Sigma = \frac{1}{\nu}I_p} = \exp(\nu^{-1}\operatorname{tr}_j R^{-1})e^{-\frac{1}{2}t^2}\operatorname{tr}\Omega\Gamma_j + 0(\nu^{-1}).$$

2.9 $$e^{-t^2\operatorname{tr}\partial^2/\tau^2}\operatorname{tr}\partial^3\,\exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\big|_{\Sigma = \frac{1}{\nu}I_p} = \exp(\nu^{-1}\operatorname{tr}_j R^{-1})e^{-\frac{1}{2}t^2}\operatorname{tr}\Gamma_j^3 + 0(\nu^{-1}) .$$

2.10 $$e^{-t^2\operatorname{tr}\partial^2/\eta^2}\operatorname{tr}\partial^3(I+6\Theta)\exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\big|_{\Sigma = \frac{1}{\nu}(I+2\Theta)} = \exp(\nu^{-1}\operatorname{tr}_j R^{-1}(I+2\Theta))e^{-\frac{1}{2}t^2}\operatorname{tr}\Gamma_j^{*3}(I+2\Theta) + 0(\nu^{-1}) .$$

Proof:

From 2.2 it is clear that

$$\mathrm{tr}\Omega\partial\ \exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \nu^{j-1}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\Sigma_{r,s}\omega_{rs}\mathrm{tr}\Lambda_j E^*_{sr}\Big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \nu^{j-1}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\mathrm{tr}\Omega\Lambda_j\Big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \exp(\nu^{-1}\mathrm{tr}_j R^{-1})\mathrm{tr}\Omega\Gamma_j\ .$$

Hence

$$\mathrm{tr}\partial^2\mathrm{tr}\Omega\partial\ \exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \Sigma_{r,s}\frac{\partial^2}{\partial\sigma^*_{rs}\sigma^*_{sr}}\{\nu^{j-1}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\mathrm{tr}\Omega\Lambda_j\}\Big|_{\Sigma = \frac{1}{\nu}I}$$

$$= \Sigma_{r.s}\frac{\partial}{\partial\sigma^*_{rs}}\{\nu^{2(j-1)}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\mathrm{tr}\Lambda_j E^*_{sr}\mathrm{tr}\Omega\Lambda_j + \nu^{j-1}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\frac{\partial\mathrm{tr}\Omega\Lambda_j}{\partial\sigma^*_{sr}}\}\Big|_{\Sigma = \frac{1}{\nu}I}$$

$$= \Sigma_{r,s}\nu^{3(j-1)}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\mathrm{tr}\Lambda_j E^*_{rs}\mathrm{tr}\Lambda_j E^*_{sr}\mathrm{tr}\Omega\Lambda_j\Big|_{\Sigma = \frac{1}{\nu}I} + 0(\nu^{-1})$$

$$= \exp(\nu^{-1}\mathrm{tr}_j R^{-1})\mathrm{tr}\Gamma_j^2\mathrm{tr}\Omega\Gamma_j + 0(\nu^{-1})$$

$$\mathrm{tr}^2\partial^2\mathrm{tr}\Omega\partial\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \Sigma_{m,n}\Sigma_{r,s}\frac{\partial^4}{\partial\sigma^*_{mn}\partial\sigma^*_{nm}\partial^*_{rs}\partial^*_{sr}}\{\nu^{j-1}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\mathrm{tr}\Omega\Lambda_j\}\Big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \Sigma_{m,n}\Sigma_{r,s}\ \nu^{5(j-1)}\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)\mathrm{tr}\Lambda_j E^*_{mn}\mathrm{tr}\Lambda_j E^*_{nm}\,\mathrm{tr}\Lambda_j E^*_{rs}\mathrm{tr}\Lambda_j E^*_{sr}\mathrm{tr}\Omega\Lambda_j\Big|_{\Sigma = \frac{1}{\nu}I} + 0(\nu^{-1})$$

$$= \exp(\nu^{-1}\mathrm{tr}_j R^{-1})\mathrm{tr}^2\Gamma_j^2\mathrm{tr}\Omega\Gamma_j + 0(\nu^{-1})\ .$$

Continuing in the same manner it is quite clear that

$$\operatorname{tr}^k \partial^2 \operatorname{tr}\Omega\partial \exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \exp(\nu^{-1}\operatorname{tr}_j R^{-1})\operatorname{tr}^k\Gamma_j^2\operatorname{tr}\Omega\Gamma_j + 0(\nu^{-1})$$

and hence 2.8 follows.

Consider

$$\operatorname{tr}\partial^3 \exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\big|_{\Sigma = \frac{1}{\nu}I_p}$$

$$= \Sigma_{r,s,e} \frac{\partial^3}{\partial\sigma^*_{rs}\partial\sigma^*_{se}\partial\sigma^*_{er}} \exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}I}$$

$$= \Sigma_{r,s,e}\nu^{3(j-1)}\exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\operatorname{tr}\Lambda_j E^*_{er}\operatorname{tr}\Lambda_j E^*_{se}\operatorname{tr}\Lambda_j E^*_{rs}\Big|_{\Sigma = \frac{1}{\nu}I} + 0(\nu^{-1})$$

$$= \exp(\nu^{-1}\operatorname{tr}_j R^{-1})\operatorname{tr}\Gamma_j^3 + 0(\nu^{-1}) \ .$$

It is therefore quite obvious that 2.8 is true.

Let $B = (I + 6\Theta)$, then

$$\operatorname{tr}\partial^3 B \exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\big|_{\Sigma = \frac{1}{\nu}(I+2\Theta)}$$

$$= \Sigma_{r,s,e,k} b_{rs} \frac{\partial^3}{\partial\sigma^*_{se}\partial\sigma^*_{ek}\partial\sigma^*_{kr}} \exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}(I+2\Theta)}$$

$$= \Sigma_{r,s,e,k} b_{rs}\nu^{3(j-1)}\exp(\nu^{j-1}\operatorname{tr}_j R^{-1}\Sigma)\operatorname{tr}\Lambda_j E^*_{se}\operatorname{tr}\Lambda_j E^*_{ek}$$

$$\operatorname{tr}\Lambda_j E^*_{rs}\Big|_{\Sigma = \frac{1}{\nu}(I+2\Theta)} + 0(\nu^{-1})$$

$$= \operatorname{tr}\Gamma_j^{*3}(I+6\Theta)\exp(\operatorname{tr}_j R^{-1}(I+2\Theta)) + 0(\nu^{-1})$$

and hence 2.10 follows easily.

3. The asymptotic distributions of $tr_j R^{-1} S$

From the result by Anderson (1946) we have that if nS is distributed $W(I,n,\Omega)$, then for B p.d.s. and scalar λ

3.1
$$E\ \mathrm{etr}(\sqrt{n}\ \lambda BS) = \left|I - \frac{2\lambda B}{\sqrt{n}}\right|^{-\frac{1}{2}n} \mathrm{etr}(-\Omega)\,\mathrm{etr}\left(\Omega\left(I - \frac{2\lambda B}{\sqrt{n}}\right)^{-1}\right) .$$

Using the expansions

$$\left|I - \frac{2\lambda B}{\sqrt{n}}\right|^{-\frac{1}{2}n} = \mathrm{etr}\left(\frac{1}{2}\sum_{i=1}^{6} n^{1-i/2}(2\lambda B)^i/i + 0(n^{-7/2})\right)$$

and
$$\mathrm{etr}\left(I - \frac{2\lambda B}{\sqrt{n}}\right)^{-1}\Omega = \mathrm{etr}\left(\sum_{i=0}^{6} n^{-i/2}(2\lambda B)^i\Omega + 0(n^{-7/2})\right) ,$$

3.1 can be written as

3.2
$$E\ \mathrm{etr}(\sqrt{n}\ \lambda BS)$$
$$= \mathrm{etr}(\sqrt{n}\ \lambda B)\,\mathrm{etr}(\tfrac{1}{2}\lambda^2B^2)\left(1 + \frac{4}{3\sqrt{n}}\lambda^3 \mathrm{tr}B^3 + 0(n^{-1})\right)$$
$$\mathrm{etr}\left(\frac{2\lambda}{\sqrt{n}} B\Omega + \frac{(2\lambda)^2}{n} B^2\Omega + \frac{(2\lambda)^3}{n\sqrt{n}} B^3\Omega + 0(n^{-2})\right) .$$

<u>Theorem 3.1</u> Let nS be distributed as $W(\Sigma,n,\Omega)$ and let $\Omega = 0(1)$, then the asymptotic distribution of

$$\gamma = \frac{\sqrt{n}}{\tau}(tr_j R^{-1}S - tr_j R^{-1}\Sigma)$$

is given by

3.3
$$P(\gamma < z) = \Phi(z) - \frac{2}{\sqrt{n}\tau}\left(\mathrm{tr}\Omega\Gamma_j\Phi'(z) + \frac{2}{3\tau^2}\mathrm{tr}\Gamma_j^3\Phi^{(3)}(z)\right) + 0(n^{-1})$$

where
$$\tau^2 = 2\mathrm{tr}\Gamma_j^2$$

and
$$\Gamma_j = (-1)^{j-1} \sum_{i=1}^{j} (-1^{j-i}(\Sigma^{1/2}R^{-1}\Sigma^{1/2})^i tr_{j-i}R^{-1}\Sigma .$$

$\Phi^{(r)}(z)$ denotes the r-th derivative of the standard normal distribution function $\Phi(z)$.

Proof: Without loss of generality assume $\Sigma = I$.

Expanding
$$\exp\left(\frac{it\sqrt{n}}{\tau} tr_j R^{-1} S\right)$$
as a Taylor series at
$$\frac{it\sqrt{n}}{\tau} S = \frac{it\sqrt{n}}{\tau} I_p$$
i.e.
$$\exp\left(\frac{it\sqrt{n}}{\tau} tr_j R^{-1} S\right)$$
$$= etr\left(\frac{it\sqrt{n}}{\tau}(S-I)\partial\right)\exp(\nu^{j-1} tr_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}I_p}$$
where
$$\nu = \frac{\tau}{it\sqrt{n}} ,$$
the characteristic function of
$$\gamma = \frac{\sqrt{n}}{\tau}(tr_j R^{-1} S - tr_j R^{-1})$$
can be written as

3.4
$$\phi_\gamma(t) = E\, e^{it\gamma}$$
$$= \exp\left(-\frac{it\sqrt{n}}{\tau} tr_j R^{-1}\right) E \exp\left(\frac{it\sqrt{n}}{\tau} tr_j R^{-1} S\right)$$
$$= \exp\left(-\frac{it\sqrt{n}}{\tau} tr_j R^{-1}\right) E\, etr\left(\frac{it\sqrt{n}}{\tau} S\partial\right)$$
$$etr\left(-\frac{it\sqrt{n}}{\tau}\partial\right)\exp(\nu^{j-1} tr_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}I_p}$$
$$= \exp(-\nu^{-1} tr_j R^{-1}) etr(-t^2\partial^2/\tau^2)(1 - \frac{2it}{\sqrt{n}\tau}(tr\partial\Omega$$
$$+ \frac{2(it)^2}{3\tau^2} tr\partial^3) + O(n^{-1}))\exp(\nu^{j-1} tr_j R^{-1}\Sigma)\Big|_{\Sigma = \frac{1}{\nu}I}$$

Hence using 2.8 and 2.9 $\phi_\gamma(t)$ becomes

3.5 $$\phi_\gamma(t) = e^{-\frac{1}{2}t^2}(1 + \frac{2it}{\sqrt{n}\tau}(\mathrm{tr}\Omega\Gamma_j + \frac{2(it)^2}{3\tau^2}\mathrm{tr}\Gamma_j^3) + 0(n^{-1})) .$$

Inverting 3.5 gives the theorem

Theorem 3.2 Let nS be distributed as $W(\Sigma,n,\Omega)$ and let $\Omega = 0(n) = n\Theta$ say, then the asymptotic distribution of

$$\zeta = \frac{\sqrt{n}}{\sigma}(\mathrm{tr}_j R^{-1}S - \mathrm{tr}_j \Sigma^{1/2}R^{-1}\Sigma^{1/2}(I+2\Theta))$$

is given by

3.6 $$P(\zeta < z) = \Phi(z) - \frac{4}{3\sqrt{n}\sigma^3}\mathrm{tr}\Gamma_j^{*3}(I+6\Theta)\Phi^{(3)}(z) + 0(n^{-1})$$

where
$$\sigma^2 = 2\mathrm{tr}\Gamma_j^{*2}$$

and
$$\Gamma_j^* = (-1)^{j-1}\sum_{i=1}^{j}(-1)^{j-i}\Sigma^{1/2}R^{-1}\Sigma^{1/2}((I+2\Theta)\Sigma^{1/2}R^{-1}\Sigma^{1/2})^{i-1}$$
$$\mathrm{tr}_{j-i}\Sigma^{1/2}R^{-1}\Sigma^{1/2}(I+2\Theta) .$$

Proof: Again we assume $\Sigma = I$.

From 3.2 under the assumption $\Omega = n\Theta$, the characteristic function of ζ can be written as

$$\phi_\zeta(t) = \exp(-\nu^{-1}\mathrm{tr}_j R^{-1}(I+2\Theta))\mathrm{etr}(-t^2\partial^2/\sigma^2)$$
$$(1 - \frac{4(it)^3}{3\sqrt{n}\sigma^3}\mathrm{tr}\partial^3(I+6\Theta) + 0(n^{-1}))$$
$$\exp(\nu^{j-1}\mathrm{tr}_j R^{-1}\Sigma)|_{\Sigma = \frac{1}{\nu}(I+2\Theta)}$$

where $\nu = \dfrac{\sigma}{it\sqrt{n}}$.

Using 2.10, the characteristic function becomes

$$\phi_\zeta(t) = \mathrm{etr}(-\tfrac{1}{2}t^2)\left(1 - \frac{4(it)^3}{3\sqrt{n}\sigma^3}\,\mathrm{tr}\Gamma_j^{*3}(I+6\Theta) + O(n^{-1})\right) . \qquad 3.7$$

Inverting 3.7 gives the theorem.

An attempt has been made to derive these two theorems to higher order terms and will be given in a later communication.

It is interesting to note that if $j = 1$, the results coincide with the results of Fujikoshi (1970).

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