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AN ASYMPTOTIC DISTRIBUTION FOR THE j-th esf
OF THE GENERALISED HOTELLING'S BETA MATRIX

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An Asymptotic Distribution for the j-th esf
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1. INTRODUCTION. Let $X_{\alpha}^{(i)}$ ($p \times 1$), $\alpha = 1, \dots, N_i$, be i.i.d. $N(\mu^{(i)}, \Sigma)$, $i = 1, \dots, r$. Let

$$\bar{X}^{(i)} = \sum_{\alpha=1}^{N_i} X_{\alpha}^{(i)} / N_i$$

$$A_i = \sum_{\alpha=1}^{N_i} (X_{\alpha}^{(i)} - \bar{X}^{(i)}) (X_{\alpha}^{(i)} - \bar{X}^{(i)})'$$

$$A = \sum_{i=1}^r A_i, \quad \bar{X} = \sum_{i=1}^r N_i \bar{X}^{(i)} / N, \quad \bar{\mu} = \sum_{i=1}^r N_i \mu^{(i)} / N,$$

$$\bar{X}^* = (\sqrt{N_1} (\bar{X}^{(1)} - \bar{X}), \dots, \sqrt{N_q} (\bar{X}^{(r)} - \bar{X})) ,$$

$$B = \bar{X}^* \bar{X}^{*'} ,$$

$$M^* = (\sqrt{N_1} (\mu^{(1)} - \bar{\mu}), \dots, \sqrt{N_q} (\mu^{(r)} - \bar{\mu})) .$$

Let $M(p \times q)$ be a transform of M^* and let $\Omega = \Sigma^{-1} M M'$, $q = r - 1$, then it follows that for $p \leq r \leq N$, A is distributed $W(\Sigma, n)$, $n = N - q$ independent of B which is distributed $W(\Sigma, q, \Omega)$. Let

$$V = B^{\frac{1}{2}} A^{-1} B^{\frac{1}{2}} ,$$

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$\frac{1}{B^2}$ being the symmetric square root of B , then $ntrV$ is the well known Hotelling's generalised T_0^2 statistic for testing the hypothesis

$$H: \mu^{(i)} = \mu^{(j)} \quad \text{for all } i, j$$

against $\mu^{(i)} \neq \mu^{(j)}$ for some i, j , $i \neq j$. Fujikoshi (1970) has derived the asymptotic distribution for T_0^2 in the noncentral case to order terms n^{-2} , which also holds if $q < p \leq N$. The equivalent of V in the case $q < p \leq N$ is defined as

$$F = Z'A^{-1}Z$$

where $Z(p \times q) \sim N(M, \Sigma \otimes I_q)$. For this case Ω is defined as $\Omega = M'\Sigma^{-1}M$.

We shall consider here the asymptotic distribution of the j -th esf of $tr_j F$. The result reduces to the result given by Fujikoshi if $j = 1$. We shall refer to V and F as Hotelling's generalised beta matrices, since they are distributed as multivariate beta distributions and Hotelling's generalised T_0^2 are related to them.

2. PRELIMINARY RESULTS.

Lemma 2.1 (de Waal (1974))

$$2.1 \quad \frac{\partial tr_j \Sigma}{\partial \sigma_{rs}^*} = tr \Lambda_j E_{rs}^*$$

where $E_{rs}^* = \frac{\partial \Sigma}{\partial \sigma_{rs}^*}$

$$\Lambda_j = \frac{\partial tr_j \Sigma}{\partial \Sigma} = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} \Sigma^{i-1} tr_{j-1} \Sigma$$

and

$$\left(\frac{\partial}{\partial \sigma_{rs}^*} \right) = \left(\frac{1}{2}(1 + \delta_{rs}) \frac{\partial}{\partial \sigma_{rs}} \right) = \partial ; \quad r, s = 1, \dots, q .$$

Lemma 2.2 Let

$$\Gamma_j = \Lambda_j \Big|_{\Sigma=R^{-1}} ,$$

then

$$2.2 \quad \text{etr}(B\partial) \exp(v^{1-j} \lambda \text{tr}_j \Sigma) \Big|_{\Sigma=vR^{-1}} = \text{etr}(\lambda B \Gamma_j) \exp(v \lambda \text{tr}_j R^{-1}) \{1 + o(v^{-1})\} .$$

Proof

$$\begin{aligned} \text{tr}(B\partial) \exp(v^{1-j} \lambda \text{tr}_j \Sigma) \Big|_{\Sigma=vR^{-1}} &= \sum_{r,s} b_{rs} \frac{\partial}{\partial \sigma_{sr}^*} \exp(v^{1-j} \lambda \text{tr}_j \Sigma) \Big|_{\Sigma=vR^{-1}} \\ &= v^{1-j} \lambda \exp(v \lambda \text{tr}_j \Lambda_j) \sum_{r,s} b_{rs} \text{tr} \Lambda_j E_{sr}^* \Big|_{\Sigma=vR^{-1}} \\ &= \lambda \exp(v \lambda \text{tr}_j \Gamma_j) \text{tr}(B \Gamma_j) . \end{aligned}$$

Also

$$\begin{aligned} \text{tr}^2(B\partial) \exp(v^{1-j} \lambda \text{tr}_j \Sigma) \Big|_{\Sigma=vR^{-1}} &= \sum_{m,n} \sum_{r,s} b_{mn} b_{rs} \frac{\partial^2}{\partial \sigma_{nm}^* \partial \sigma_{sr}^*} \exp(v^{1-j} \lambda \text{tr}_j \Sigma) \Big|_{\Sigma=vR^{-1}} \\ &= \sum_{m,n} \sum_{r,s} b_{mn} b_{rs} \frac{\partial}{\partial \sigma_{nm}^*} \left\{ v^{1-j} \exp(v^{1-j} \lambda \text{tr}_j \Sigma) \right. \\ &\quad \left. \text{tr} \Lambda_j E_{sr}^* \right\} \Big|_{\Sigma=vR^{-1}} \end{aligned}$$

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$$\begin{aligned}
 &= \sum_{m,n} \sum_{r,s} b_{mn} b_{rs} v^{2(1-j)} \lambda^2 \exp(v^{1-j} \lambda \text{tr}_j \Sigma) \text{tr} \Lambda_j E_{nm}^* \text{tr} \Lambda_j E_{sr}^* \Big|_{\Sigma=vR^{-1}} + o(v^{-1}) \\
 &= \lambda^2 \text{tr}^2(B\Gamma_j) \exp(v\lambda \text{tr}_j \Gamma_j) + o(v^{-1}) .
 \end{aligned}$$

Continuing in this way, it is clear that the lemma holds.

Lemma 2.3 (Crowther (1974))

Let $Z(p \times q) \sim N(M, I_p \otimes I_q)$, $q < p$, then for $A(p \times p)$ fixed, the density of $S = Z'A^{-1}Z$ is given by

$$\begin{aligned}
 2.3 \quad & \frac{1}{2^{\frac{1}{2}qp} \Gamma_q(\frac{1}{2}p)} |A|^{\frac{1}{2}q} \text{etr}(-\frac{1}{2}S) |S|^{\frac{1}{2}(p-q-1)} \\
 & \sum_{k=0}^{\infty} \sum_K \frac{1}{(\frac{1}{2}p)_K k!} E_{Y C_K} \left[-\frac{1}{2} S \left(Y - \frac{i}{\sqrt{2}} M \right)' A \left(Y - \frac{i}{\sqrt{2}} M \right) \right], \quad S > 0
 \end{aligned}$$

where the expectation is taken w.r.t. the density

$$2.4 \quad \pi^{-\frac{1}{2}pq} \text{etr}(-YY'), \quad Y(p \times q) .$$

Crowther (1974) proved this lemma for the case $\Sigma \neq I_p$.

Lemma 2.4 (Fujikoshi (1970))

$$\begin{aligned}
 2.5 \quad & \frac{\text{etr}(-\frac{1}{2}n'n) \Gamma_p(\frac{1}{2}(n+q))}{(\text{int}) \frac{1}{2}^{pq} \Gamma_p(\frac{1}{2}n)} E_Y |I_q + \frac{1}{itn} (Y - \frac{i\eta}{\sqrt{2}})' (Y - \frac{i\eta}{\sqrt{2}})|^{-\frac{1}{2}(n+q)} \\
 & = (1 - 2it)^{-\frac{1}{2}pq} \text{etr}(\frac{it}{1-2it} n'n) \left[1 + \frac{1}{4n} \{ pq(q-p-1) \right. \\
 & \quad - \frac{2q}{1-2it} (pq - \text{tr}n'n) + \frac{1}{(1-2it)^2} (pq(p+q+1) - 2(p+2q+1)\text{tr}n'n) \\
 & \quad \left. + \text{tr}(n'n)^2 + \frac{2}{(1-2it)^3} ((p+q+1)\text{tr}n'n - \text{tr}(n'n)^2) \right]
 \end{aligned}$$

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$$+ \frac{\text{tr}(n'n)^2}{(1-2it)^4} \left\} + \frac{1}{96n^2} \left\{ \sum_{\alpha=0}^8 (1-2it)^{-\alpha} B_{\alpha} \left(\frac{1}{2}n'n \right) \right\} + o(n^{-3}) \right]$$

$$= g(t|n'n) , \text{ say}$$

where the expectation is taken w.r.t. the density 2.4, $n(p \times q)$ any fixed matrix and $B_{\alpha} \left(\frac{1}{2}n'n \right)$, $\alpha = 0, 1, \dots, 8$, given in Fujikoshi (1970) p. 104.

Proof Fujikoshi (1970) showed that the characteristic function of Hotelling's generalised $T_0^2 = ntrF$ is given by

$$2.6 \quad C(t) = g(t|\Omega) .$$

On the other hand for $Z(p \times q) \sim N(M, I_p \otimes I_q)$ and $A(p \times p) \sim W(I, n)$ independent of each other and using Lemma 2.3, the character function of $ntrRF$, $R(q \times q)$ fixed nonsingular

$$2.7 \quad E \text{etr}(intrRF) = \frac{2^{-\frac{1}{2}p(n+q)}}{\Gamma_q \left(\frac{1}{2}p \right) \Gamma_p \left(\frac{1}{2}n \right)} \text{etr} \left(-\frac{1}{2}\Omega \right) \int_{A>0} \int_{S>0} \text{etr}(intrRS) |S|^{\frac{1}{2}(p-q-1)} \\ \text{etr} \left(-\frac{1}{2}A \right) |A|^{\frac{1}{2}(n+q-p-1)} \sum_{k=0}^{\infty} \sum_K \frac{1}{\left(\frac{1}{2}p \right)_K k!} E_Y C_K \left(-\frac{1}{2}S \left(Y - \frac{iM}{\sqrt{2}} \right) \right) \\ A \left(Y - \frac{iM}{\sqrt{2}} \right) dS dA \\ = \frac{\text{etr} \left(-\frac{1}{2}\Omega \right) \Gamma_p \left(\frac{1}{2}(n+q) \right)}{\left(\text{int} \right) \frac{1}{2^{pq}} \Gamma_p \left(\frac{1}{2}n \right)} E_Y |I_q + \frac{1}{\text{int}} \left(Y - \frac{iM}{\sqrt{2}} \right) \left(Y - \frac{iM}{\sqrt{2}} \right)^{-\frac{1}{2}} \left| -\frac{1}{2} \right|^{-\frac{1}{2}(n+q)} .$$

Equating 2.6 and 2.7 for $R = I_q$ proves the lemma.

The following theorem generalises the result given by Fujikoshi (1970):

Theorem 2.1 For $R(q \times q)$ fixed p.d.s. and $F(q \times q)$ Hotelling's generalised beta matrix

$$\begin{aligned}
 2.8 \quad P(\text{ntnrRF} < x) &= \text{etr}\left(-\frac{1}{2}\Omega(I - R^{-1})\right) \left[P(\chi_f^2(\delta^2) < x) + \frac{1}{4n} \left\{ pq(q - p - 1) \right. \right. \\
 &\quad - 2q(pq - \text{tr}\Omega R^{-1}) P(\chi_{f+2}^2(\delta^2) < x) + (pq(p + q + 1) \\
 &\quad - 2(p + 2q + 1)\text{tr}\Omega R^{-1} + \text{tr}(\Omega R^{-1})^2) P(\chi_{f+4}^2(\delta^2) < x) \\
 &\quad + 2((p + q + 1)\text{tr}\Omega R^{-1} - \text{tr}(\Omega R^{-1})^2) P(\chi_{f+6}^2(\delta^2) < x) \\
 &\quad \left. \left. + \text{tr}(\Omega R^{-1})^2 P(\chi_{f+8}^2(\delta^2) < x) \right\} + \frac{1}{96n^2} \left\{ \sum_{\alpha=0}^8 B_{\alpha} \left(\frac{1}{2}\Omega R^{-1}\right) \right. \right. \\
 &\quad \left. \left. P(\chi_{f+2\alpha}^2(\delta^2) < x) \right\} + O(n^{-3}) \right] \quad \text{where } \delta^2 = \text{tr}\left(\frac{1}{2}\Omega R^{-1}\right), \\
 f &= \frac{1}{2}pq \quad \text{and } \chi_f^2(\delta^2) \text{ denotes the noncentral chi-square}
 \end{aligned}$$

variable with f degrees of freedom and noncentrality parameter δ^2 .

Proof From 2.7 and Lemma 2.4 the c.f. of $\text{ntnr}^{-1}F$ is given by

$$\begin{aligned}
 2.9 \quad E \text{etr}(it\text{ntnrRF}) &= \frac{\text{etr}\left(-\frac{1}{2}\Omega\right) \Gamma_p\left(\frac{1}{2}(n+q)\right)}{(\text{int})^{\frac{1}{2}pq} \Gamma_p\left(\frac{1}{2}n\right)} E_Y | I_q + \frac{1}{\text{int}} \left(Y - \frac{i}{\sqrt{2}} MR^{-\frac{1}{2}} \right), \left(Y - \frac{i}{\sqrt{2}} MR^{-\frac{1}{2}} \right) \Big|^{-\frac{1}{2}(n+q)} \\
 &= \text{etr}\left(-\frac{1}{2}\Omega\right) \text{etr}\left(\frac{1}{2}\Omega R^{-1}\right) g(t | \Omega R^{-1}).
 \end{aligned}$$

Inverting 2.9 gives the theorem. This result holds for $q \geq p$ using the argument given by Fujikoshi (1970).

3. THE ASYMPTOTIC DISTRIBUTION OF $\text{tr}_j F$. We shall now consider the asymptotic distribution of

$$\gamma = n \text{tr}_j (F/\tau)$$

where τ is some fixed constant.

Theorem 3.1 Let R be any fixed p.d.s. matrix and $\xi = n(\text{tr}_j (F/\tau) + \text{tr} R^{-1} \Gamma_j - \text{tr}_j R^{-1})$, then

$$3.1 \quad P(\xi < x) = P(\chi_f^2(\delta^2) < x) + o(n^{-1})$$

where

$$f = \frac{1}{2}pq, \quad \delta^2 = \tau \text{tr} \left(\frac{1}{2} \Omega \Gamma_j^{-1} \right), \quad \tau = \text{tr} \Omega / \text{tr} \Omega \Gamma_j^{-1} \quad \text{and}$$

$$\Gamma_j = (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} R^{1-i} \text{tr}_{j-i} R^{-1}.$$

Proof The characteristic function of

$$\gamma = n \text{tr}_j (F/\tau)$$

is given by

$$\phi(t) = E \exp(i \text{tr}_j (F/\tau) t) .$$

Expanding $\exp(i \text{tr}_j F/\tau)$ as a Taylor series at

$$\frac{i \text{tr}_j F}{\tau} = i \text{tr}_j R^{-1}$$

for some p.d.s. matrix R

i.e. for $v = \text{int}$

$$\begin{aligned} \exp(v \text{tr}_j(F/\tau)) &= \exp(v^{1-j} \text{tr}_j(vF/\tau)) \\ &= \text{etr}(v(F/\tau - R^{-1})\theta) \exp(v^{1-j} \text{tr}_j \Sigma) \Big|_{\Sigma=vR^{-1}} . \end{aligned}$$

Let $B = v(F/\tau - R^{-1})$ in Lemma 2.2, then $\phi(t)$ can be written as

$$\begin{aligned} 3.2 \quad \phi(t) &= E \text{etr}(v(F/\tau - R^{-1})\Gamma_j) \exp(v \text{tr}_j R^{-1}) + o(n^{-1}) \\ &= \exp(v(\text{tr}_j R^{-1} - \text{tr} R^{-1} \Gamma_j)) E \text{etr}\left(\frac{v}{\tau} F \Gamma_j\right) + o(n^{-1}) \end{aligned}$$

where Γ_j is given in Lemma 2.2.

Let

$$R = \frac{1}{\tau} \Gamma_j$$

in 2.9, then using 2.5 the characteristic function becomes

$$\begin{aligned} 3.3 \quad \phi(t) &= \exp(v(\text{tr}_j R^{-1} - \text{tr} R^{-1} \Gamma_j)) \text{etr}\left(-\frac{1}{2}\Omega\right) \\ &\quad \text{etr}\left(\frac{1}{2}\tau \Omega \Gamma_j^{-1}\right) (1 - 2it)^{-\frac{1}{2}pq} \text{etr}\left(\frac{it\tau}{1-2it} \Omega \Gamma_j^{-1}\right) + o(n^{-1}) . \end{aligned}$$

Let

$$3.4 \quad \tau = \text{tr} \Omega / \text{tr} \Omega \Gamma_j^{-1}$$

then it follows from 3.3 that the characteristic function of

$$\xi = n \left\{ \text{tr}_j(F/\tau) + \text{tr} R^{-1} \Gamma_j - \text{tr}_j R^{-1} \right\}$$

can be written as

$$3.5 \quad h(t) = (1 - 2it)^{-\frac{1}{2}pq} \operatorname{etr}\left(\frac{-it\tau}{1-2it} \Omega \Gamma_j^{-1}\right) + o(n^{-1})$$

which proves the theorem.

Since R is arbitrary, it may be useful to let $R = I$. In this case Γ_j becomes (see Gradshteyn and Ryzhik (1965))

$$\begin{aligned} 3.6 \quad \Gamma_j &= \Lambda_j \Big|_{\Sigma = I_q} \\ &= (-1)^{j-1} \sum_{i=1}^j (-1)^{j-i} \binom{q}{j-i} I_q \\ &= (-1)^{j-1} \sum_{x=0}^{j-1} (-1)^x \binom{q}{x} I_q \\ &= \binom{q-1}{j-1} I_q . \end{aligned}$$

Then

$$\tau = \binom{q-1}{j-1}$$

and ξ becomes

$$3.7 \quad \xi = n \left\{ \operatorname{tr}_j \left[F / \binom{q-1}{j-1} \right] + (j-1) \binom{q}{j} \right\} .$$

The noncentrality parameter will then be

$$\delta^2 = \operatorname{tr} \frac{1}{2} \Omega .$$

In the special case $j = 1$ it follows that

$$\xi = n \operatorname{tr} F$$

and the result by Fujikoshi (1979) follows immediately.

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