MEASURE REPRESENTATIONS FOR EVENTS IN A PARTIAL ORDER: AN AXIOMATIZATION

By

Roger A. Blau, Richard H. Shachtman, Thomas S. Wallsten

School of Business Administration
Department of Biostatistics
Curriculum in Operations Research and Systems Analysis
L. L. Thurstone Psychometric Laboratory

University of North Carolina at Chapel Hill

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ABSTRACT

Currently all the axiom systems for probability representation include the assumption that the ordering of events is total. However, empirical systems exist for which this condition is too strong. In particular, there may exist events A_1, A_2 in an algebra $\mathbf{A}$ for which the ordering is not connected; i.e., it is not possible to judge with complete confidence whether $A_1 \precsim_{\mathbf{A}} A_2$ or $A_2 \precsim_{\mathbf{A}} A_1$.

One approach to deriving representations for such sets proceed as follows: assume the existence of a uniformly distributed random variable which induces a σ-field $\mathbf{C}$ containing ordered events C_i, $i \in I$. For each $A \in \mathbf{A}$, half-open intervals are formed, $S_{ij} \equiv (C_i <_0 A \precsim_0 C_j)$ for all i, j such that $i < j$, where $\precsim_0$ is an ordering between the event A and the events C_i, $C_j \in \mathbf{C}$. Consider for each $A \in \mathbf{A}$ the triple $(\mathbf{S}, \precsim_{\mathbf{S}}, \square)$, where $\mathbf{S}$ is a collection of S_{ij}, for all i and j, $\precsim_{\mathbf{S}}$ is a total order to be read as "not more confident that" and $\square$ is a concatenation operation. Plausible axioms about the triple lead to a distribution of probabilities $\mathbb{P}_A$ for each $A \in \mathbf{A}$.

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ROGER A. BLAU[1], RICHARD H. SHACHTMAN[2], THOMAS S. WALLSTEN[3]

1. Introduction and Background

Consider a decision maker (DM) at a race track trying to decide whether horse A_1, A_2 or A_3 is more likely to win. He may have sufficient knowledge to judge that A_2 is at least as likely to win as A_1 ($A_1 \lesssim_A A_2$), but know little or nothing about A_3. Therefore he is unable to decide conclusively that either $A_3 \lesssim_A A_i$ or $A_i \lesssim_A A_3$, i = 1, 2. One approach to providing a solution for the DM's dilemma is to invoke the "principle of insufficient reason", Savage [5], which instructs the DM to behave as if $A_i \sim_A A_3$. This would entail, however, claiming that $A_1 \sim_A A_2$, which the DM may not be willing to do. We are forced to conclude that the set $\{A_1, A_2, A_3\}$ is partially ordered, and therefore not capable of being represented by a unique probability measure.

All current axiom systems sufficient to insure a probability representation over an algebra of sets (e.g., Chapter 6 in DeGroot [1], Fine [2], Luce [4]). Assume that the ordering between pairs of events is total, i.e., reflexive, connected and transitive. Generally, each system also has four additional axioms concerning properties of the algebra, which in conjunction with the first condition yield the required measure.

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[1]Graduate School of Business Administration and Curriculum in Operations Research and Systems Analysis.

[2]Department of Biostatistics and Curriculum in Operations Research and Systems Analysis.

[3]L. L. Thurstone Psychometric Laboratory

The above example may be taken as prototypical of many empirical or practical contexts in which a subjective probability representation is desired, but cannot be constructed, because the events are partially, but not totally ordered. Our goal is to develop a set of axioms sufficient to yield a numerical representation of $(\mathbf{A}, \lesssim_{\mathbf{A}})$, where $\mathbf{A}$ is an algebra of sets and $\lesssim_{\mathbf{A}}$ is a partial ordering on $\mathbf{A}$. That is, $\lesssim_{\mathbf{A}}$ is reflexive and transitive, but not connected. In addition, we want this representation to be of such a form that a class of probability measures can be derived, each element of which is consistent with the partial ordering.

Our solution to the problem of numerically representing $(\mathbf{A}, \lesssim_{\mathbf{A}})$ is to be achieved in two stages. The first step consists of exhibiting a set of axioms sufficient to insure the existence of a probability distribution over a probability space defined specifically for <u>each</u> $A \in \mathbf{A}$. Hence, we end up with a set of probability distributions indexed over $\mathbf{A}$, $\{\mathbb{P}_A: \; A \in \mathbf{A}\}$. The second stage of the work consists of developing rational techniques for integrating the collection of probability distributions $\{\mathbb{P}_A: \; A \in \mathbf{A}\}$ into specific probability distributions over $\mathbf{A}$. The present paper is concerned with the first stage.

To obtain the desired individual representation for any $A \in \mathbf{A}$, we need a set of axioms that are somewhat different in nature from those normally employed, but are empirically reasonable nevertheless. First of all, we will incorporate into the notion of a partial ordering the idea, alluded to in the opening example, of differing levels of knowledge about events. Given sufficient knowledge about a pair of events A_1 and A_2, the DM knows definitely that either $A_1 \lesssim_{\mathbf{A}} A_2$ or $A_2 \lesssim_{\mathbf{A}} A_1$. Given no knowledge about A_1 or A_2, the DM cannot judge at all whether $A_1 \lesssim_{\mathbf{A}} A_2$ or $A_2 \lesssim_{\mathbf{A}} A_1$. However, given at least some knowledge about A_1 and A_2, the DM will have more confidence either in the statement $A_1 \lesssim_{\mathbf{A}} A_2$ or in the statement $A_2 \lesssim_{\mathbf{A}} A_1$.

Consequently, rather than assuming either that events are comparable or that they are not, we postulate a totally ordered continuum of confidence along which statements of the form $A_1 \lesssim_{\mathbf{A}} A_2$ are ordered. Conceptually, then, a pair of events from $\mathbf{A}$ are ordered with respect to $\lesssim_{\mathbf{A}}$ if the DM has "complete confidence" in one of the two possible orders. They are unordered if he has less than complete confidence in either of the statements.

In addition, we will assume the existence of a random variable which has a uniform distribution on the interval [0,1]. It should be noted that the existence of such a random variable can be proven given a set of purely qualitative axioms. This has been done by Villegas [6]. Furthermore, this assumption is identical to SP_5 used by DeGroot [1] in his development of a subjective probability representation.

However, our use of this axiom differs somewhat from DeGroot's in a manner now to be specified. The existence of the uniform random variable implies the existence of a totally ordered class of events $(\mathbf{C}, \lesssim_{\mathbf{C}})$. DeGroot assumes that for every $A \in \mathbf{A}$, there exists a $C \in \mathbf{C}$ such that $A \sim_o C$. We will assume, instead, that for every $A \in \mathbf{A}$ and all $C_i, C_j, C_k, C_\ell \in \mathbf{C}$, such that $C_i <_{\mathbf{C}} C_j$ and $C_k <_{\mathbf{C}} C_\ell$, the half-open intervals, $C_i <_o A \lesssim_o C_j$ and $C_k <_o A \lesssim_o C_\ell$ can be formed. Making use of the confidence property assumed for the ordering of pairs of events from $\mathbf{A}$, it is reasonable to assume that the DM has more confidence either in the statement $C_i <_o A \lesssim_o C_j$ or in the statement $C_k <_o A \lesssim_o C_\ell$. In other words, the DM is uncertain of the likelihood ordering of A with respect to the other events in $\mathbf{A}$, but for example, is more confident that A falls into the interval $(C_i, C_j]$ than into the interval $(C_k, C_\ell]$.

The notions outlined above will be developed formally in the next section. For each $A \in \mathbf{A}$, the set of all half-open intervals $\{(C_i, C_j]\}$, together with a confidence ordering on the set, and a concatenation operation different from the union operation, will lead to what we call a comparison algebra. Two structural axioms, in addition to the two assumptions discussed above, will be introduced. The four axioms are then used to prove a variety of properties about the system.

The representation theorem is proved in Section 3. We show that the algebra, with the properties developed in Section 2, is an open, extensive structure, as defined by Krantz, Luce, Suppes, and Tversky [3]. Therefore, the representation proved by them applies to our structure. It is then a small step to show that the representation is, in fact, a unique, finitely additive probability measure.

The probability measure, it should be kept in mind, is not over $\mathbf{A}$, but exists independently for each $A \in \mathbf{A}$. The final section notes the remaining task for obtaining a class of probability measures consistent with the original partial ordering.

2. The Structural Development of a Comparison Algebra

In order to establish a representation for $A \in \mathbf{A}$, a comparison structure is required, similar to that developed from qualitative considerations by Villegas [6] and employed by DeGroot [1]. We therefore state:

<u>Axiom I</u>: <u>There exists a uniform random variable</u>.

The existence of a uniform random variable implies the existence of a totally ordered collection of events $(\mathbf{C}, \lesssim_{\mathbf{C}})$ distinct from $(\mathbf{A}, \lesssim_{\mathbf{A}})$. Hence for any collection $\{C_j\}_{j=1}^{n} \subset \mathbf{C}$ and the binary relation $\lesssim_{\mathbf{C}}$, there exists a permutation

$(m_1,\ldots,m_n)$ of $(1,\ldots,n)$ such that

$$C_{m_1} \lesssim_{\mathbf{C}} \cdots \lesssim_{\mathbf{C}} C_{m_n}.$$

The connectedness of the ordering allows us to index the collection $\mathbf{C}$ with a closed interval I in the real line $\mathbb{R}$; i.e., for $I \subset \mathbb{R}$

$$\mathbf{C} = \{C_\alpha: \ \alpha \in I\}.$$

Since we have assumed that $\lesssim_{\mathbf{A}}$ is a partial order, it makes little sense to compare an $A \in \mathbf{A}$ directly with a $C_j \in \mathbf{C}$, as was done in [1] and [6]. Instead we will compose the events from $\mathbf{A}$ and $\mathbf{C}$ into new events which are totally ordered as follows.

<u>Definition 2.1</u>: Let $A \in \mathbf{A}$ and C_i, $C_j \in \mathbf{C}$ such that $C_i \lesssim_{\mathbf{C}} C_j$, $i < j$; $i,j \in I$. We define the half-open intervals

$$S_{ij} \equiv (C_i <_o A \lesssim_o C_j)$$

where $<_o$ and $\lesssim_o$ are a pair of likelihood orderings between any ordered pair C_i, $C_j \in \mathbf{C}$ and any $A \in \mathbf{A}$.

<u>Definition 2.2</u>: $S(A) = \{S_{ij}: \ C_i, C_j \in \mathbf{C},\ i < j;\ i,j \in I\}$.

<u>Definition 2.3</u>: Let $\lesssim_{S(A)}$ be an ordering on $S(A)$.

<u>Axiom II</u>: <u>$(S(A), \lesssim_{S(A)})$ is totally ordered.</u>

Note that $S(A)$ is a collection of sets, each of which is itself a triple. Each set, S_{ij}, represents a likelihood comparison between A, which is only partially ordered in $\mathbf{A}$, and each of C_i and C_j, where $C_i \lesssim_{\mathbf{C}} C_j$. We desire a measure of the likelihood that A is, in fact, in the interval $(C_i, C_j]$. Thus, the triple S_{ij} is ordered by $\lesssim_{S(A)}$, which represents the confidence ordering on half-open intervals discussed in Section 1. The measure obtained for S(A) from the ordering and the axioms will induce a probability distribution over the interval [0,1] for a given A.

When there is no confusion, the dependence on A will be suppressed.

One motivation for Definitions 2.1, 2.2, and 2.3 arises from imagining that there is mass associated with the events S_{ij}. This can be illustrated geometrically by the following: let $\{C_k\colon\ k \in I = [a, b]\}$ correspond to an interval on $\mathbb{R}$ and let $i,j \in I$ with $i < j$.

[Figure 1 goes here]

The shaded area beneath the curve in Figure 1 represents the density associated with the event $S_{ij} \equiv (C_i <_0 A \lesssim_0 C_j)$. The ordering for the S_{ij}'s in conjunction with our axioms will determine a measure like the one indicated by the shaded area. In this sense, a particular ordering on **S** determines a particular curve for a given A.

The equivalence and strict ordering relations for all the orders $\lesssim_{\mathbf{A}}$, $\lesssim_{\mathbf{C}}$, $\lesssim_0$ for **S** and $\lesssim_{\mathbf{S}}$ are defined in the usual way and denoted by ~ and < with the appropriate subscripts where necessary.

For unions of the events in **S** to be well-defined, and to facilitate subsequent developments, it is useful to construct an element definition of the sets $S_{ij} \in \mathbf{S}$.

<u>Definition 2.4</u>: For $S_{ij} \equiv (C_i <_0 A \lesssim_0 C_j)$, where $C_i, C_j \in \mathbf{C}$, $i < j$; $i,j \in I$ and $A \in \mathbf{A}$, the event S_{ij} is composed of a collection of elementary events $\{(C_\alpha, A_{C_\alpha})\}$, $\alpha \in I$, $i < \alpha \leq j$.

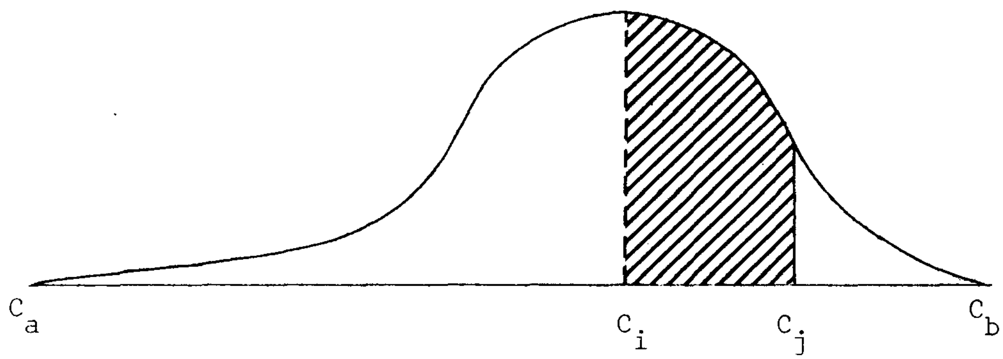

Figure 1: Hypothetical density over $\mathcal{S}(A)$

Therefore $S_{ij} = (C_i <_o A \lesssim_o C_j) = \bigcup_{i < \alpha \leq j} \{(C_\alpha, A_{C_\alpha})\}$. When there is no confusion with other A's in $\mathbf{A}$, we use A_α instead of A_{C_α}.

Definition 2.4 can be understood geometrically by imagining the element (C_α, A_α) to represent the line segment originating at the point C_α on the abscissa and extending vertically to the curve as indicated in Figure 2.

[Figure 2 goes here]

Definition 2.4 provides a basis for constructing the union operation. For $S_{ij}, S_{k\ell} \in \mathbf{S}$,

$$(2.1) \qquad S_{ij} \cup S_{k\ell} = \{(C_\alpha, A_\alpha): \; i < \alpha \leq j\} \cup \{(C_\beta, A_\beta): \; k < \beta \leq \ell\}$$

$$= \{(C_\gamma, A_\gamma): \; \gamma \in (i, j] \cup (k, \ell]\}.$$

As presently defined, the union operation on $\mathbf{S}$ is not closed. For example, in equation (2.1), if $j < k$ or $\ell < i$, then the right hand side is not in $\mathbf{S}$; i.e., is not a half-open interval.

Hence we will define a more general binary operation that is closed on $\mathbf{S}$. Furthermore this operation will allow constructions sufficiently strong to satisfy axioms needed in Section 3.

Prior to introducing this binary operation, however, further structure must be imposed on $\mathbf{S}$ in the form of an additional axiom. We will assume an "atomless" property for $\mathbf{S}$. That is, for the example illustrated in Figure 1, the curve generated by the binary orderings $\lesssim_o$ and $\lesssim_S$ is the density function of an absolutely continuous distribution function. This means that for the likelihood ordering in $\mathbf{S}$, A is never associated with any particular C_j with a positive mass.

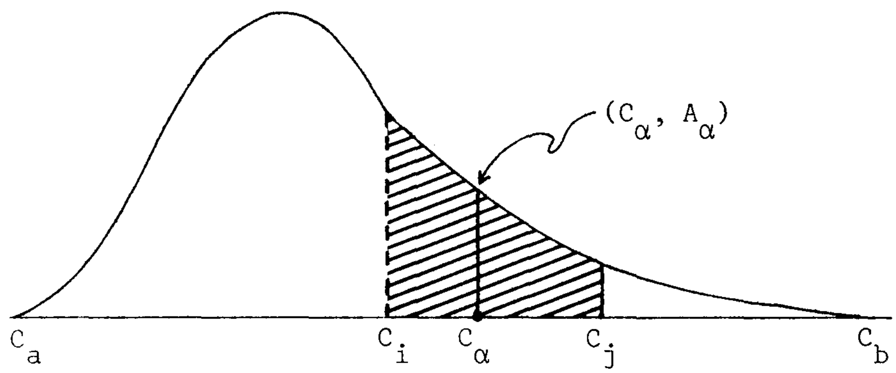

Figure 2: Illustration of element definition of S_{ij}

Let $S_\emptyset$ be the null set for **S**; i.e., $S_\emptyset \cap S_{ij} = S_\emptyset$ for all $i,j \in I$, $i < j$.

<u>Definition 2.5</u>: S_{ij} is an atom of $(\mathbf{S}, \precsim_{\mathbf{S}})$ if for any $S_{k\ell} \in \mathbf{S}$ with $S_{k\ell} \precsim_{\mathbf{S}} S_{ij}$, either $S_{k\ell} \sim_{\mathbf{S}} S_\emptyset$ or $S_{k\ell} \sim_{\mathbf{S}} S_{ij}$.

<u>Axiom III</u>: <u>**S** contains no atoms</u>.

Henceforth, Axioms I, II and III will be assumed.

<u>Theorem 2.6</u>: For all S_{ij} and $S_{k\ell} \in \mathbf{S}$, there exists $S_{hm} \in \mathbf{S}$ such that

$$(S_{ij} \cup S_{k\ell}) \sim_{\mathbf{S}} S_{hm}.$$

<u>Proof</u>: We will show that there exists a $C_m \in \mathbf{C}$ such that $(S_{ij} \cup S_{k\ell}) \sim_{\mathbf{S}} (C_{i \wedge k} <_o A \precsim_o C_m)$ where $i \wedge k \equiv \min(i, k)$ and the right hand side is equal to $\{(C_\gamma, A_\gamma)\colon\ i \wedge k < \gamma \leq m\}$.

Let $\delta^* = \sup\{\delta \in I\colon\ T_\delta \precsim_{\mathbf{S}} (S_{ij} \cup S_{k\ell})\}$ where $T_\delta \equiv (C_{i \wedge k} <_o A \precsim_o C_\delta)$. The set of δ's is non-empty and the supremum exists. Let $\zeta^* = \inf\{\zeta \in I\colon\ S_{ij} \cup S_{k\ell} \precsim_{\mathbf{S}} T_\zeta\}$; then $\delta^* \leq \zeta^*$.

Suppose $\delta^* < \zeta^*$; then there exists $\theta \in (\delta^*, \zeta^*)$ and by Axioms I and III, there exists $C_\theta \in \mathbf{C}$ and $T_\theta \in \mathbf{S}$ such that

$$(S_{ij} \cup S_{k\ell}) <_{\mathbf{S}} T_\theta <_{\mathbf{S}} (S_{ij} \cup S_{k\ell})$$

since $\delta^* < \theta < \zeta^*$. Contradiction. Hence, $\delta^* = \zeta^*$ and $T_{\delta^*} \sim_{\mathbf{S}} (S_{ij} \cup S_{k\ell})$. We finish by selecting $m = \delta^*$ and $h = i \wedge k$. ||

<u>Corollary 2.7</u>: Let $S_{ij} \in \mathbf{S}$.

(i) For all $a \in I$ such that $a \leq i$, there exists $b \in I$ with $b \leq j$ such that $S_{ab} \sim_{\mathbf{S}} S_{ij}$.

(ii) For all $b \in I$ such that $b \geq j$, there exists $a \in I$ with $a \geq i$ such that $S_{ab} \sim_{\mathbf{S}} S_{ij}$.

The more general binary operation can be introduced now.

<u>Definition 2.8</u>: For any S_{ij}, $S_{k\ell} \in \mathbf{S}$ we define a binary operation, $\square$, on $\mathbf{S}$ by

$$S_{ij} \square S_{k\ell} = S_{hm}$$

where S_{hm} is given by Theorem 2.6.

We note that S_{hm} is not necessarily a uniquely determined set. However, we can, by convention, uniquely determine the result of the composition, $\square$, by using the h and m defined in the proof of Theorem 2.6. It should also be pointed out that $\square$ is commutative, as evidenced by the proof of Theorem 2.6.

<u>Corollary 2.9</u>: If $(i, j) \cap (k, \ell) \neq \emptyset$, then $S_{ij} \square S_{k\ell} = S_{ij} \cup S_{k\ell} = S_{i \wedge k,\, j \vee \ell}$, where $j \vee \ell \equiv \max(j, \ell)$.

<u>Lemma 2.10</u>: $S_{ij} \square S_{\emptyset} = S_{ij}$, for all $S_{ij} \in \mathbf{S}$.

We will construct the universal set, S_{Ω}, $\Omega = \Omega(A)$, where

$$S_{ij} \square S_{\Omega} = S_{\Omega}$$

for all $S_{ij} \in \mathbf{S}$.

Let $\gamma \equiv \inf\{\alpha \in I\colon\ C_{\alpha} \in \mathbf{C}\}$. γ exists and is in I. We can then define $\tau \equiv \inf\{\beta \in I\colon\ S_{ij} \lesssim_{\mathbf{S}} (C_{\gamma} <_{o} A \lesssim_{o} C_{\beta})$, for all $S_{ij} \in \mathbf{S}\}$, which exists and is in I, where $\tau = \tau(A)$.

<u>Lemma 2.11</u>: $S_{ij} \lesssim_{\mathbf{S}} S_{\gamma\tau}$, for all $S_{ij} \in \mathbf{S}$.

There exists $\sigma \equiv \sup\{\eta \in I\text{: } S_{ij} \lesssim_{\mathbf{S}} (C_\eta <_o A \lesssim_o C_\tau)$, for all $S_{ij} \in \mathbf{S}\} \in I$, with $\sigma = \sigma(A)$. By Axiom III:

<u>Theorem 2.12</u>:

(i) $\sigma < \tau$.

(ii) there exists $S_{\sigma\tau} \in \mathbf{S}$.

(iii) $S_\emptyset \lesssim_{\mathbf{S}} S_{ij} \lesssim_{\mathbf{S}} S_{\sigma\tau}$, for all $S_{ij} \in \mathbf{S}$.

(iv) $S_\emptyset <_{\mathbf{S}} S_{\sigma\tau}$.

<u>Corollary 2.13</u>: For all $S_{ij} \in \mathbf{S}$,

(i) $S_{ij} \square S_{\sigma\tau} = S_{\sigma\tau}$.

(ii) $S_{ij} \cup S_{\sigma\tau} = S_{\sigma\tau}$.

<u>Corollary 2.14</u>: $S_\Omega \equiv S_{\sigma\tau}$ is unique and σ, τ are independent of the order of computation.

Note that $\sim_{\mathbf{S}}$ is an equivalence relation on $\mathbf{S}$ and we may construct sets which are equivalent $\sim_{\mathbf{S}}$ to any S_{ij} using $\square$.

Neither $(\mathbf{S}, \square)$ nor $(\mathbf{S}, \cup)$ is an algebra: the former does not contain complements and the latter does not contain finite disjoint unions. Proceeding to a collection of finite disjoint unions would yield an algebra, but is not as advantageous to us as working with a collection of equivalence classes, $[\mathbf{S}]$. The latter is introduced in the next section with an operation, $\square^*$, defined on the equivalence classes, yielding the algebra $([S], \square^*)$.

To complete our system, we introduce a fundamental axiom.

<u>Axiom IV</u>: <u>If $S_{ij} \subset S_{k\ell}$, then $S_{ij} \lesssim_{\mathbf{S}} S_{k\ell}$.</u>

For our purposes, there is no advantage in using the convention of having other axioms imply the above likelihood property.

3. A Probability Representation

It is convenient to collect and restate the axioms introduced in Section 2, which will be assumed in this section.

<u>Axiom I</u>: There exists a uniformly distributed random variable (and thus a totally ordered class of events $(\mathbf{C}, \lesssim_{\mathbf{C}})$).

<u>Axiom II</u>: $(\mathbf{S}, \lesssim_{\mathbf{S}})$ is totally ordered, where

$\mathbf{S} \equiv \{S_{ij}\colon \; C_i, C_j \in \mathbf{C};\; i < j;\; i, j \in I\}$ and

$S_{ij} \equiv (C_i <_o A \lesssim_o C_j)$.

<u>Axiom III</u>: $\mathbf{S}$ contains no atoms.

<u>Axiom IV</u>: If $S_{ij} \subset S_{k\ell}$, then $S_{ij} \lesssim_{\mathbf{S}} S_{k\ell}$.

In as much as the events $A \in \mathbf{A}$ are not totally ordered with respect to their likelihood, there does not exist a unique probability measure over these events. However, we shall show that under our assumptions, for each $A \in \mathbf{A}$ there exists a unique probability measure over the events in $\mathbf{S}(A)$.

<u>Theorem 3.1</u>: For each $A \in \mathbf{A}$, there exists a unique order-preserving function $\mathbb{P}_A$ from $\mathbf{S}(A)$ into the unit interval [0,1] such that $(S_{\Omega(A)}, \mathbf{S}(A), \mathbb{P}_A)$ is a finitely additive probability space.

In order to prove Theorem 3.1, it is first necessary to construct an ordered algebra based on $\mathbf{S}(A)$. Suppressing A in the notation, let $[S_{ij}]$ denote the equivalence class defined by the equivalence relation $\sim_{\mathbf{S}}$ that includes S_{ij}, and let $[\mathbf{S}]$ be the set of all equivalence classes, excluding $[S_\emptyset]$.

<u>Definition 3.2</u>: The operation $\square^*$ is defined on $[\mathbf{S}] \times [\mathbf{S}]$ as

$$[S_{ij}] \boxdot^* [S_{k\ell}] = [S_{mn}],$$

where $S_{i'j'} \boxdot S_{k'\ell'} = S_{m'n'}$, for some $S_{i'j'} \in [S_{ij}]$, $S_{k'\ell'} \in [S_{k\ell}]$ and $S_{m'n'} \in [S_{mn}]$, such that $S_{i'j'} \cap S_{k'\ell'} = S_{\emptyset}$.

Using Definition 2.8 and Corollary 2.7, the following lemma can be proved:

Lemma 3.3: ($[\mathbf{S}]$, $\boxdot^*$) is an algebra.

Consider, $[\mathbf{T}] = \{([S_{ij}], [S_{k\ell}])$: $S_{\emptyset} <_{\mathbf{S}} S_{ij}$, $S_{\emptyset} <_{\mathbf{S}} S_{k\ell}$ and there exists $S_{i'j'} \in [S_{ij}]$, $S_{k'\ell'} \in [S_{k\ell}]$ with $S_{i'j'} \cap S_{k'\ell'} = S_{\emptyset}\}$. We now show that $[\mathbf{S}]$ and $[\mathbf{T}]$ are non-empty. $S_{\Omega} = S_{\sigma\tau}$, and Axiom III assures us that $\sigma < \tau$. Choose any $\theta \in (\sigma,\tau)$. By Axioms I and III, $C_{\theta} \in \mathbf{C}$; $S_{\sigma\theta}$, $S_{\theta\tau} \in \mathbf{S}$. Thus $[S_{\sigma\theta}] \in [\mathbf{S}]$ and $([S_{\sigma\theta}], [S_{\theta\tau}]) \in [\mathbf{T}]$.

If we let $\lesssim^*$ be the induced total order on $[\mathbf{S}]$, the proof of Theorem 3.1 essentially involves demonstrating that the six axioms stated in Section 3.4 of Krantz et al. [3] for an open extensive structure with no essential maximum are satisfied by the quadruple ($[\mathbf{S}]$, $\lesssim^*$, $[\mathbf{T}]$, $\boxdot^*$). We then invoke Theorem 3.3, [3], which proves the existence of a function ψ: $[\mathbf{S}] \to \mathbb{R}^+$ such that for all $[S_{ij}]$, $[S_{k\ell}] \in [\mathbf{S}]$,

(i) $[S_{k\ell}] \lesssim^* [S_{ij}]$ iff $\psi([S_{k\ell}]) \leq \psi([S_{ij}])$;

(ii) If $([S_{ij}], [S_{k\ell}]) \in [\mathbf{T}]$ then $\psi([S_{ij}] \boxdot^* [S_{k\ell}]) = \psi([S_{ij}]) + \psi([S_{k\ell}])$.

Furthermore, if another function ψ' satisfies (i) and (ii), then there exists $\alpha > 0$ such that, for all nonmaximal $[S_{ij}] \in [\mathbf{S}]$, $\psi'([S_{ij}]) = \alpha\psi([S_{ij}])$.

The method of proving Theorem 3.1 by introducing equivalence classes and demonstrating that the resulting system is an open extensive structure with no essential maximum is similar to that employed in the proof of Theorem 5.2 in [3].

Given a set of five axioms, they prove the existence of a unique, order preserving probability measure for a set of totally ordered events in a specified algebra. However, although the general approach for the present system and for their system is similar, the system axioms, and thus the specifics of the proofs differ considerably.

We now restate from [3] the six axioms for an open extensive structure with no essential maximum, in notation appropriate to the present system. For all $[S_{ij}]$, $[S_{k\ell}]$, $[S_{mn}] \in [\mathbf{S}]$,

1. $([\mathbf{S}], \lesssim^*)$ is a total order.

2. (Associativity). If $([S_{ij}], [S_{k\ell}]) \in [\mathbf{T}]$ and $([S_{ij}] \square^* [S_{k\ell}], [S_{mn}]) \in [\mathbf{T}]$, then $([S_{k\ell}], [S_{mn}]) \in [\mathbf{T}]$, $([S_{ij}], [S_{k\ell}] \square^* [S_{mn}]) \in [\mathbf{T}]$, and $[S_{ij}] \square^* ([S_{k\ell}] \square^* [S_{mn}]) \lesssim^* ([S_{ij}] \square^* [S_{k\ell}]) \square^* [S_{mn}]$.

3. (Monotonicity and commutativity). If $([S_{ij}], [S_{mn}]) \in [\mathbf{T}]$ and $[S_{k\ell}] \lesssim^* [S_{ij}]$, then $([S_{mn}], [S_{k\ell}]) \in [\mathbf{T}]$ and $[S_{mn}] \square^* [S_{k\ell}] \lesssim^* [S_{ij}] \square^* [S_{mn}]$.

4. (Solvability). If $[S_{k\ell}] <^* [S_{ij}]$, then there exists $[S_{pq}] \in [\mathbf{S}]$ such that $([S_{k\ell}], [S_{pq}]) \in [\mathbf{T}]$ and $[S_{k\ell}] \square^* [S_{pq}] \lesssim^* [S_{ij}]$.

5. (Positivity). If $([S_{ij}], [S_{k\ell}]) \in [\mathbf{T}]$, then $[S_{ij}] <^* [S_{ij}] \square^* [S_{k\ell}]$.

6. (Archimedean). Every strictly bounded standard sequence is finite, where $[S_{i_1,j_1}], \ldots, [S_{i_n,j_n}]$ is a standard sequence if $[S_{i_n,j_n}] = [S_{i_{n-1},j_{n-1}}] \square^* [S_{i_1,j_1}]$, for $n \geq 2$, and it is strictly bounded if for some $[S_{k\ell}] \in [\mathbf{S}]$ and for all $[S_{i_n,j_n}]$ in the sequence $[S_{i_n,j_n}] <^* [S_{k\ell}]$.

The following eight lemmas demonstrate that Axioms 1-6 for an open extensive structure are satisfied by $([S], \lesssim^*, [T], \square^*)$.

Lemma 3.4: $([S], \lesssim^*)$ is a total order. (Axiom 1 obtains.)

A stronger version of Axiom 2 can be shown, where associativity holds as an equivalence relationship.

Lemma 3.5: If $([S_{ij}], [S_{k\ell}]) \in [T]$ and $([S_{ij}] \square^* [S_{k\ell}], [S_{mn}]) \in [T]$, then $([S_{k\ell}], [S_{mn}]) \in [T]$, $([S_{ij}], [S_{k\ell}] \square^* [S_{mn}]) \in [T]$ and $[S_{ij}] \square^* ([S_{k\ell}] \square^* [S_{mn}]) \sim^* ([S_{ij}] \square^* [S_{k\ell}]) \square^* [S_{mn}]$. (Axiom 2 obtains.)

Proof. By assumption, there exists $S_{i'j'} \in [S_{ij}]$, $S_{k'\ell'} \in [S_{k\ell}]$, and $S_{m'n'} \in [S_{mn}]$ such that $S_{i'j'} \cap S_{k'\ell'} = S_{\emptyset}$. Also $S_{ac} \sim_S S_{i'j'} \square S_{k'\ell'}$, and $S_{ac} \cap S_{m'n'} = S_{\emptyset}$. Suppose $i' \leq k' \leq m'$. By Theorem 2.6 and the convention for the operation $\square$, $S_{ac} = S_{i'c} = S_{ab} \square S_{bc}$ where $S_{ab} \in [S_{ij}]$, and $S_{bc} \in [S_{k\ell}]$. Consider $S_{ac} \square S_{m'n'} = S_{ac} \square S_{cd} = S_{ad}$, where $S_{cd} \in [S_{mn}]$. Since $S_{ab} \cap S_{bc} \cap S_{cd} = S_{\emptyset}$, it follows that $([S_{k\ell}], [S_{mn}]) \in [T]$ and $([S_{ij}], [S_{k\ell}] \square^* [S_{mn}]) \in [T]$. Furthermore, $([S_{ij}] \square^* [S_{k\ell}]) \square^* [S_{mn}] \sim^* [S_{ad}] \sim^* [S_{ij}] \square^* ([S_{k\ell}] \square^* [S_{mn}])$. The result is shown in a similar manner when the other possible orders of i', k', and m' are considered. ||

Lemma 3.6: If $([S_{ij}], [S_{mn}]) \in [T]$ and $[S_{k\ell}] \lesssim^* [S_{ij}]$, then $([S_{mn}], [S_{k\ell}]) \in [T]$ and $[S_{mn}] \square^* [S_{k\ell}] \lesssim^* [S_{ij}] \square^* [S_{mn}]$. (Axiom 3 obtains.)

Proof. Select $S_{i'j'} \in [S_{ij}]$, $S_{m'n'} \in [S_{mn}]$, where $S_{i'j'} \cap S_{m'n'} = S_{\emptyset}$. Suppose $j' < m'$. If $S_{k'\ell'} \in [S_{k\ell}]$, and $j' < k'$, by Corollary 2.7, there exists p such that $S_{k'\ell'} \sim_S S_{i'p}$. By Axiom IV, $p \leq j'$. Again by Corollary 2.7, there exist r and s such that $S_{m'n'} \sim_S S_{j'r} \sim_S S_{ps}$ and by Axiom IV, $s \leq r$. Since

$S_{i'p} \cap S_{ps} = S_{\emptyset}$, $([S_{mn}], [S_{k\ell}]) \in \mathbf{T}$. Furthermore, $S_{k'\ell'} \square S_{m'n'} = S_{m'n'} \square S_{k'\ell'} \sim_{\mathbf{S}} S_{i's} \lesssim_{\mathbf{S}} S_{i'r} \sim_{\mathbf{S}} S_{i'j'} \square S_{m'n'}$, by the commutativity of $\square$. Similar arguments can be made for other orderings of the primed subscripts. ||

A stronger version of Axiom 4 can be shown, where solvability holds as an equivalence relationship.

<u>Lemma 3.7</u>: If $[S_{k\ell}] <^* [S_{ij}]$, then there exists $[S_{pq}] \in [\mathbf{S}]$ such that $([S_{k\ell}], [S_{pq}]) \in [\mathbf{T}]$ and $[S_{k\ell}] \square^* [S_{pq}] \lesssim^* [S_{ij}]$. (Axiom 4 obtains.)

<u>Proof</u>: Suppose $S_{i'j'} \in [S_{ij}]$, $S_{k'\ell'} \in [S_{k\ell}]$ and $i' \leq k'$. By Corollary 2.7 there exists r such that $S_{k'\ell'} \sim_{\mathbf{S}} S_{i'r}$ and by Axiom IV, $r \leq j'$. Consider $S_{pq} = S_{rj'}$. Thus, $([S_{k\ell}], [S_{pq}]) \in [\mathbf{T}]$ and $S_{i'j'} = S_{i'r} \square S_{rj'}$, so that $[S_{ij}] \sim^* [S_{k\ell}] \square^* [S_{pq}]$. A similar argument can be used when $k' < i'$. ||

<u>Lemma 3.8</u>: If $([S_{ij}], [S_{k\ell}]) \in \mathbf{T}$ then $[S_{ij}] <^* [S_{ij}] \square^* [S_{k\ell}]$. (Axiom 5 obtains.)

<u>Proof</u>: Clearly $[S_{ij}] \lesssim^* [S_{ij}] \square^* [S_{k\ell}]$. Suppose $[S_{ij}] \square^* [S_{k\ell}] \sim^* [S_{ij}]$. Select $S_{i'j'} \in [S_{ij}]$, $S_{k'\ell'} \in [S_{k\ell}]$ such that $S_{i'j'} \cap S_{k'\ell'} = S_{\emptyset}$. Then $S_{k'\ell'}$ must be equivalent to $S_{\emptyset}$. However in that case, $[S_{k\ell}] \notin [\mathbf{S}]$, and thus $([S_{ij}], [S_{k\ell}]) \notin [\mathbf{T}]$. Contradiction. ||

<u>Lemma 3.9</u>: For all $n = 2^m$ and all $S_{ij} \in \mathbf{S}$, where $S_{\emptyset} <_{\mathbf{S}} S_{ij}$, there exists a partition (mutually exclusive and exhaustive), $\{T_{ij}^{(\alpha)}\}_{\alpha=1}^{n}$ of S_{ij}, such that $T_{ij}^{(\alpha)} \sim_{\mathbf{S}} T_{ij}^{(\beta)}$ for all $\alpha, \beta = 1, 2, \ldots, n$.

<u>Proof</u>. Suppose $m = 1$ $(n = 2)$ and let $\gamma \equiv \sup\{\alpha \in I\colon\ S_{i\alpha} \lesssim_{\mathbf{S}} S_{\alpha j}\}$ and $\delta = \inf\{\beta \in I\colon\ S_{\beta j} \lesssim_{\mathbf{S}} S_{i\beta}\}$. Clearly, $\gamma \leq \delta$. We claim that $\gamma = \delta$, for suppose that $\gamma < \delta$. Then there exists $\theta \in (\gamma, \delta)$. By Axioms I and III, there exists $S_{i\theta} \in \mathbf{S}$ with $\theta > \gamma$, implying that $S_{\theta j} <_{\mathbf{S}} S_{i\theta}$, and $\theta < \delta$, implying that $S_{i\theta} <_{\mathbf{S}} S_{\theta j}$, which is a contradiction.

Thus, $T_{ij}^{(1)} = S_{i\gamma} \sim_S S_{\gamma j} = T_{ij}^{(2)}$. Suppose the lemma is true for $n = 2^m$, i.e., there exists a partition $\{T_{ij}^{(\alpha)}\}_{\alpha=1}^{2^m}$ of S_{ij} such that $T_{ij}^{(\alpha)} \sim_S T_{ij}^{(\beta)}$ for all $\alpha, \beta, = 1, 2, \ldots, 2^m$. The lemma can be shown to be true for 2^{m+1} by considering the case $m = 1$ to subdivide each $T_{ij}^{(\alpha)}$, $\alpha = 1, \ldots, 2^m$, into two equally likely sets. ||

__Lemma 3.10__: For all $S_{ij}, S_{k\ell} \in \mathbf{S}$, where $S_\emptyset <_S S_{ij}$, $S_\emptyset <_S S_{k\ell}$, there exists an n and a partition of S_{ij}, $\{T_{ij}^{(\alpha)}\}_{\alpha=1}^{n}$ such that $T_{ij}^{(\alpha)} \lesssim_S S_{k\ell}$, $\alpha = 1, \ldots, n$.

__Proof__: Assume $i \leq k$. By Corollary 2.7, there exists a p such that $S_{k\ell} \sim_S S_{ip}$. From Lemma 3.9, for all $n = 2^m$, there exists a partition of S_{ij}, $\{T_{ij}^{(\alpha)}\}_{\alpha=1}^{n}$, such that $T_{ij}^{(\alpha)} \sim_S T_{ij}^{(\beta)}$ for all $\alpha, \beta = 1, 2, \ldots, n$. Thus associated with each $n = 2^m$ is a partition whose first element we denote by $T_{i,j(n)}$. Either there exists an $n = 2^m$ which yields a partition where $T_{i,j(n)} \lesssim_S S_{ip}$ or not. Suppose it is not true, i.e., for all $n = 2^m$,

$$S_{ip} <_S T_{i,j(n)}. \tag{3.1}$$

Note that as m ↑, j(n) ↓ and since by Axiom IV, $p < j(n)$, the limit of j(n), say q, is in I and is such that $p \leq q$. Therefore there exists $S_{ip} \lesssim_S T_{iq}$. If $p = q$, then $T_{iq} = S_{ip}$, which contradicts (3.1). Suppose $p < q$. By Lemma 3.8, T_{iq} can be partitioned into $S_{iq'}$ and $S_{q'q}$, $i < q' < q$, where $S_{iq'} \sim_S S_{q'q}$. If $p < q'$, then this contradicts the fact that q is the limit of j(n). On the other hand, if $q' \leq p$, a partition of S_{ij} has been found such that $T_{ij}^{(1)} = S_{iq'} \lesssim_S S_{ip}$, contradicting (3.1). The proof for $k < i$ is similar. ||

__Lemma 3.11__: Every strictly bounded standard sequence is finite, where a standard sequence is defined in the statement of Axiom 6. (Axiom 6 obtains.)

<u>Proof:</u> Either every strictly bounded standard sequence is finite or there exists at least one which is infinite. Assume the latter case; i.e., suppose $\{[S_{i_r,j_r}]\}_{r=1}^{\infty}$ is strictly bounded, i.e., there is some $[S_{k\ell}] \in [S]$ such that $[S_{i_r,j_r}] <^* [S_{k\ell}]$ for all r. Select $S_{k',\ell'} \in [S_{k\ell}]$, $S_{i_1',j_1'} \in [S_{i_1,j_1}]$. By Lemma 3.10, there exists an n such that $\{T_{k',\ell'}^{(\alpha)}\}_{\alpha=1}^{n}$ is a partition of $S_{k',\ell'}$ and $T_{k',\ell'}^{(\alpha)} \lesssim_S S_{i_1',j_1'}$, $\alpha = 1, \ldots, n$. Thus, $[T_{k\ell}] \lesssim^* [S_{i_1,j_1}]$. Select $S_{i_r'',j_r''} \in [S_{i_r,j_r}]$, $r = 1, \ldots, n-1$, such that $S_{i_1',j_1'} \cap S_{i_r'',j_r''} = S_{\emptyset}$, $r = 1, \ldots, n-1$. Thus, by Lemma 3.6,

$$T_{k'\ell'}^{(1)} \,\square\, T_{k'\ell'}^{(2)} \lesssim_S T_{k'\ell'}^{(2)} \,\square\, S_{i_1',j_1'}$$

$$\lesssim_S S_{i_1'',j_1''} \,\square\, S_{i_1',j_1'}$$

$$\sim_S S_{i_2'',j_2''} .$$

By induction, we are led to the following contradiction:

$$S_{k'\ell'} = T_{k'\ell'}^{(1)} \,\square\, \ldots \,\square\, T_{k'\ell'}^{(n)} \lesssim_S S_{i_{n-1}'',j_{n-1}''} \,\square\, S_{i_1',j_1'} <_S S_{k'\ell'}.$$

Therefore $[S_{i_r,j_r}]$ is not strictly bounded for all r, i.e., there exists some m such that $[S_{k\ell}] \lesssim^* [S_{i_m,j_m}]$. By Lemma 3.7, $[S_{k\ell}] \lesssim^* [S_{i_r,j_r}]$ for all $r \geq m$. Hence, every strictly bounded standard sequence is finite. ||

We now proceed with the proof of Theorem 3.1. By Lemmas 3.4 through 3.8 and 3.11, $([S], \lesssim^*, [T], \square^*)$ is an extensive structure with no essential maximum. Thus by Theorem 3.3, in [3], there is a positive-valued ratio scale ψ on [S] such that $[S_{k\ell}] \lesssim^* [S_{ij}]$ iff $\psi([S_{k\ell}]) < \psi([S_{ij}])$ and for $([S_{ij}], [S_{k\ell}]) \in [T]$,

$$\psi([S_{ij}] \,\square^*\, [S_{k\ell}]) = \psi([S_{ij}]) + \psi([S_{k\ell}]).$$

Note that $\psi([S_\Omega]) \in \mathbb{R}^+$. For any $S_{ij} \in \mathbf{S}$, let

$$\mathbb{P}(S_{ij}) = \begin{cases} \dfrac{\psi([S_{ij}])}{\psi([S_\Omega])}, & \text{if} \quad S_\emptyset <_{\mathbf{S}} S_{ij} \\ 0 \quad , & \text{if} \quad S_\emptyset \sim_{\mathbf{S}} S_{ij} \; . \end{cases}$$

$\mathbb{P}(S_\Omega) = 1$ and if $[S_{ij}] <^* [S_\Omega]$, $0 \leq \mathbb{P}(S_{ij}) < 1$. $\mathbb{P}$ is a unique finitely additive probability representation, for if another function $\mathbb{P}'$ existed, then $\psi'([S_{ij}]) \equiv \mathbb{P}'(S_{ij})$ would be a representation of $([\mathbf{S}], \lesssim^*, [\mathbf{T}], \square^*)$. Since $\psi' = \alpha\psi$ and $\psi'([S_\Omega]) = \psi([S_\Omega]) = 1$, $\psi' = \psi$ and $\mathbb{P}' = \mathbb{P}$. ||

4. Subsequent Work: A measure on A

As we previously indicated, our approach to representing $(A, \lesssim_{\mathbf{A}})$ consists of two stages. The first step, contained in this paper, has been to derive an individual probability representation $\mathbb{P}_A$ for each $A \in \mathbf{A}$. An individual $\mathbb{P}_A$ does not assign a unique probability to A but reflects a relationship between A and the class $\mathbf{C}$; i.e., $\mathbb{P}_A$ is defined on $\mathbf{S}(A)$. If there were to exist $j \in I$ and a sequence $\{i_n\} \subset I$, $i_n < j$, such that $i_n \uparrow j$ and:

$$\lim_{n\to\infty} \mathbb{P}_A(S_{i_n j}) = \lim_{n\to\infty} \mathbb{P}_A(C_{i_n} <_o A \lesssim_o C_j) = 1,$$

then we could construct a probability value for the individual A. Let μ be a probability measure on $\mathbf{C}$; i.e., μ measures sets which are inverse images of a uniform random variable. A probability value for A, say Q(A), is defined by $Q(A) = \mu(C_j)$.

However, it will not be possible to derive a probability value Q(A) in this manner, because the limit defined in the previous paragraph will be zero when the atomless property applies. In this case other techniques must be employed to obtain a unique value Q(A) from the probability representation $\mathbb{P}_A$.

Regardless of the method employed, care will have to be taken in deriving Q(A) for all $A \in \mathbf{A}$, so that the resulting representation of $\mathbf{A}$ will itself be a probability measure. That is, the representation must satisfy the three basic Kolmogorov axioms concerning nonnegativity, boundedness, and finite additivity. Any set of Q(A) for all $A \in \mathbf{A}$ can meet the first two requirements, given a probability measure on $\mathbf{C}$. But constraints will have to be imposed on the Q(A) in order to insure that the set is finitely additive.

Thus, the second stage of the initial representation problem is to integrate the $\mathbb{P}_A$ for all $A \in \mathbf{A}$ in such a manner that the result is a probability measure over $\mathbf{A}$, and is consistent with $\lesssim_{\mathbf{A}}$. For example, we may wish to derive a first order approximation to each Q(A), $A \in \mathbf{A}$, which "best agrees with" the derived distribution $\mathbb{P}_A$, $A \in \mathbf{A}$, constrained only to satisfy the Kolmogorov axioms. We then have the problem of providing and justifying a definition of "best agrees with" and of specifying the constraints in a workable manner. There exist a variety of numerical techniques for reducing the $\mathbb{P}_A$ for all $A \in \mathbf{A}$ to a probability representation of $(\mathbf{A}, \lesssim_{\mathbf{A}})$, each with its own motivation. These will be examined in a subsequent paper.

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