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A GENERALIZED NOTION OF MONOTONE FAILURE RATE WITH LIFE TESTING APPLICATIONS

by

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ABSTRACT

New notions related to monotone failure rate are defined. Sufficient conditions are found which guarantee that these notions hold. Applications result in inequalities and sometimes equalities which relate power and significance level for certain hypothesis tests. These results generalize similar equalities found by S. D. Dubey in connection with the exponential failure law and are closely related to equalities found by E. Paulson, again in connection with the exponential failure law. All of the tests considered involve a minimum life parameter in the presence of a nuisance scale parameter. Other applications are anticipated but are not discussed.

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1. INTRODUCTION. This paper is motivated by a remarkably simple formula,

$$\beta(m) = 1 - e^{-n(m_o - m)/\theta}(1 - \alpha) , \quad m \le m_o , \tag{1}$$

which relates the power function β to the significance level α for a number of tests concerned with the location parameter m of the two-parameter exponential distribution. $\theta > 0$ is a scale parameter, known or unknown, and m_o arises in the definitions of the hypotheses. This formula appears in the work of Paulson (1941) for two different tests and in the work of Dubey (1962, 1963) for several classes of tests. In each instance, the formula is a consequence of well-known properties of the exponential distributions:

(i) The exponential distributions "lack memory" ("aging" does not occur) when the location parameter is zero.

(ii) The minimal order statistic in a random sample is exponentially distributed.

(iii) The spacings between order statistics are independently distributed.

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The location parameter m of the two-parameter exponential distributions represents the minimum possible life length in a life testing context. Analogs of (1) can be obtained for certain other location-scale parameter families of distributions when m has the same minimum life interpretation. We shall illustrate this for a simple class of tests considered by Dubey (1963).

Stated roughly, if a distribution has a linear Mill's ratio[1], an analog of (1) will hold for the family of distributions produced by introducing location and scale parameters. If the Mill's ratio is convex or concave, an *inequality* analog of (1) will hold. The sense of the inequality will depend on whether convexity or concavity holds and, also, upon the nature of the null hypothesis, whether it specifies a single value m_0 for m or an interval of values $[m_0, \infty)$.

In demonstrating these results, we find it convenient to introduce and study a generalized notion of monotone failure rate. This is done in Section 3.

Section 2 reviews the relevant literature associated with testing for the location parameter of the two-parameter exponential distribution. In addition, two inequality analogs of (1) are derived. This material helps to motivate our new notion of monotone failure rate, but it is not a logical prerequisite for Section 3.

Analogs of (1) are obtained in Section 4 for more general location-scale parameter families. Here, the new notion of monotone failure rate is used.

In Section 5, it is shown that the Mill's ratio of each two-parameter Weibull distribution and of each beta distribution (types one and two) is convex, linear

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[1]By Mill's ratio, we mean the ratio of the upper tail ($1 - F(x)$, say) of a distribution to the density ($f(x)$, say). This is the reciprocal of the hazard function. The term Mill's ratio originally was applied to the standard normal distribution, but has been used more generally in the statistical literature.

or concave. This sets the stage for explicit applications.

We hope to discuss applications outside the life testing context in a later paper.

2. TESTING FOR THE MINIMUM LIFE OF AN EXPONENTIAL DISTRIBUTION.

Let $x_1, x_2, \ldots, x_n$ be the order statistics based on a random sample whose common density is

$$f(u) = \frac{1}{\theta} e^{-(u-m)/\theta}$$

on its support $[m,\infty)$. The problem of testing the hypothesis $m = m_o$ against the alternative $m \neq m_o$ has been extensively studied with the following conclusions obtained:

When $\theta > 0$ is known, x_1 is a minimal sufficient statistic and the likelihood ratio tests reject the hypothesis $m = m_o$ when $x_1 - m_o \notin [0,c]$, where the positive constant c determines the significance level. These tests are strictly unbiassed (see Paulson (1941)), and, in fact, are uniformly most powerful (U.M.P.) unbiased tests for their respective significance levels.

When θ is unknown, $(x_1, \sum_1^n x_i)$ is a minimal sufficient statistic and the likelihood ratio tests reject the hypothesis $m = m_o$ when the test statistic $s = (x_1 - m_o)/\sum_1^n (x_i - x_1) \notin [0,c]$, where the positive constant c determines the significance level. These tests are U.M.P. unbiased tests for their respective significance levels. (See Lehmann (1959), page 202.)

The power function β for both cases is given by (1) for $m < m_o$, where α is the significance level. When $m > m_o$, the forms of the power function are distinct for the two cases. They are given by Paulson, but are unimportant here.

In a life testing context, it is frequently convenient to work with test statistics which depend on the smallest observations, say $x_1, x_2, \ldots, x_r$, $2 \le r < n$. For these, the likelihood ratio tests in the unknown θ case are given by Dubey (1962), and are based upon the statistic

$$(3) \qquad s = (x_1 - m_o) \Big/ \left\{ \sum_1^r (x_i - x_1) + (n - r)(x_r - x_1) \right\} .$$

Again the tests are U.M.P. unbiased tests (for the data used), and again (1) is applicable. Carlson (1958) considers the simple statistic $s = (x_1 - m_o)/(x_n - x_1)$ which is a special case of the statistics

$$(4) \qquad s = (x_a - m_o)/(x_b - x_a) , \quad 1 \le a < b \le n ,$$

studied by Dubey (1963). Equation (1) is applicable whenever $a = 1$ (for tests which reject $m = m_o$ when $s \notin [0,c]$). While Dubey has derived formulas for the power function when $a \ne 1$, they are fairly complicated, and it would be instructive to derive the following analog to (1):

$$(5) \qquad \beta(m) < 1 - \Pr(x_a > m_o)(1 - \alpha) , \quad m < m_o .$$

Since $\Pr(x_a > m_o) > \Pr(x_1 > m_o) = e^{-n(m_o-m)/\theta}$ for $m < m_o$, inequality (5) implies that a strict loss in power occurs to the left of m_o when $a \ne 1$.

<u>Derivation of (5)</u>. Let $z_1, z_2, \ldots, z_n$ denote a generic sequence of standard exponential order statistics, arising under a fixed probability measure P, corresponding to a random sample of size n whose common density is e^{-z}, $z \ge 0$. Then, without loss in generality, we may assume that $x_i = \theta z_i + m$ when m and θ are the true parameter values. In terms of the z's ,

$$1 - \beta(m) = P\big((m_o - m)/\theta < z_a \le (m_o - m)/\theta + c(z_b - z_a)\big)$$
$$= P\big(z_a > (m_o - m)/\theta\big) - P\big(z_a > (m_o - m)/\theta + c(z_b - z_a)\big) ,$$

$Pr(x_a > m_o) = P\big(z_a > (m_o - m)/\theta\big)$, and $1 - \alpha = P\big(z_a \le c(z_b - z_a)\big)$. Thus, (5) is equivalent to

(5') $P\big(z_a > u + c(z_b - z_a)\big) < P(z_a > u)P\big(z_a > c(z_b - z_a)\big)$, $m < m_o$,

where $u = (m_o - m)/\theta$. Since z_a and $z_b - z_a$ are independent, as property (iii) in Section 1 makes clear, inequality (5') holds providing it can be shown that

(6) $P(z_a > u + v) < P(z_a > u)P(z_a > v)$, $u > 0$, $v > 0$.

This is the case, in fact, when $a \ne 1$. If the strict inequality between the probabilities were replaced by a non-strict inequality, (6) would say, precisely, that the distribution of z_a satisfies the "new (is) better than used" property. (cf., Hollander and Proschan (1972).) This property is easily deduced for any distribution which has the more familiar increasing failure rate property. It is not difficult to show that the hazard function for z_a is *strictly* increasing, from which (6) easily follows. □

It probably should be pointed out that the marginal distributions of the order statistics have increasing failure rates when the parent distribution does. See Barlow and Proschan (1965), page 36. We suspect that the new better than used property is preserved by order statistics as well.

Consider now the one-sided hypothesis testing problem $m \geq m_o$ versus $m < m_o$, with θ unknown, and tests which reject $m \geq m_o$ when $s < c$, where s is any one of the above test statistics and the positive constant c controls the significance level. These tests are unbiased and quite good for the test statistics defined in (3) and (4) when $a = 1$. Formula (1) holds for these tests. When the test statistic in (4) is used with $a \neq 1$, (1) does not hold but an analog does:

$$(7) \qquad \beta(m) > 1 - \Pr(x_a > m_o)(1 - \alpha) \; , \quad m < m_o \; .$$

(Compare this with (5).) This can be derived in much the same way (5) is. Specifically, (6) implies (7).

The right side of (1) provides an *upper* bound for $\beta(m)$. This is because the test statistic $s = (x_1 - m_o)/\sum_1^n (x_i - x_1)$ produces a U.M.P. test within the class of tests which have constant power on the boundary set $\{(m_o, \theta), \theta > 0\}$, and, for this, (1) holds.

3. A GENERALIZED NOTION OF MONOTONE FAILURE RATE.

Let x be a random variable with distribution function F and, if they exist, with density f and hazard function $h = f/\bar{F}$, where $\bar{F}$ denotes $1 - F$. There are a number of essentially equivalent definitions of the notion that F has an increasing failure rate. One simple definition states that h is an increasing function on the set of points u for which $\bar{F}(u) > 0$.[2] Other definitions appear in Barlow and Proschan (1965),

[2]The convention that "increasing" means "non-decreasing" rather than "strictly increasing" is being followed. Likewise, "decreasing" means "non-increasing".

pages 23 - 25. One suggestive definition requires $\overline{F}(u+v)/\overline{F}(u)$ to be a *decreasing* function of u for each $v > 0$. This can be reworded in terms of conditional probabilities as " $\Pr(x > u + v | x > u)$ is decreasing in u when $\Pr(x > u) > 0$ for each $v > 0$." Here, of course, v is a constant which assumes positive values. We shall want to view v as a fixed function of x, in the next section, and be concerned with whether $\Pr(x > u + v(x) | x > u)$ is a decreasing function of u.

Let w be any real (measurable) function of a single real variable. The random variable x and its distribution function F will be said to have a *monotone failure rate* (MFR) *on* $S \subset \{u: \overline{F}(u) > 0\}$ *with respect to* w if $\Pr\big(w(x) > u | x > u\big)$ is a monotone function on S. The terminology *increasing failure rate* (IFR), *decreasing failure rate* (DFR) and *constant failure rate* (CFR) *on* S *with respect to* w will be used according as $\Pr\big(w(x) > u | x > u\big)$ is decreasing, increasing or constant on S.

The value of these new concepts is tied to the next two theorems. Suppose F is continuous everywhere and strictly increasing on an interval I whose image under F is $(0,1)$ or $[0,1)$. For simplicity, let $S = I$.

THEOREM 1 F *has a* MFR *on* S *with respect to* w *if the ratio* $\overline{F}\big(w(t)\big)/\overline{F}(t)$ *is monotone on* I. *The failure rate is increasing, decreasing or constant on* S *with respect to* w *according as the ratio is increasing, decreasing or constant on* I.

PROOF.

Let $F^{-1}(s) = \inf\{t: F(t) \geq s\}$, $0 \leq s \leq 1$, denote the "quantile function" associated with F and let z denote a standard exponential variable. Then

$1 - e^{-z}$ is a uniform variable on $[0,1]$ and $F^{-1}(1 - e^{-z})$ has distribution function F. We can, without loss of generality, take the random variable x, used in the above definitions, to be $F^{-1}(1 - e^{-z})$. Then the condition $x > u$ is equivalent to $z > -\log \bar{F}(u)$. Thus

$$\Pr\big(w(x) > u \,|\, x > u\big) = \Pr\big(w(F^{-1}(1 - e^{-z})) > u \,|\, z > \log \bar{F}(u)\big) .$$

Now the conditional distribution of z given $z > -\log \bar{F}(u)$ is the same as the unconditional distribution of $z - \log \bar{F}(u)$. Hence

$$\text{(8)} \qquad \Pr\big(w(x) > u \,|\, x > u\big) = \Pr\Big\{w\big(F^{-1}(1 - e^{-z}\bar{F}(u))\big) > u\Big\} .$$

Suppose, for definiteness, that $\bar{F}\big(w(t)\big)/\bar{F}(t)$ is increasing on I and let u_1 and u_2, $u_1 < u_2$, be any two points in S. Set $x_i = F^{-1}\big(1 - e^{-z}\bar{F}(u_i)\big)$, $i = 1,2$. It easily follows that $x_i \in I$ and $\bar{F}(x_i) = e^{-z}\bar{F}(u_i)$ for $i = 1,2$ and $x_1 \le x_2$. Thus $\bar{F}(x_1)\bar{F}(u_2) = \bar{F}(x_2)\bar{F}(u_1) > 0$.

The following implications hold:

$$\begin{aligned} w(x_2) > u_2 &\iff \bar{F}\big(w(x_2)\big) < \bar{F}(u_2) \quad \text{(even if } w(x_2) \notin I \text{)} \\ &\iff \bar{F}(x_1)F\big(w(x_2)\big) < \bar{F}(x_1)\bar{F}(u_2) \\ &\iff \bar{F}(x_1)\bar{F}\big(w(x_2)\big) < \bar{F}(x_2)\bar{F}(u_1) \\ &\implies \bar{F}\big(w(x_1)\big) < \bar{F}(u_1) \\ &\iff w(x_1) > u_1 . \end{aligned}$$

The unidirectional implication uses the assumption that $\bar{F}\big(w(t)\big)/\bar{F}(t)$ is increasing on I. Consequently, from (8),

$$\begin{aligned} \Pr\big(w(x) > u_2 \,|\, x > u_2\big) &= \Pr\big(w(x_2) > u_2\big) \le \Pr\big(w(x_1) > u_1\big) \\ &= \Pr\big(w(x) > u_1 \,|\, x > u_1\big) . \end{aligned}$$

I.e., F has an IFR on S with respect to w. The remaining DFR and CFR cases follow similarly. □

REMARKS

1. If $\overline{F}(w(t))/\overline{F}(t)$ is constant or decreasing in t, $w(t)$ must be increasing in t. If the ratio is increasing in t, $w(t)$ does not need to be monotone. Whenever the ratio is monotone on I, the interval I can be partitioned into two intervals I_1 and I_2 with $\overline{F}(w(t)) > 0$ on I_1 and $= 0$ on I_2. Either interval may be empty.

2. If the hazard function exists and w is differentiable, then $\overline{F}(w(t))/\overline{F}(t)$ is increasing, decreasing or constant on I_1 according as

$$h(t) \geq, \leq \text{ or } = h(w(t))w'(t), \quad t \in I_1. \tag{9}$$

3. While the assumptions that F is continuous and strictly increasing are explicitly used in our proof of Theorem 1, we suspect they are unnecessary. The crucial equivalence "$x > u \Longleftrightarrow z > \log \overline{F}(u)$" does not depend on either one of them.

4. For purposes of applications, it is desirable to weaken the assumption in Theorem 1 that $\overline{F}(w(t))/\overline{F}(t)$ is monotone on I. In doing so, we find it necessary to assume that w is continuous. Whether it is really necessary is unknown.

<u>THEOREM 2</u> *F has a MFR on S with respect to w if w is continuous on I and the ratio $\overline{F}(w(t))/\overline{F}(t)$ is monotone on $I^* = \{t \in I: w(t) \in I\}$. The failure rate is increasing, decreasing or constant on S with respect to w according as the ratio is increasing, decreasing or constant on I^*.*

<u>PROOF.</u>

Refer to the proof of Theorem 1. Assume the ratio is increasing on I^*. The objective is still to show that $w(x_2) > u_2 \Rightarrow w(x_1) > u_1$. The argument used in Theorem 1 to show this implication would apply here *if* $w(x_i) \in I$, $i = 1,2$.

Suppose $w(x_2) \notin I$ and $w(x_1) \le u_1$. Then there exists a new u_2, say u_2^*, with $u_2^* > u_1$ and such that the new x_2, say x_2^*, defined by $x_2^* = F^{-1}(1 - e^{-z}\overline{F}(u_2^*))$, satisfies $w(x_2^*) > u_2^*$ and $w(x_2^*) \in I$. To see this, let $x(u) = F^{-1}(1 - e^{-z}\overline{F}(u))$, with $z > 0$ held fixed. Then $x_i = x(u_i)$, $i = 1,2$. $x(u)$ is continuous on S and $x(u) \in I$ for $u \in S$. Thus $w(x(u))$ is continuous on S. But $w(x(u_2)) = w(x_2) > u_2 > u_1 \ge w(x_1) = w(x(u_1))$. Consequently, there exists a u_2^*, $u_1 < u_2^* < u_2$, such that $w(x(u_2^*)) = u_2$. Then, for $x_2^* = x(u_2^*)$, $w(x_2^*) = u_2 \in I$ and $w(x_2^*) = u_2 > u_2^*$.

By working with the new u_2 and x_2 with the superscripts deleted, the required implication simplifies to:

(10) $$w(x_2) > u_2 \text{ and } w(x_2) \in I \Rightarrow w(x_1) > u_1 .$$

Now, it has been noted that (10) holds if, in addition, $w(x_1) \in I$. Thus it suffices to show that the conditions

(11) $$w(x_2) > u_2 , \quad w(x_2) \in I , \quad w(x_1) \le u_1 , \quad w(x_1) \notin I$$

lead to a contradiction. The argument again uses $x(u)$ and the fact that $w(x(u))$ is continuous on S in order to find a new u_1 and x_1, say u_1^* and x_1^*, such that $w(x_1^*) = u_1$ and $u_1 < u_1^* < u_2$. The details are essentially the same, and are omitted. Then $w(x_1^*) = u_1 \in S = I$. But

$$w(x_2) > u_2 \ , \quad w(x_2) \in I \ , \quad w(x_1^*) \in I \Rightarrow w(x_1^*) > u_1^* \ ,$$

which contradicts $w(x_1^*) = u_1$.

We have shown that if w is continuous on I and the ratio $\overline{F}(w(t))/\overline{F}(t)$ is increasing on I^*, then F has an IFR on S with respect to w. It remains to show that if the ratio is decreasing or constant on I^*, $w(x_1) > u_1 \Rightarrow w(x_2) > u_2$. For then it will follow, as in the proof of Theorem 1, that F has a DFR on S with respect to w.

But if the ratio is decreasing or constant on I^*, w is increasing on I^* (see Remark 1), and it follows from the continuity of w that the interval I can be partitioned into three subintervals I', I^* and I'', where $I' < I^* < I''$, $w(t) < I$ on I' and $w(t) > I$ on I''. (A set A is less than a set B if each point of A is less than each point of B. A point is less than a set if it is less than each point in the set, etc.) Since $u_1 \geq S = I$, the condition $w(x_1) > u_1$ implies $x_1 \in I^*$ or I''. Since $x_2 \geq x_1$, $x_2 \in I^*$ or I''. In the first case, $w(x_i) \in I$, $i = 1,2$, and the implication "$w(x_1) > u_1 \Rightarrow w(x_2) > u_2$" follows as in the proof of Theorem 1, while in the second case, $w(x_2) > I$ and, consequently, $w(x_2) > u_2$. $\square$

5. The set S is set equal to I in Theorems 1 and 2 in order to maximize the strength of the conclusions. Their proofs only require S to be a subset of I. In the next section, two point sets $S = \{0,u\}$, where u is allowed to range within I, would suffice for the conclusions we obtain. Expressed another way, our mathematics is most favorably described in terms of a generalized notion of monotone failure rate, but our applications are in the spirit of the new better than used property. "New" corresponds to the (boundary of the) null hypothesis and "used" corresponds to alternatives.

4. EQUATIONS AND INEQUALITIES INVOLVING POWER. Our objective in this section is to illustrate in a fairly simple but substantive context a specific type of of application. The context will be that of testing for the value of a minimum life parameter m in the presence of a nuisance scale parameter θ. Observations are assumed to have a distribution function of the form $F\big((t - m)/\theta\big)$, where F is known. It is further assumed that $F(0) = 0$, that F is absolutely continuous with density f, that f is strictly positive on some interval I whose left limit point is zero and that f is zero elsewhere. We shall begin with the testing problem $m \geq m_o$ versus $m < m_o$.

It is convenient to introduce three sets of order statistics which are functionally related. If $z_1, \ldots, z_n$ is a set of order statistics for a standard exponential random sample, then $x_i = F^{-1}(1 - e^{-z_i})$, $i = 1, \ldots, n$, is a set of order statistics for the parent distribution function F and

$y_i = \theta x_i + m$, $i = 1, \ldots, n$ is a set of order statistics for the parent distribution function $F\big((t - m)/\theta\big)$. The latter set represents actual observations.

Following Dubey's (1963) lead, we shall consider the simple test T which rejects the hypothesis $m \geq m_o$ when $y_1 - m_o \leq c(y_b - y_1)$, where $b \in \{2, 3, \ldots, n\}$ and $c > 0$ are fixed.

<u>THEOREM 3</u> *The test* T *is unbiased. If the Mill's ratio* $g(t) = \overline{F}(t)/f(t)$ *is convex for* $t \in I$, *the power function* β *satisfies*

$$(12) \qquad \beta(m) \geq 1 - \overline{F}^{\,n}\big((m_o - m)/\theta\big)(1 - \alpha) , \quad m \leq m_o ,$$

where α *is the significance level of the test. Equality holds in* (12) *if the Mill's ratio is linear.*

<u>PROOF.</u>

$\beta(m) = \Pr\big(y_1 - m_o \leq c(y_b - y_1)\big) = \Pr\big(x_1 \leq (m_o - m)/\theta + c(x_b - x_1)\big)$. Thus $\beta(m)$ is a decreasing function of m and equals α when $m = m_o$. This implies that the test T is unbiased.

Since $1 - \overline{F}^n$ is the distribution function of x_1 , (12) is equivalent to

$$(12') \quad \Pr\big(x_1 > u + c(x_b - x_1)\big) \leq \Pr(x_1 > u)\Pr\big(x_1 > c(x_b - x_1)\big) ,$$

where $u = (m_o - m)/\theta$. Now, because $x_1 = F^{-1}(1 - e^{-z_1})$, $e^{-z_1} = \overline{F}(x_1)$ and

$$(13) \qquad x_b = F^{-1}(1 - e^{-z_b}) = F^{-1}\big(1 - \overline{F}(x_1)e^{-(z_b - z_1)}\big) .$$

But $z_b - z_1$ is independent of z_1 by property (iii) of Section 1. Consequently, $z_b - z_1$ is independent of x_1 . In view of (13), inequality (12') holds if, for each $\varepsilon > 0$,

$$\Pr\left(x_1 > u + c\left(F^{-1}(1 - \overline{F}(x_1)e^{-\varepsilon}) - x_1\right)\right) \le \Pr(x_1 > u)\Pr\left(x_1 > c\left(F^{-1}(1 - \overline{F}(x_1)e^{-\varepsilon}) - x_1\right)\right) .$$

Thus (12) holds if for each $\varepsilon > 0$, $\Pr\left(w(x_1) > u | x_1 > u\right)$ is a decreasing function of u , where

$$w(r) = (1 + c)r - cF^{-1}\left(1 - \overline{F}(r)e^{-\varepsilon}\right) .$$

In words, it suffices to show that x_1 has a IFR on I with respect to w . This will be shown by validating the first inequality in (9). To be precise, Theorem 2 will be used. It will be shown that w is differentiable. Since x_1 has distribution function $F_1 = 1 - \overline{F}^n$, x_1 has the hazard function $h_1 = nf/\overline{F}$. It follows that $\overline{F}_1\left(w(t)\right)/\overline{F}_1(t)$ is a decreasing function of t on $I^* = \{t \in I: w(t) \in I\}$ if

$$h_1(t) \ge h_1\left(w(t)\right)w'(t) , \quad t \in I , \quad w(t) \in I .$$

This is equivalent to showing

$$g(r)q' \le g(q) , \quad r \in I , \quad q \in I , \tag{14}$$

where $q = w(r)$ is a function of r and $g = \overline{F}/f$ is the Mill's ratio associated with the distribution function F .

Let $s = F^{-1}\left(1 - \overline{F}(r)e^{-\varepsilon}\right)$, a function of r . Then

$$\overline{F}(s) = \overline{F}(r)e^{-\varepsilon} , \quad f(s)s' = f(r)e^{-\varepsilon} , \quad g(r)s' = g(s) . \tag{15}$$

Since $q = (1 + c)r - cs$, $q' = (1 + c) - s'$ (in particular, w is differentiable) and (14) can be expressed as

$$(16) \qquad (1 + c)g(r) \leq g(q) + cg(s) \ , \quad r \in I \ , \quad q \in I \ .$$

But $c = (r - q)/(s - r)$ and, hence, (16) can be expressed as

$$(17) \qquad g(r) \leq \frac{s-r}{s-q}\, g(q) + \frac{r-q}{s-q}\, g(s) \ , \quad r \in I \ , \quad q \in I \ .$$

It remains to show that (17) follows from the assumed convexity of g on I . It is easily seen from the first equality in (15) that $s \in I$ if $r \in I$. Therefore (17) holds providing $q < r < s$. But the first equality in (15) implies $r < s$ and, consequently, $q = r + c(r - s) < r$. Thus (12) holds.

Finally, if g is linear, equality holds in (17). This translates into an equality in (12). □

If g is concave, the reverse inequality in (12) holds. Such an inequality is not very useful. However, it is useful to know that g is concave for a different testing problem, namely $m = m_o$ versus $m \neq m_o$. Again, following Dubey's (1963) lead, we shall consider the simple test T^* which accepts $m = m_o$ if $0 \leq y_1 - m_o \leq c(y_b - y_1)$. The power function $\beta(m)$ is given by

$$(18) \quad \beta(m) = \Pr\big(x_1 > (m_o - m)/\theta + c(x_b - x_1)\big) + \Pr\big(x_1 \leq (m_o - m)/\theta\big) \ .$$

Suppose g is concave. The reverse of inequality (12') holds. I.e.,

$$(19) \qquad \Pr\big(x_1 > u + c(x_b - x_1)\big) \geq \Pr(x_1 > u)\Pr\big(x_1 > c(x_b - x_1)\big) \ ,$$

where $u = (m_o - m)/\theta \geq 0$. Combining (18) and (19) yields

$$\beta(m) \geq 1 - \bar{F}^n\big((m_o - m)/\theta\big)(1 - \alpha) \ , \quad m \leq m_o \ , \tag{20}$$

where $\alpha = \beta(0)$ is the significance level of the test. From (18) it follows that $\beta(m) > \alpha$ when $m > m_o$, and from (20) it follows that $\beta(m) > \alpha$ when $m < m_o$. This means that T^* is a strictly unbiased test. Equality holds in (20) if g is a linear function.

5. MILL'S RATIOS. In this section we are concerned with showing that some common distributions have a Mill's ratio which is convex, concave or linear. The importance of such a property is explained in Section 4.

The simplest example involves the two-parameter Weibull distributions, whose densities are

$$f(t) = \lambda\gamma t^{\gamma-1}e^{-\lambda t^{\gamma}} \ , \quad t > 0 \ , \quad \lambda > 0 \ , \quad \gamma > 0 \ .$$

The Mill's ratio is $g(t) = \bar{F}(t)/f(t) = t^{1-\gamma}/(\lambda\gamma)$. g is linear when $\gamma = 1$, i.e., when f is an exponential density. g is convex when $\gamma \geq 1$ and concave when $0 < \gamma \leq 1$.

THEOREM 4 *The Mill's ratio for the beta-one densities*

$$f(t) = \frac{\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\nu)} t^{\mu-1}(1 - t)^{\nu-1} \ , \quad 0 < t < 1 \ , \quad \mu > 0 \ , \quad \nu > 0 \ ,$$

are convex when $\mu \geq 1$ *and concave when* $0 < \mu \leq 1$. *The Mill's ratio for the beta-two densities*

$$f(t) = \frac{\Gamma(\mu+\nu)}{\Gamma(\mu),\Gamma(\nu)} t^{\mu-1}/(1+t)^{\mu+\nu}, \quad 0 < t < \infty, \quad \mu > 0, \quad \nu > 0,$$

are convex when $\mu \ge 1$ *and concave when* $0 < \mu \le 1$.

PROOF.

Consider any one of the above densities f and its corresponding distribution function F . Form the Mill's ratio $g = \overline{F}/f$. By direct calculus, $g'' = \left(f^2f' + (2f'^2 - ff'')\overline{F}\right)/f^3$. Set

$$k = (2f'^2 - ff'')/f^2 \quad \text{and} \quad \ell = fg''/k = f'/k + \overline{F},$$

defined on the support of f except when $k(t) = 0$. Observe that g is convex if $k\ell \ge 0$ and concave if $k\ell \le 0$.

After some algebra, one obtains

$$\ell'(t) = \frac{-2f(t)(\mu-1)(\nu-1)}{k^2(t)t^3(1-t)^3}, \quad k(t) = \frac{\mu(\mu-1)}{t^2} - \frac{2(\mu-1)(\nu-1)}{t(1-t)} + \frac{\nu(\nu-1)}{(1-t)^2}$$

for the beta-one densities, and

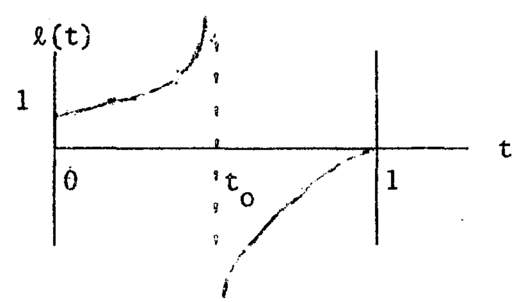
$$\ell'(t) = \frac{-2f(t)(\mu-1)(\mu+\nu)}{k^2(t)t^3(1+t)^3}, \quad k(t) = \frac{\mu(\mu-1)}{t^2} - \frac{2(\mu-1)(\mu+\nu)}{t(1+t)} + \frac{(\mu+\nu)(\mu+\nu-1)}{(1+t)^2}$$

for the beta-two densities. Thus ℓ' always has a constant sign and ℓ is monotone except across roots of k , where ℓ has asymptotes. Except for the trivial case of $k(t) \equiv 0$, the function k clearly can have at most two roots in the support of f . By conventional algebraic methods, it can be shown that there is one such root when $\ell' > 0$ and no such roots when $\ell' < 0$. (For instance, for the beta-one densities, there are two real roots when $\mu + \nu < 1$, one on each side of the interval of support.) The case $\ell' = 0$

corresponds to $\mu = 1$ or $\nu = 1$ for the beta-one densities, and to $\mu = 1$ for the beta-two densities. This case is easily treated by direct methods. Except when $\ell' = 0$, $\ell(0+) = 1$ and ℓ converges to zero at the other end of the interval of support. There remain two basic types of cases to consider, $\ell' < 0$ and $\ell' > 0$. We shall illustrate them for the beta-one densities.

Type I: $\ell' < 0$ and $\ell(1-) = 0$ and sign $(k) = \text{sign}(\mu - 1) \neq 0$. Consequently, $\ell > 0$ and g is convex or concave according as $\mu > 1$ or $\mu < 1$.

Type II: $\ell' > 0$, $\ell(0+) = 1$, $\ell(1-) = 0$, k has a root at some point t_o within the interval of support and $\text{sign}\big(k(t)\big) = \text{sign}\big((t_o - t)(\mu - 1)\big)$. Consequently, ℓ has the general shape:



and g is convex or concave according as $\mu > 1$ or $\mu < 1$. □

Observe that $\overline{F} = \ell - f'/k$. In the Type I situation (which is easily characterized in terms of the parameters), $\ell > 0$ and one obtains the lower bound $\overline{F} > -f'/k$. We have not investigated this bound to see whether it is ever a good bound, nor have we searched the literature to see how it might compare with other known bounds.

Both families of densities have a linear Mill's ratio when $\mu = 1$, for every value of ν. It will be recalled that when the Mill's ratio is linear certain inequalities in the last section, relating power to significance level, become equalities.

There is no simple relationship between the convexity or concavity of a Mill's ratio and the monotonicity of a hazard function. The Weibull distributions suggest a close relationship but other examples do not. For instance, when $\mu = 1$, both the beta-one and beta-two distributions have convex and concave Mill's ratios. But, when $\mu = 1$, the beta-one densities have strictly increasing hazard functions while the beta-two densities have strictly decreasing hazard functions. It is not hard to see that a distribution with support $[0,\infty)$ and a concave Mill's ratio must have a decreasing hazard function. Likewise, a distribution with support $[0,\infty)$ and a convex Mill's ratio must have an increasing hazard function unless some of its moments are infinite. (If the hazard function ever strictly decreases, then $g(t) \geq at + b$ for some $a > 0$ and all $t \geq 0$. It follows that the distribution will have infinite moments of all orders $\geq a^{-1}$.)

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