

ASYMPTOTIC DISTRIBUTIONS FOR HOTELLING'S T_0^2 AND PILLAI'S STATISTIC
FOR TESTING INDEPENDENCE - A DIFFERENT APPROACH

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Asymptotic Distributions for Hotelling's T_0^2 and Pillai's Statistic for Testing Independence - A Different Approach

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1. Introduction. Let $A(p \times p)$ be distributed Wishart with covariance matrix Σ and n degrees of freedom. Partition A and Σ as

$$A = \begin{pmatrix} A_{11} & (q \times q) & A_{12} \\ A_{21} & & A_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & (q \times q) & \Sigma_{12} \\ \Sigma_{21} & & \Sigma_{22} \end{pmatrix}.$$

Let $P = \Sigma_{11}^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$ define the generalized population correlation matrix and $R = A_{11}^{-1} A_{12} A_{22}^{-1} A_{21}$ define the generalized sample correlation matrix.

Fujikoshi [2] derived the asymptotic distributions under the alternative of the two test statistics

(i) $n \operatorname{tr} R$ - Pillai's-statistic,

(ii) $m \operatorname{tr} R(I - R)^{-1}$ - Hotelling's T_0^2 , $m = n - p + q$

for testing the hypotheses $H_0: P = 0$ against the alternative $H_1: P \neq 0$.

We shall derive the same two noncentral distributions in this paper but will use a different approach. The approach will be to consider A_{22} as fixed.

The asymptotic characteristic functions of the two test statistics conditional on A_{22} is known from Fujikoshi [2] and by averaging out over A_{22} the unconditional characteristic functions will be derived. The asymptotic distributions will be considered under the assumption that $P = \frac{1}{n}\theta$ as pointed out by Sugiura [3]. The following preliminary results will be

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necessary:

2. Preliminary Results:

Lemma 2.1. Let A and Σ be partitioned as in Section 1 with A distributed $W(\Sigma, n)$, then $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$ is distributed $W(\Sigma_{11.2}, n - p + q)$ independently of $G = A_{12}A_{22}^{-1}A_{21}$ and G conditional on A_{22} is distributed $W(\Sigma_{11.2}, p - q, \Omega)$, i.e. a noncentral Wishart with noncentrality parameter $\Omega = \Sigma_{11.2}^{-\frac{1}{2}} \beta A_{22} \beta' \Sigma_{11.2}^{-\frac{1}{2}}$, $\beta = \Sigma_{12} \Sigma_{22}^{-1}$.

Proof: The result is well known and can be proved from Anderson [1], Theorem 4.3.2.

Lemma 2.2. If $A_{11.2}$ is distributed $W(\Sigma_{11.2}, n - p + q)$ and G conditional on A_{22} is distributed $W(\Sigma_{11.2}, p - q, \Omega)$, then conditional on A_{22} the characteristic function of $n \operatorname{tr} R = n \operatorname{tr}(A_{11.2} + G)^{-1}G$ is given by (Fujikoshi [2], Equation 5.17).

$$\begin{aligned}
 (1) \quad g_1(t|A_{22}) &= (1 - 2it)^{-\frac{1}{2}q(p-q)} \operatorname{etr}\left\{\frac{it}{1-2it}\Omega\right\} \left[1 - \frac{1}{4n} \left\{ q(p - q)(p + 1) \right. \right. \\
 &\quad \left. \left. (1 - 2(1 - 2it)^{-1}) + (q(p - q)(p + 1) - 4(p + 1)\operatorname{tr}\frac{1}{2}\Omega - 4\operatorname{tr}(\frac{1}{2}\Omega)^2) \right. \right. \\
 &\quad \left. \left. (1 - 2it)^{-2} + 4(p + 1)(1 - 2it)^{-3}\operatorname{tr}\frac{1}{2}\Omega + 4(1 - 2it)^{-4}\operatorname{tr}(\frac{1}{2}\Omega)^2 \right\} + \right. \\
 &\quad \left. \frac{1}{96n^2} \left\{ q(p - q)(h_0 - h_1(1 - 2it)^{-1}) + \sum_{\alpha=2}^8 A_{\alpha}(\frac{1}{2}\Omega)(1 - 2it)^{-\alpha} \right\} + \right. \\
 &\quad \left. 0(n^{-3}) \right]
 \end{aligned}$$

where

$$A_2\left(\frac{\Omega}{2}\right) = q(p - q)h_2 - 2h_1 \operatorname{tr}\left(\frac{\Omega}{2}\right) - 24q(p - q)(p + 1) \operatorname{tr}\left(\frac{\Omega}{2}\right)^2$$

$$A_3\left(\frac{\Omega}{2}\right) = -q(p - q)h_3 + 4h_2 \operatorname{tr}\left(\frac{\Omega}{2}\right) + 48\{(p - q)q^2 + ((p - q)^2 + (p - q) + 4)q + 4(p - q + 1)\} \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 + 128 \operatorname{tr}\left(\frac{\Omega}{2}\right)^3$$

$$A_4\left(\frac{\Omega}{2}\right) = q(p - q)h_4 - 6h_3 \operatorname{tr}\left(\frac{\Omega}{2}\right) + 48\{q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) - 1\} \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 - 96(p + 2) \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 + 96(p + 1) \operatorname{tr}\left(\frac{\Omega}{2}\right) \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 + 48(\operatorname{tr}\left(\frac{\Omega}{2}\right)^2)^2$$

$$A_5\left(\frac{\Omega}{2}\right) = 8h_4 \operatorname{tr}\left(\frac{\Omega}{2}\right) - 96\{q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 3\} \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 - 48\{(p - q)q^2 + ((p - q)^2 + (p - q) + 12)q + 4(3(p - q) + 4)\} \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 - 96(p + 1) \operatorname{tr}\left(\frac{\Omega}{2}\right) \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 - 384 \operatorname{tr}\left(\frac{\Omega}{2}\right)^3$$

$$A_6\left(\frac{\Omega}{2}\right) = 48\{q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 7\} \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 + 24\{(p - q)q^2 + ((p - q)^2 + (p - q) + 20)q + 4(5(p - q) + 8)\} \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 - 96(p + 1) \operatorname{tr}\left(\frac{\Omega}{2}\right) \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 - 128 \operatorname{tr}\left(\frac{\Omega}{2}\right)^3 - 96(\operatorname{tr}\left(\frac{\Omega}{2}\right)^2)^2$$

$$A_7\left(\frac{\Omega}{2}\right) = 96(p + 1) \operatorname{tr}\left(\frac{\Omega}{2}\right) \operatorname{tr}\left(\frac{\Omega}{2}\right)^2 + 384 \operatorname{tr}\left(\frac{\Omega}{2}\right)^3$$

$$A_8\left(\frac{\Omega}{2}\right) = 48(\operatorname{tr}\left(\frac{\Omega}{2}\right)^2)^2$$

and

$$h_0 = 3(p - q)q^3 + 2(3(p - q)^2 + 3(p - q) - 4)q^2 + 3(p - q + 1)((p - q)^2 + (p - q) - 4)q - 4(2(p - q)^2 + 3(p - q) - 1)$$

$$h_1 = 12q(p - q)(p + 1)^2$$

$$h_2 = 6\{3(p - q)q^3 + 2(3(p - q)^2 + 3(p - q) + 4)q^2 + (p - q + 1)(3(p - q)^2 + 3(p - q) + 16)q + 8(p - q + 1)^2\}$$

$$h_3 = 4\{3(p - q)q^3 + 2\{3(p - q)^2 + 3(p - q) + 8\}q^2 + 3(p - q + 1)\{(p - q)^2 + (p - q) + 12\}q + 4\{4(p - q)^2 + 9(p - q) + 7\}\}$$

$$h_4 = 3\{(p - q)q^3 + 2\{(p - q)^2 + (p - q) + 4\}q^2 + (p - q + 1)\{(p - q)^2 + (p - q) + 20\}q + 4\{2(p - q)^2 + 5(p - q) + 5\}\}$$

$$\Omega = \beta' \Sigma_{11.2}^{-1} \beta A_{22}, \quad \beta = \Sigma_{12} \Sigma_{22}^{-1}$$

Lemma 2.3. If $A_{11.2}$ is distributed $W(\Sigma_{11.2}, n - p + q)$ and G conditional on A_{22} is distributed $W(\Sigma_{11.2}, p - q, \Omega)$, $\Omega = \beta' \Sigma_{11.2}^{-1} \beta A_{22}$, $\beta = \Sigma_{12} \Sigma_{22}^{-1}$, then conditional on A_{22} the characteristic function of $m \operatorname{tr} R(I - R)^{-1} = m \operatorname{tr} A_{11.2}^{-1} G$, $m = n - p + q$ is given by (Fujikoshi [2], Equation 6.16).

$$(2) \quad g_2(t|A_{22}) = (1 - 2it)^{-q(p-q)} \operatorname{etr}\left\{\frac{it}{1-2it} \Omega\right\} \left[1 + \frac{1}{4m} \{q(p - q) (p - 2q - 1) - 2(p - q) \{q(p - q) - \operatorname{tr}(\Omega)\} (1 - 2it)^{-1} + \{q(p - q)(p + 1) - 2(-q + 2p + 1)\operatorname{tr}(\Omega) + \operatorname{tr}(\Omega)^2\} (1 - 2it)^{-2} + 2\{(p + 1)\operatorname{tr}(\Omega) - \operatorname{tr}(\Omega)^2\} (1 - 2it)^{-3} + \operatorname{tr}(\Omega)^2 (1 - 2it)^{-4}\} + \frac{1}{96m^2} \sum_{\alpha=0}^8 B_{\alpha}(\frac{1}{2}\Omega) (1 - 2it)^{-\alpha} + o(m^{-3}) \right]$$

where $B_0(\frac{1}{2}\Omega) = q(p - q) \ell_0$

$$B_1(\frac{1}{2}\Omega) = -\ell_1 \{q(p - q) - \operatorname{tr}(\Omega)\}$$

$$B_2(\frac{1}{2}\Omega) = q(p - q) \ell_2 - (\ell_1 + 2\ell_2) \operatorname{tr}(\Omega) + 12(p - q)^2 \operatorname{tr}^2(\Omega) - 6(p - q) \{q^2 - q(p - q - 1) - 4\} \operatorname{tr}(\Omega)^2$$

$$B_3(\frac{1}{2}\Omega) = -q(p - q) \ell_3 + (2\ell_2 + 3\ell_3) \operatorname{tr}(\Omega) - 24\{q(p - q) + 2(p - q)^2 + p - q + 2\} \operatorname{tr}^2(\Omega) + 12\left\{q^2(p - q) - \{2(p - q)^2 - p + q + 4\}q - 8(2(p - q) + 1)\right\} \operatorname{tr}(\Omega)^2 + 12(p - q) \operatorname{tr}(\Omega) \operatorname{tr}(\Omega)^2 + 16 \operatorname{tr}(\Omega)^3$$

$$B_4\left(\frac{1}{2}\Omega\right) = q(p - q)\ell_4 - (3\ell_3 + 4\ell_4)\text{tr}(\Omega) + 12\left\{q^2 + 2(3(p - q) + 1)q + 3(2(p - q)^2 + 2(p - q) + 5)\right\}\text{tr}^2(\Omega) + 12\left\{3((p - q)^2 + 6)q + 4(9(p - q) + 8)\right\}\text{tr}(\Omega)^2 - 12(q + 4(p - q) + 1)\text{tr}(\Omega)\text{tr}(\Omega)^2 - 96\text{tr}(\Omega)^3 + 3\text{tr}^2(\Omega)^2$$

$$B_5\left(\frac{1}{2}\Omega\right) = 4\ell_4\text{tr}(\Omega) - 24\left\{q^2 + q(3(p - q) + 2) + 2(p - q)^2 + 3(p - q) + 9\right\}\text{tr}^2(\Omega) - 12\left\{q^2(p - q) + (2(p - q)^2 + p - q + 24)q + 8(4(p - q) + 5)\right\}\text{tr}(\Omega)^2 + 36(q + 2(p - q) + 1)\text{tr}(\Omega)\text{tr}(\Omega)^2 + 192\text{tr}(\Omega)^3 - 12\text{tr}^2(\Omega)^2$$

$$B_6\left(\frac{1}{2}\Omega\right) = 12\left\{q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 7\right\}\text{tr}^2(\Omega) + 6\left\{q^2(p - q) + q(p + q)^2 + q(p - q) + 20q + 4(5(p - q) + 8)\right\}\text{tr}(\Omega)^2 - 12(3q + 4(p - q) + 3)\text{tr}(\Omega)\text{tr}(\Omega)^2 - 160\text{tr}(\Omega)^3 + 18\text{tr}^2(\Omega)^2$$

$$B_7\left(\frac{1}{2}\Omega\right) = 12(p + 1)\text{tr}(\Omega)\text{tr}(\Omega)^2 + 48\text{tr}(\Omega)^3 - 12\text{tr}^2(\Omega)^2$$

$$B_8\left(\frac{1}{2}\Omega\right) = 6\text{tr}^2(\Omega)^2$$

$$\ell_0 = 3(p - q)q^3 - 2(3(p - q)^2 - 3(p - q) + 4)q^2 + 3q(p - q - 1)((p - q)^2 - p + q + 4) - 4(2(p - q)^2 - 3(p - q) - 1)$$

$$\ell_1 = -12q(p - q)^2(q + 1)$$

$$\ell_2 = -6(p - q)\left\{q^3 + 2q^2 - 3q((p - q)^2 + 1) - 4(2(p - q) + 1)\right\}$$

$$\ell_3 = 4\left\{(3(p - q)^2 + 4)q^2 + 3q((p - q)^3 + (p - q)^2 + 8(p - q) + 4) + 8(2(p - q)^2 + 3(p - q) + 2)\right\}$$

$$\ell_4 = 3\{q^3(p - q) + 2((p - q)^2 + (p - q) + 4)q^2 + (p - q + 1)((p - q)^2 + (p - q) + 20)q + 4(2(p - q)^2 + 5(p - q) + 5)\}$$

Lemma 2.4. Let $A_{22}(p - q \times p - q)$ be distributed $W(\Sigma_{22}, n)$, then for $\beta = \Sigma_{12}\Sigma_{22}^{-1}$, $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$ and $P = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} = \frac{1}{n}\theta$

$$\begin{aligned}
 (3) \quad E \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} &= \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[1 + \frac{1}{n} \left\{ \frac{1}{2} \operatorname{tr} \theta^2 ((1-2it)^{-1} - 1) + \right. \right. \\
 &\quad \left. \frac{1}{2} \operatorname{tr} \theta^2 ((1-2it)^{-2} - 2(1-2it)^{-1} + 1) \right] + \\
 &\quad \frac{1}{2} \left[\frac{1}{n} \operatorname{tr} \theta^3 ((1-2it)^{-1} - 1) + \frac{1}{n} \operatorname{tr} \theta^3 ((1-2it)^{-2} - \right. \\
 &\quad \left. 2(1-2it)^{-1} + 1) + \frac{2}{3} \operatorname{tr} \theta^3 ((1-2it)^{-3} - \right. \\
 &\quad \left. 3(1-2it)^{-2} + 3(1-2it)^{-1} - 1) \right] + \left\{ \right. \\
 &\quad \frac{1}{8} ((1-2it)^{-2} - 2(1-2it)^{-1} + 1) \operatorname{tr}^2 \theta^2 + \\
 &\quad \left. \frac{1}{8} \operatorname{tr}^2 \theta^2 ((1-2it)^{-3} - 3(1-2it)^{-2} + \right. \\
 &\quad \left. 3(1-2it)^{-1} - 1) + \frac{1}{32} \operatorname{tr}^2 \theta^2 ((1-2it)^{-4} - \right. \\
 &\quad \left. 4(1-2it)^{-3} + 6(1-2it)^{-2} - 4(1-2it)^{-1} + \right. \\
 &\quad \left. 1) \right] + o(n^{-3}) \quad \left. \right]
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad E \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} \operatorname{tr} \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right) &= \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[\frac{1}{2} \operatorname{tr} \theta + \frac{1}{2n} \left\{ \operatorname{tr} \theta^2 + \right. \right. \\
 &\quad \left. \operatorname{tr} \theta^2 ((1-2it)^{-1} - 1) + \frac{1}{2} \operatorname{tr} \theta \operatorname{tr} \theta^2 ((1-2it)^{-1} - \right. \\
 &\quad \left. 1) + \frac{1}{2} \operatorname{tr} \theta \operatorname{tr} \theta^2 ((1-2it)^{-2} - 2(1-2it)^{-1} + 1) \right] + \\
 &\quad o(n^{-2}) \quad \left. \right]
 \end{aligned}$$

$$\begin{aligned}
 (5) \quad E \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} \operatorname{tr} \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right)^2 &= \\
 &= \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[\frac{1}{4} \operatorname{tr} \theta^2 + \frac{1}{4n} \left\{ \frac{1}{2} \operatorname{tr}^2 \theta^2 \left((1-2it)^{-1} - 1 \right) + \right. \right. \\
 &\quad \left. \frac{1}{2} \operatorname{tr}^2 \theta^2 \left((1-2it)^{-2} - 2(1-2it)^{-1} + 1 \right) + \operatorname{tr} \theta^2 + \right. \\
 &\quad \left. \left. 2 \operatorname{tr} \theta^3 + 2 \operatorname{tr} \theta^3 \left((1-2it)^{-1} - 1 \right) + \operatorname{tr}^2 \theta \right\} + o(n^{-2}) \right]
 \end{aligned}$$

$$(6) \quad E \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} \operatorname{tr} \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right)^3 = \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[\frac{1}{8} \operatorname{tr} \theta^3 + o(n^{-1}) \right]$$

$$(7) \quad E \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} \operatorname{tr}^2 \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right) = \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[\frac{1}{4} \operatorname{tr}^2 \theta + o(n^{-1}) \right]$$

$$\begin{aligned}
 (8) \quad E \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} \operatorname{tr} \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right) \operatorname{tr} \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right)^2 &= \\
 &= \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[\frac{1}{8} \operatorname{tr} \theta \operatorname{tr} \theta^2 + o(n^{-1}) \right]
 \end{aligned}$$

$$(9) \quad E \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} \operatorname{tr}^2 \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right)^2 = \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[\frac{1}{16} \operatorname{tr}^2 \theta^2 + o(n^{-1}) \right]$$

Proof: Since the zonal polynomial $C_K \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right)$ can be expressed in terms of sums of powers of the characteristic roots of the matrix and since the inverse is also true, the key integral will be

$$\begin{aligned}
 (10) \quad T_K &= \frac{1}{\Gamma_{p-q} \left(\frac{1}{2} n \right)} |2\Sigma_{22}|^{-\frac{1}{2}n} \int_{A_{22} > 0} \operatorname{etr} \left(-\frac{1}{2} \Sigma_{22}^{-1} A_{22} \right) |A_{22}|^{\frac{1}{2}(n-p+q-1)} \\
 &\quad \operatorname{etr} \left\{ \frac{it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right\} C_K \left(\frac{1}{2} \beta' \Sigma_{11.2}^{-1} \beta A_{22} \right) dA_{22} \\
 &= \left(\frac{1}{2} n \right)_K |I - \frac{2it}{1-2it} \Sigma_{22} \beta' \Sigma_{11.2}^{-1} \beta|^{-\frac{1}{2}n} C_K \left(\beta' \Sigma_{11.2}^{-1} \beta \left(\Sigma_{22}^{-1} - \frac{2it}{1-2it} \beta' \Sigma_{11.2}^{-1} \beta \right)^{-1} \right)
 \end{aligned}$$

But

$$(11) \quad \Sigma_{11.2}^{-1} = \Sigma_{11}^{-\frac{1}{2}} \left(I - \frac{1}{n} \theta \right) \Sigma_{11}^{-\frac{1}{2}}$$

$$= \Sigma_{11}^{-\frac{1}{2}} \left(I + \frac{1}{n} \theta + \frac{1}{n^2} \theta^2 + o(n^{-3}) \right) \Sigma_{11}^{-\frac{1}{2}}$$

Let $\frac{1}{\sqrt{n}} \theta_1 = \Sigma_{22}^{\frac{1}{2}} \beta' \Sigma_{11}^{-\frac{1}{2}}$ such that $\theta_1' \theta_1 = nP = \theta$, then it follows that

$$(12) \quad T_K = \text{etr} \left\{ \frac{it\theta}{1-2it} \right\} \left[1 + \frac{1}{n} \left\{ \alpha_1(K) + \left(\frac{it}{1-2it} \right) \text{tr} \theta^2 + \left(\frac{it}{1-2it} \right)^2 \text{tr} \theta^2 \right\} + \frac{1}{n^2} \left\{ \left(\frac{it}{1-2it} \right) \text{tr} \theta^3 + \right. \right.$$

$$2 \left(\frac{it}{1-2it} \right)^2 \text{tr} \theta^3 + \frac{4}{3} \left(\frac{it}{1-2it} \right)^3 \text{tr} \theta^3 + \frac{1}{2} \left(\frac{it}{1-2it} \right)^2 \text{tr}^2 \theta^2 + \left(\frac{it}{1-2it} \right)^3 \text{tr}^2 \theta^2 +$$

$$\left. \frac{1}{2} \left(\frac{it}{1-2it} \right)^4 \text{tr}^2 \theta^2 + \frac{1}{6} (k - \alpha_2(K) + 3\alpha_1^2(K)) + \alpha_1(K) \left\{ \frac{it}{1-2it} + \left(\frac{it}{1-2it} \right)^2 \right\} \text{tr} \theta^2 \right\} +$$

$$o(n^{-2}) \left. \right] C_K \left(\frac{1}{2} \theta + \frac{1}{2n} (\theta^2 + \frac{2it}{1-2it} \theta^2) + o(n^{-2}) \right)$$

where $\alpha_1(K) = \sum_{i=1}^q k_i (k_i - 1)$

$\alpha_2(K) = \sum_{i=1}^q k_i (4k_i^2 - 6k_i + 3)$. Use was made of the binomial expansion

into a series and

$$\left(\frac{it}{1-2it} \right)_K = \left(\frac{it}{1-2it} \right)^k \left[1 + \frac{1}{n} \alpha_1(K) + \frac{1}{6n^2} (k - \alpha_2(K) + 3\alpha_1^2(K)) + o(n^{-3}) \right]$$

3. Asymptotic Distributions.

Theorem 3.1. The asymptotic distribution of Pillai's statistic $n \text{tr} R$ if

$P = \frac{1}{n} \theta$, is given by

$$(13) \quad P(n \operatorname{tr} R < z] = P[\chi_f^2(\delta^2) < z] - \frac{1}{4n} \left\{ \sum_{\alpha=0}^4 a_\alpha P \left[\chi_{f+2\alpha}^2(\delta^2) < z \right] \right\} + \frac{1}{96n^2} \left\{ \sum_{\alpha=0}^8 b_\alpha P \left[\chi_{f+2\alpha}^2(\delta^2) < z \right] \right\} + O(n^{-3})$$

where a_α , $\alpha = 0, \dots, 4$ and b_α , $\alpha = 0, \dots, 8$ are given in (15) and (16) and $\delta^2 = \operatorname{tr} \frac{1}{2} \theta$, the noncentrality parameter of the noncentral $\chi_{f+2\alpha}^2$ distribution. $f = q(p - q)$.

Proof: Applying Lemma 2.4 on Lemma 2.2, the unconditional characteristic function of $n \operatorname{tr} R$ is given by

$$(14) \quad q_1(t) = E_{A_{22}} g_1(t|A_{22}) \\ = \operatorname{etr} \left\{ \frac{it\theta}{1-2it} \right\} (1 - 2it)^{-\frac{1}{2}q(p-q)} \left[1 - \frac{1}{4n} \left\{ \sum_{\alpha=0}^4 a_\alpha (1 - 2it)^{-\alpha} \right\} + \frac{1}{96n^2} \left\{ \sum_{\alpha=0}^8 b_\alpha (1 - 2it)^{-\alpha} \right\} + O(n^{-3}) \right]$$

where

$$(15) \quad a_0 = q(p - q)(p + 1) + 4\operatorname{tr}(\frac{1}{2}\theta)^2 \\ a_1 = -2q(p - q)(p + 1) \\ a_2 = q(p - q)(p + 1) - 4(p + 1)\operatorname{tr} \frac{1}{2}\theta - 8\operatorname{tr}(\frac{1}{2}\theta)^2 \\ a_3 = 4(p + 1)\operatorname{tr} \frac{1}{2}\theta \\ a_4 = 4\operatorname{tr}(\frac{1}{2}\theta)^2$$

and

$$(16) \quad b_0 = q(p - q)h_0 + 24q(p - q)(p + 1)\operatorname{tr}(\frac{1}{2}\theta)^2 - 128\operatorname{tr}(\frac{1}{2}\theta)^3 + 48\operatorname{tr}^2(\frac{1}{2}\theta)^2 \\ b_1 = -q(p - q)h_1 - 48q(p - q)(p + 1)\operatorname{tr}(\frac{1}{2}\theta)^2 \\ b_2 = q(p - q)h_2 - 2h_1\operatorname{tr}(\frac{1}{2}\theta) + 96\operatorname{tr}^2(\frac{1}{2}\theta) - 24\{q(p - q)(p + 1) - 4\}\operatorname{tr}(\frac{1}{2}\theta)^2 - 96(p + 1)\operatorname{tr} \frac{1}{2}\theta \operatorname{tr}(\frac{1}{2}\theta)^2 - 192\operatorname{tr}^2(\frac{1}{2}\theta)^2$$

$$b_3 = -q(p - q)h_3 + 4h_2 \text{tr}(\frac{1}{2}\theta) + 96[(p - q)q^2 + q(p - q)^2 + q(p - q) + 4q + 4(p - q + 1)]\text{tr}(\frac{1}{2}\theta)^2 + 96(p + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 + 640\text{tr}(\frac{1}{2}\theta)^3$$

$$b_4 = q(p - q)h_4 - 6h_3 \text{tr}(\frac{1}{2}\theta) + 48(q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) - 3)\text{tr}^2(\frac{1}{2}\theta) - 24\left\{(p - q)q^2 + [(p - q)^2 + (p - q) + 12]q + 4(3(p - q) + 5)\right\}\text{tr}(\frac{1}{2}\theta)^2 + 192(p + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 + 288\text{tr}^2(\frac{1}{2}\theta)^2$$

$$b_5 = 8h_4 \text{tr}(\frac{1}{2}\theta) - 96(q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 3)\text{tr}^2(\frac{1}{2}\theta) - 48\{(p - q)q^2 + q(p - q)^2 + q(p - q) + 12q + 12(p - q) + 16\}\text{tr}(\frac{1}{2}\theta)^2 - 192(p + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 - 768\text{tr}(\frac{1}{2}\theta)^3$$

$$b_6 = 48(q^2 + 2(p - q + 1)q + (p - q)^2 + 2(p - q) + 7)\text{tr}^2(\frac{1}{2}\theta) + 24(q^2(p - q) + q(p - q)^2 + q(p - q) + 20q + 20(p - q) + 32)\text{tr}(\frac{1}{2}\theta)^2 - 96(p + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 - 128\text{tr}(\frac{1}{2}\theta)^3 - 192\text{tr}^2(\frac{1}{2}\theta)^2$$

$$b_7 = 96(p + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 + 384\text{tr}(\frac{1}{2}\theta)^3$$

$$b_8 = 48\text{tr}^2(\frac{1}{2}\theta)^2$$

and h_i , $i = 1, \dots, 4$ given in Lemma 2.2.

Theorem 3.2. The asymptotic distribution of Hotelling's- T_0^2 statistic $m \text{tr} R(I - R)^{-1}$, $m = n - p + q$, if $P = \frac{1}{n}\theta$, is given by

$$(17) \quad P[m \text{tr} R(I - R)^{-1} < z] = P[X_f^2(\delta^2) < z] + \frac{1}{4m} \left\{ \sum_{\alpha=0}^4 c_\alpha P[X_{f+2\alpha}^2(\delta^2) < z] \right\} + \frac{1}{96m^2} \left\{ \sum_{\alpha=0}^8 d_\alpha P[X_{f+2\alpha}^2(\delta^2) < z] \right\} + o(m^{-3})$$

where $\delta^2 = \frac{1}{2}\text{tr}\theta$, $f = q(p - q)$ and c_α , $\alpha = 0, \dots, 4$ and d_α , $\alpha = 0, \dots, 8$ given in (19) and (20).

Proof: Applying Lemma 2.4 on Lemma 2.3, the unconditional characteristic function of $m \operatorname{tr} R(I - R)^{-1}$ is given by

$$(18) \quad q_2(t) = E_{A_{22}} g_2(t|A_{22}) \\ = \operatorname{etr}\left\{\frac{it\theta}{1-2it}\right\} (1-2it)^{-\frac{1}{2}q(p-q)} \left[1 + \frac{1}{4m} \left\{ \sum_{\alpha=0}^4 c_{\alpha} (1-2it)^{-\alpha} \right\} + \frac{1}{96m^2} \left\{ \sum_{\alpha=0}^8 d_{\alpha} (1-2it)^{-\alpha} \right\} + o(m^{-3}) \right]$$

where

$$(19) \quad c_0 = q(p-q)(p-2q-1) - 4\operatorname{tr}(\frac{1}{2}\theta)^2 \\ c_1 = -2(p-q)(q(p-q) - 2\operatorname{tr}(\frac{1}{2}\theta)) \\ c_2 = q(p-q)(p+1) - 4(2p-q+1)\operatorname{tr}(\frac{1}{2}\theta) + 8\operatorname{tr}(\frac{1}{2}\theta)^2 \\ c_3 = 4\{(p+1)\operatorname{tr}\frac{1}{2}\theta - 2\operatorname{tr}(\frac{1}{2}\theta)^2\} \\ c_4 = 4\operatorname{tr}(\frac{1}{2}\theta)^2$$

and

$$(20) \quad d_0 = q(p-q)\ell_0 + 24(p-q)\{q^2 - q(p-q-1) + 4\}\operatorname{tr}(\frac{1}{2}\theta)^2 - 128\operatorname{tr}(\frac{1}{2}\theta)^3 + 48\operatorname{tr}^2(\frac{1}{2}\theta)^2 \\ d_1 = -\ell_1\{q(p-q) - 2\operatorname{tr}(\frac{1}{2}\theta)\} + 48q(p-q)^2\operatorname{tr}(\frac{1}{2}\theta)^2 - 96(p-q)\operatorname{tr}(\frac{1}{2}\theta)\operatorname{tr}(\frac{1}{2}\theta)^2 \\ d_2 = q(p-q)\ell_2 - 2(\ell_1 + 2\ell_2)\operatorname{tr}(\frac{1}{2}\theta) + 48\{(p-q)^2 + 2\}\operatorname{tr}^2(\frac{1}{2}\theta) - 24\{3(p-q)q^2 - (p-q)(p-q-3)q - 4\{3(p-q) + 1\}\}\operatorname{tr}(\frac{1}{2}\theta)^2 + 96(2p-q+1)\operatorname{tr}(\frac{1}{2}\theta)\operatorname{tr}(\frac{1}{2}\theta)^2 - 192\operatorname{tr}^2(\frac{1}{2}\theta)^2$$

$$d_3 = -q(p - q)\lambda_3 + 2(2\lambda_2 + 3\lambda_3)\text{tr}(\frac{1}{2}\theta) - 96\{(p - q)q + 2(p - q)^2 + p - q + 4\} \\ \text{tr}^2(\frac{1}{2}\theta) + 48\{(p - q)q^2 - (3(p - q)^2 - p + q + 8)q - 8(3(p - q) + 2)\} \\ \text{tr}(\frac{1}{2}\theta)^2 - 96(2q - p + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 + 640\text{tr}(\frac{1}{2}\theta)^3 + 192\text{tr}^2(\frac{1}{2}\theta)^2$$

$$d_4 = q(p - q)\lambda_4 - 2(3\lambda_3 + 4\lambda_4)\text{tr}(\theta) + 48\{q^2 + 2q(3p - 3q + 1) + 6(p - q)^2 + \\ 6(p - q) + 17\}\text{tr}^2(\frac{1}{2}\theta) + 24\{q^2(p - q) + (7(p - q)^2 + (p - q) + 44)q + \\ 4(20(p - q) + 19)\text{tr}(\frac{1}{2}\theta)^2 - 192(q + 3(p - q) + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 - \\ 1536\text{tr}(\frac{1}{2}\theta)^3 + 96\text{tr}^2(\frac{1}{2}\theta)^2$$

$$d_5 = 8\lambda_4\text{tr}(\frac{1}{2}\theta) - 96\{q^2 + q(3p - 3q + 2) + 2(p - q)^2 + 3(p - q) + 9\}\text{tr}^2(\frac{1}{2}\theta) - \\ 48\{q^2(p - q) + q(2(p - q)^2 + p - q + 24) + 8(4p - 4q + 5)\text{tr}(\frac{1}{2}\theta)^2 + \\ 96(4q + 7p - 7q + 4)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 + 1920\text{tr}(\frac{1}{2}\theta)^3 - 384\text{tr}^2(\frac{1}{2}\theta)^2$$

$$d_6 = 48\{q^2 + 2q(p - q + 1) + (p - q)^2 + 2(p - q) + 7\}\text{tr}^2(\frac{1}{2}\theta) + 24\{q^2(p - q) + \\ q((p - q)^2 + p - q + 20) + 4(5p - 5q + 8)\}\text{tr}(\frac{1}{2}\theta)^2 - 96(3q + 4p - 4q + 3) \\ \text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 - 1280\text{tr}(\frac{1}{2}\theta)^3 + 384\text{tr}^2(\frac{1}{2}\theta)^2$$

$$d_7 = 96\{(p + 1)\text{tr}(\frac{1}{2}\theta)\text{tr}(\frac{1}{2}\theta)^2 + 4\text{tr}(\frac{1}{2}\theta)^3 - 2\text{tr}^2(\frac{1}{2}\theta)^2\}$$

$$d_8 = 48\text{tr}^2(\frac{1}{2}\theta)^2$$

$\lambda_1, \lambda_2, \lambda_3$ and λ_4 are given in Lemma 2.3.

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