THE HIGGS FACTORIZATION OF A GEOMETRIC STRONG MAP

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ABSTRACT

The *Higgs factorization* of a strong map between matroids on a fixed set is that factorization into elementary maps in which each matroid is the Higgs lift of its successor. This factorization is characterized by properties of the modular filters which induce the elementary maps of the factorizations in two different ways. It is also shown to be minimal in a natural order on factorizations arising from the weak-map partial order on matroids.

The notion of *essential nullity* of flats of a matroid is introduced; this quantity is nonzero precisely for the cyclic flats, and is shown to be related to the minimal flats of the modular filters inducing the maps of the Higgs factorization.

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Key Words and Phrases: Matroid, combinatorial geometry, strong map, elementary strong map, strong map factorization, Higgs lift.

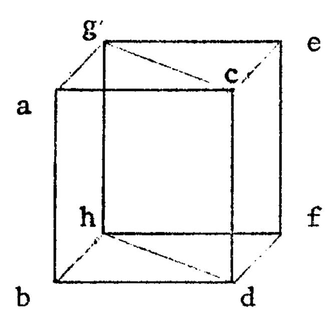
1. Introduction.

The factorization $H = G_0 \to G_1 \to \ldots \to G_n = G$ of a strong map $H \to G$ between matroids on the same set X into *elementary maps*, or strong maps that reduce the rank by one, was first discovered by Higgs ([9]; see also [2]) using a construction we call the *Higgs lift* of the map. T. Brown [1] later studied a special type of elementary map, called "F-products" by him and *principal maps* in a subsequent paper of Dowling and Kelly [7]. Matroids whose canonical strong maps $B \to G$ (B is the free matroid on X) can be factored into principal maps are shown to be the duals of transversal matroids. In a second paper [8], Dowling and Kelly investigated general elementary strong maps between matroids on the same set, using extensively the notions of modular cuts and modular filters introduced by Crapo [3], as well as the dual notion of modular ideals. Kennedy [11] later studied strong map factorizations, introducing the notion of the *major* of a factorization, and proving several results about the *Higgs factorization*, in which every matroid G_j is the Higgs lift of the strong map $H \to G_{j+1}$. This factorization was studied also in [8].

The work in [7], [8], [10], and [11] was partly motivated by Higgs' tantalizing notion of the *essential flats* of a matroid. These are the flats which, as submatroids, are truncations of matroids of higher rank, and thus whose existence as flats cannot be predicted from the flats they contain. The essential flats of a matroid, together with their ranks, determine the matroid; this was first noticed by Higgs, according to Crapo [4]; for a proof see [8].

Dowling and Kelly [8] noticed an apparent connection between factorizations of the canonical strong map $B \to G$ and essential flats of G, and conjectured that every matroid admits a "proper factorization". This is one in which the minimal flats of the modular cuts determining the elementary maps are precisely the essential flats of G, each flat A appearing $e(A)$ times as a minimal flat, where $e(A)$ is the largest difference in rank between A and a matroid whose truncation is A.

This conjecture was shown by Kennedy [10] to be false; his counterexample is the rank-4 geometry on eight points a, b, c, d, e, f, g, h, whose copoints (rank-3 flats) are abcd, cdef, efgh, abgh, cdgh, and all 3-subsets not contained in one of these. It appears as "1, 8, 28, 41w, 1" in Crapo's catalog [5], and in affine 3-space is "pictured" as follows (although in fact it is not representable):



In this paper we introduce the notion of the *essential nullity* $N(A)$ of a flat, a quantity that is positive if and only if the flat is cyclic, i.e. isthmus-free. (All the essential flats of a matroid are cyclic.) We show that "proper factorizations" exist for these flats; specifically, in the Higgs factorization of $B \to G$, the minimal flats of the modular cuts are exactly the cyclic flats of G, and each such flat A appears $N(A)$ times as a minimal flat. (The result is actually proved in greater generality for an arbitrary strong map $f: H \to G$, with essential nullity generalized to "essential f-nullity.")

In addition, we prove some other results about the Higgs factorization of an arbitrary strong map, including its characterization as the only factorization in which the modular cuts are nested. Many of these results can be found in a slightly different context in [8].

The next section concludes with a more precise statement of our main results.

2. Definitions and Statements of Results.

In this paper we will consider only matroids on a fixed finite set X of size k. A matroid G will be viewed as a family of subsets of X which contains X itself and which forms a geometric lattice under inclusion. The members of this lattice are the flats of G; we will use the term *G-flats* when there are more than one matroid under consideration. The free matroid, in which every subset of X is a flat, will be denoted by $\mathcal{B}$. The rank function of G is denoted r_G, and closure by $A \mapsto \overline{A}$.

The reader is assumed to be familiar with the following notions for matroids (combinatorial pregeometries): geometric lattice, rank, nullity, closure, circuit, spanning set, dual matroid. An introduction to the subject of matroids can be found in [6]; further elaboration of the results collected below can be found in [8].

We define a relation among matroids on X by saying that $H \to G$ *is a strong map* if every G-flat is an H-flat. (This curious way of referring to a relation comes from the fact that every G-flat is an H-flat if and only if the identity on X induces a strong map (see [9]) between the geometric lattices of H and G.) If $H \to G$ is a strong map, then $r(H) \geq r(G)$, with equality if and only if $H = G$. If $r(H) = r(G) + 1$, we then say that $H \to G$ is an *elementary strong map*, or simply an *elementary map*.

Subsets A and B of X are called a *modular pair* in the matroid H if $r_H(A) + r_H(B) = r_H(A \cap B) + r_H(A \cup B)$. A family F of subsets of X (resp. H-flats) is an *order filter* if whenever A and B are subsets of X (resp. H-flats) and $B \supseteq A \in F$, then $B \in F$.

A *modular cut* of a matroid H is an order filter M of H-flats such that $A \cap B \in M$ whenever A and B are a modular pair in M . A *modular filter* of H is an order filter of subsets of X with the same property. We call a modular filter *proper* if it is nonempty and does not contain all subsets of X .

The connection between modular cuts and modular filters is simple: given a modular cut M , the family F_M of sets spanning members of M is a modular filter; given a modular filter F , the family M_F of flats in F is a modular cut; and the maps $M \mapsto F_M$ and $F \mapsto M_F$ are inverses.

There is one-one correspondence between matroids G for which $H \to G$ is an elementary map and proper modular filters of H , as follows. If F is a proper modular filter, then the rank function of G is given by

$$r_G(A) = \begin{cases} r_H(A) - 1 & \text{if } A \in F \\ r_H(A) & \text{if } A \notin F . \end{cases} \tag{1}$$

Similarly, if $H \to G$ is elementary, then the sets A for which $r_G(A) = r_H(A) - 1$ forms a modular filter in H . We say that F is the modular filter *associated* with the map $H \to G$, and we denote the situation by writing $H \xrightarrow{F} G$. If $M = M_F$ as in the previous paragraph, then it follows that the flats of H that are not flats of G are precisely those not in M but covered by flats in M .

$H \to G$ is a strong (resp. elementary) map if and only if $G^* \to H^*$ is a strong (resp. elementary) map. The relations between the modular filters associated with these maps can be found in [8], but need not concern us here.

We will also need the notion of weak maps between matroids of the same rank on X. We say that there is a *weak map* from H to H', or that $H \leq H'$ in the *weak-map partial order*, if $r_H(A) \geq r_{H'}(A)$ for all subsets of A of X.. A result of Lucas [12] is that if H and H' are matroids of the same rank on X, then $H \leq H'$ if and only if $H^* \leq H'^*$.

Let $H \to G$ be a strong map with $r(H) > r(G)$. A matroid L is a *lift* of the map $H \to G$ if $H \to L$ is strong and $L \to G$ is elementary. Lifts always exist; the *Higgs lift* is that matroid whose flats are the flats of G together with those flats A of H for which $r_H(A) = r_G(A)$. We shall deal enough with the above situation to give it a name: we will say " $H \to L \xrightarrow{F} G$ is a *lifted* (or *Higgs-lifted*) *strong map*" to signal that $H \to L$ is strong and $L \to G$ is the elementary map associated to the modular filter F of L. It follows from results in Section 7 of [8] that if $H \to L \xrightarrow{F} G$ is a lifted strong map, then it is a Higgs-lifted strong map if and only if $F = \{A: r_H(A) > r_G(A)\}$.

Another situation we will deal with is that of a *factorization* of a strong map $H \to G$, namely, a sequence

$$H = G_0 \xrightarrow{F_1} G_1 \xrightarrow{F_2} G_2 \xrightarrow{F_3} \cdots \xrightarrow{F_n} G_n = G \qquad (2)$$

of elementary maps. The *Higgs factorization* is the one in which G_{j-1} is the Higgs lift of the map $H \to G_j$ for $j = 1, \ldots, n$. The integer n is the *nullity* of the map $H \to G$. It will sometimes be convenient to denote the map

$H \to G$ by f, in order to use the notation $n_f(A)$ for $r_H(A) - r_G(A)$. This quantity is called the *f-nullity* of A; when H is the free matroid $\mathcal{B}$ on X, it reduces to the nullity of A (and n is the nullity of the matroid G). It is easy to check that for any strong map f, n_f is nondecreasing ($A \subseteq B$ implies $n_f(A) \leq n_f(B)$) and that because of (1), for any factorization of f, $n_f(A)$ equals the number of modular filters among $F_1, \ldots, F_n$ of which A is a member.

We will also denote the map $H \to G_j$ in a factorization by f_j; and for convenience we will write n_j instead of n_{f_j} and r_j instead of r_{G_j}.

We define the *essential f-nullity* of a set $A \subseteq X$ by

$$N_f(A) = n_f(\bar{A}) - \max \{ B: B \text{ is a G-flat properly contained in } A \} .$$

Because n_f is nondecreasing, the maximum is attained at some flat B covered by $\bar{A}$:

$$N_f(A) = n_f(\bar{A}) - \max \{n_f(B): B \prec \bar{A}\} .$$

When $H = \mathcal{B}$, $N_f(A)$ is called the *essential nullity* of A and denoted $N(A)$. To say that $N(A) = 0$ for a flat A is the same as saying that A covers a flat of the same nullity, i.e. that A has an isthmus (a subflat of rank 1 whose points are in no circuit of A). So the flats of positive essential nullity are just the cyclic flats (i.e. flats that are unions of circuits).

At last the main results of this paper can be stated. They are

1. $N_f(A)$ is the number of modular filters among $F_1, \ldots, F_n$ in the Higgs factorization in which $\bar{A}$ is a minimal flat. (Corollary 2)

2. A factorization (2) is the Higgs factorization if and only if $F_1 \subsetneq F_2 \subseteq \dots \subseteq F_n$. (Theorem 3)

3. In the Higgs factorization, $F_j = \{A: n_f(A) > n-j\}$, $j = 1, \dots, n$. (Theorem 4)

4. Let (2) be the Higgs factorization and $H = G_0 \to G'_1 \to \dots \to G'_{n-1} \to G_n = g$ any other factorization of $H \to G$. Then for $j = 1, \dots, n$, $G_j \le G'_j$ in the weak-map partial order. (Theorem 8)

5. $G^* = G_n^* \to G_{n-1}^* \to \dots \to G_1^* \to G_0^* = H^*$ is the Higgs factorization of $G^* \to H^*$ if (2) is the Higgs factorization of $H \to G$.

3. Properties and Construction of the Higgs Factorization.

Proposition 1. Let $H \to L \xrightarrow{F} G$ be a Higgs-lifted strong map; denote $H \to G$ by f and $H \to L$ by f' . Then for any subset A of X ,

$$N_f(A) = \begin{cases} N_{f'}(A) + 1 & \text{if } \overline{A} \text{ is a minimal flat of } F \\ N_{f'}(A) & \text{if not .} \end{cases}$$

Proof. It suffices to prove the assertion for the case in which A is a G-flat, so that $N_f(A) = n_f(A) - \max\{n_f(B): B \lessdot A\}$.

Case 1. Let A be a minimal flat of F . Then $n_f(A) = n_{f'}(A) + 1$ because A is in F ; and if $B \lessdot A$, then B is not in F , so $n_f(B) = n_{f'}(B)$. Thus $n_f(A) - n_f(B) = n_{f'}(A) - n_{f'}(B) + 1$. This is true for all $B \lessdot A$, so $N_f(A) = N_{f'}(A) + 1$.

Case 2. Suppose A is not in F . Then $n_f(A) = n_{f'}(A)$, and the same is true for any flat $B \subseteq A$. So $N_f(A) = N_{f'}(A)$.

Case 3. Suppose A is in F but is not a minimal flat of F. Then there is a flat $B \lessdot A$ with B in F. We have $n_f(A) = n_{f'}(A) + 1$ because A is in F. Moreover, $\max \{n_f(B): B \lessdot A\}$ and $\max \{n_{f'}(B): B \lessdot A\}$ are both attained at flats B in F (not necessarily at the same flat), because if B is not in F, then $n_f(B) = n_{f'}(B) = 0$. (This is the only place in the proof where properties of the Higgs lift are used.) And if B is in F, then $n_f(B) = n_{f'}(B) + 1$. Hence $N_f(A) = N_{f'}(A)$.

Corollary 2. If

$$H = G_0 \xrightarrow{F_1} G_1 \xrightarrow{F_2} \cdots \xrightarrow{F_n} G_n = G \tag{3}$$

is the Higgs factorization of the strong map $f: H \to G$, then $N_f(A)$ is the number of modular filters among $F_1, \ldots, F_n$ in which $\bar{A}$ is a minimal flat.

Theorem 3. The factorization (3) is the Higgs factorization of $f: H \to G$ if and only if $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n$.

Proof. Suppose (3) is the Higgs factorization. Then A is in F_j iff $r_j(A) < f_H(A)$. But $r_{j+1}(A) \le r_j(A)$, and so if A is in F_j, then $r_{j+1}(A) < r_H(A)$, which implies that A is in F_{j+1}. Thus $F_j \subseteq F_{j+1}$.

Conversely, suppose $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n$; we show that A is in F_j iff $r_H(A) > r_j(A)$, so that G_{j-1} is the Higgs lift of $f_j: H \to G_j$. If A is in F_j, then $r_j(A) = r_{j-1}(A) - 1 < r_{j-1}(A) \le r_H(A)$. And if A is not in F_j, then A is not in $F_1, F_2, \ldots, F_j$, so that $r_j(A) = r_H(A)$.

Theorem 4. If (3) is the Higgs factorization of $F: H \to G$, then

$$F_j = \{A: n_f(A) > n-j\}, \quad j = 1, \ldots, n.$$

Proof. If A is in F_j, then A is in F_j, $F_{j+1}, \ldots, F_n$; so $r_G(A) = r_{j-1}(A) - (n - j + 1)$. Thus $r_H(A) \geq r_{j-1}(A) > r_G(A) + n-j$, so $n_f(A) > n - j$.

On the other hand, if A is not in F_j, then A is not in $F_1, \ldots, F_{j-1}, F_j$; so $r_H(A) = r_j(A) \leq r_{j+1}(A) + 1 \leq r_{j+2}(A) + 2 \leq \ldots \leq r_n(A) + n-j$, i.e. $n_f(A) \leq n - j$.

4. Minimality of the Higgs Factorization.

The results of this section appear in a slightly different context in [8], using modular ideals and free quotients, which are dual to modular filters and Higgs (free) lifts.

Lemma 5. Let $L \xrightarrow{F} F$ and $L' \xrightarrow{F'} G$ be elementary strong maps. Then $L \leq L'$ in the weak-map partial order if and only if $F \supseteq F'$.

Proof. For an arbitrary pair of elementary maps as given, the following are true.

(1) If A is in both F and F' or in neither of F and F', then $r_L(A) = r_{L'}(A)$.

(2) If A is in F but not F', then $r_L(A) - r_G(A) = 1$ while $r_{L'}(A) - r_G(A) = 0$; thus $r_L(A) > r_{L'}(A)$.

(3) Similarly, if A is in F' but not F, then $r_L(A) < r_{L'}(A)$.

Thus $L \leq L'$ iff $r_L(A) \geq r_{L'}(A)$ for all A, i.e. iff $F' - F = \emptyset$, i.e. iff $F \supseteq F'$.

Proposition 6. If L is the Higgs lift and L' any lift of a strong map $H \to G$, then $L \le L'$.

Proof. Let F and F' be the associated modular filters in L and L', respectively. $F = \{A: r_H(A) > r_G(A)\}$. Now if A is in F', then $r_{L'}(A) = r_G(A) + 1$, so $r_H(A) > r_G(A)$, so A is in F. Thus $F' \subseteq F$, and thus $L \le L'$.

Proposition 7. Let $H \to L_1 \xrightarrow{F_1} G_1$ and $H \to L_2 \xrightarrow{F_2} G_2$ be Higgs-lifted strong maps. If $G_1 \le G_2$, then $L_1 \le L_2$.

Proof. $r_{G_1}(A) - r_{G_2}(A) \ge 0$ for all A, and

$$r_{L_i}(A) = \begin{cases} r_{G_i}(A) + 1 & \text{if } A \text{ is in } F_i \\ r_{G_i}(A) & \text{if not .} \end{cases}$$

We need to show $r_{L_1}(A) \ge r_{L_2}(A)$ for all A. But

$$r_{L_1}(A) - r_{L_2}(A) = \begin{cases} r_{G_1}(A) - r_{G_2}(A) & \text{if } A \in F_1 \cap F_2 \text{ or } A \notin F_1 \cup F_2 \\ r_{G_1}(A) - r_{G_2}(A) + 1 & \text{if } A \in F_1 - F_2 \\ r_{G_1}(A) - r_{G_2}(A) - 1 & \text{if } A \in F_2 - F_1 . \end{cases}$$

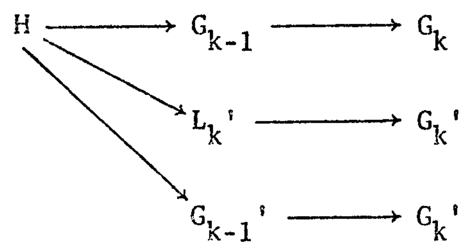
So the result will follow if we show $F_2 - F_1 = \emptyset$. But $F_2 - F_1$ is the set of all sets A for which $r_{G_2}(A) > r_H(A)$ and $r_{G_1}(A) = r_H(A)$, so that $A \in F_2 - F_1$ implies $r_{G_2}(A) > r_{G_1}(A)$, which is impossible since $G_1 \le G_2$ by hypothesis. So $F_1 - F_2 = \emptyset$.

Theorem 8. If (3) is the Higgs factorization and

$$H = G_0' \xrightarrow{F_1'} G_1' \xrightarrow{F_2'} \cdots \xrightarrow{F_{n-1}'} G_n' = G$$

is any other factorization of the strong map $H \to G$, then $G_j \le G_j'$ in the weak-map partial order, $j = 1, \dots ,n$.

<u>Proof</u>. Certainly $G_n \le G_n'$, since both equal G ; and $G_{n-1} \le G_{n-1}'$ by Proposition 6. Suppose inductively that $G_k \le G_k'$; we show that $G_{k-1} \le G_{k-1}'$. Consider the following three lifted strong maps:



where L_k' is the Higgs lift of the map $H \to G_k'$. We have $G_k \le G_k'$ by Proposition 7, since G_{k-1} and L_k' are Higgs lifts. And $L_k' \le G_{k-1}'$ by Proposition 6, since L_k' is the Higgs lift of $H \to G_k'$.

<u>Corollary 9</u>. Let (3) be the Higgs factorization of $H \to G$. Then

$$G^* = G_n^* \to G_{n-1}^* \to \dots \to G_1^* \to G_0^* = H^*$$

is the Higgs factorization of $G^* \to H^*$.

<u>Proof</u>. Let

$$G^* = G_n'^* \to G_{n-1}'^* \to \dots \to G_1'^* \to G_0'^* = H^*$$

be any other factorization of $G^* \to H^*$. Then

$$H = G_0' \to G_1' \to \dots \to G_{n-1}' \to G_n' = G$$

is a factorization of $H \to G$. By Theorem 8, $G_j \le G_j'$, $j = 1, \dots ,n$. By the theorem of Lucas mentioned in Section 2 above, $G_j^* \le G_j'^*$, $j = 1, \dots ,n$. The result follows.

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