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CONDITIONAL FAILURE TIME DISTRIBUTIONS UNDER COMPETING RISK THEORY WITH DEPENDENT FAILURE TIMES AND PROPORTIONAL HAZARD RATES*

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ABSTRACT

Suppose that death (or any non repetitive event) can occur due to various causes, each having its own failure time. Assuming *independence* of failure times and *proportional* hazard rates over the whole range of time, some authors shown that the single cause failure time distributions conditional on the cause of death, each in presence of the remaining causes, are the same as the distribution of observable failure time, regardless of the cause. It has been shown in the present article, that this result is also valid *without* the assumption of independence (Section 3).

It has also been suggested (Section 5), that in the case of *dependent* failure time, a *conditional* limiting distribution (as $T_\alpha \to \infty$) could represent the failure time distribution when cause C_α is eliminated. Three examples (trivariate exponential, bivariate Gompertz, and U.S. Life Table 1959-61 data) are given as illustrations.

1. INTRODUCTION

Consider a population in which k *causes* of death, $C_1, C_2, \ldots, C_k$, are operating. Each individual in this population is exposed to risk of dying of any one of these causes. Methods of analyzing mortality experiences in such a population are known as *competing risk analysis*. A fundamental problem of competing risk analysis is the estimation of prolongation of life expectancy where one (or more) cause(s) of death is (are) eliminated. A useful and uptodate critical review on models which have been employed in this field, has been recently given by Gail (1975). We shall refer to his article throughout this paper.

There are at least two kinds of distribution associated with failure due to cause C_α, which are important in competing risk analysis:

(i) The failure-time distribution due to cause C_α, conditionally that C_α is the cause of death, *in presence* of other causes;

(ii) the failure-time distribution due to C_α, if C_α is acting *alone*. Of course, this latter distribution cannot be observed.

A central assumption in the analysis of competing causes is that the force of mortality due to cause C_α in the *absence* of other causes is the same as in their *presence*. It is sometimes called an assumption of *identity of forces of mortality* (see (2.18)).

To be able to treat the problem in a more formal (mathematical) way, the idea of a "due time", T_α, for death from cause C_α is often introduced. It is supposed that each individual, presumably at birth, is endowed with a set of such times, $T_1, T_2, \ldots, T_k$, one for each cause. The actual time of death is the minimum of $T_1, T_2, \ldots, T_k$. It is often assumed that the random variables, $T_1, T_2, \ldots, T_k$, are mutually *inde-*

pendent. Under this assumption, stochastic models in competing risk analysis become much simpler.

The purpose of this article is to discuss some mathematical and practical consequences when neither the assumption of "identity of forces" nor the assumption of independence hold. In particular in Section 3, we shall prove that in the class of distributions with *proportional* forces of mortality, the failure distribution due to C_α (conditional that the death occur due to C_α), in presence of other causes is identical with that due to other (unspecified) cause(s), even though the assumption of independence may not be valid.

For a joint failure time distribution of $T_1, T_2, \ldots, T_k$, when it is expressed in a parametric form, we will also suggest (in Section 5) a method of deriving failure time distributions when C_α is acting *alone*, and when C_α is *eliminated*.

To illustrate the results and the techniques, three examples: joint trivariate exponential distribution, joint bivariate Gompertz distribution and a multiple decrement life table model will be discussed in some detail.

2. BASIC DEFINITIONS AND NOTATION

To make the paper selfcontained, we shall introduce some basic concepts of competing risk analysis. We will use an approach essentially similar to that used by Gail (1975) but with a somewhat different notation which resembles (but is not identical with) the actuarial notation used in multiple decrement tables. We will be mainly concerned with the consequences of certain assumptions on the mathematical models of failure-time distributions rather than on the comparison of existing models under different assumptions.

If T_α is the failure time due to cause C_α alone, then

$$F_\alpha(t) = \Pr\{T_\alpha \leqslant t\} \quad (2.1)$$

is the cumulative *failure time distribution* due to C_α, and

$$S_\alpha(t) = \Pr\{T_\alpha > t\} = 1-F_\alpha(t) \quad (2.1a)$$

is the *survival function* from C_α, where C_α is the *only* cause of death.

In most derivations in this paper, where it is convenient, the concept of the survival function rather than of the failure time distribution will be used.

The function

$$\mu_\alpha(t) = -\frac{dS_\alpha(t)}{dt} \Big/ S_\alpha(t) = -\frac{d\log S_\alpha(t)}{dt} \quad (2.2)$$

is known as the force of mortality or the *hazard rate* (also the intensity function) of the failure time distribution.

When k causes, $C_1, C_2, \ldots, C_k$, are operating, with 'hypothetical' failure times $T_1, T_2, \ldots, T_k$ respectively, one may introduce the *joint failure time distribution*

$$F_{12\ldots k}(t_1,t_2,\ldots,t_k) = \Pr\{\bigcap_{\alpha=1}^{k}(T_\alpha<t)\} \quad (2.3)$$

and the corresponding *joint survival function*

$$S_{12\ldots k}(t_1,t_2,\ldots,t_k) = \Pr\{\bigcap_{\alpha=1}^{k}(T_\alpha>t_\alpha)\} \quad (2.4)$$

(when no ambiguity arises, we use simply the notation $S(t_1,t_2,\ldots,t_k)$). We notice that $S(0,0,\ldots,0) = 1$ and $S(\infty,\infty,\ldots,\infty) = 0$.

Methods of analyzing mortality experiences of populations when many causes of death are operating usually imply a common assumption, which we will call the *fundamental assumption* of survivorship analysis, and which is

EACH DEATH IS DUE TO A SINGLE CAUSE.

This assumption has important consequences on the underlying models of failure.

2.1. First, we cannot simultaneously observe the failure times, $T_1, T_2, \ldots, T_k$, so that $S(t_1,t_2,\ldots,t_k)$ cannot be observed nor can its form tested, without introducing further assumptions. What, in fact, can be observed is

$$W = \min(T_1,T_2,\ldots,T_k) \ . \tag{2.5}$$

The corresponding survival function is

$$S_W(t) = \Pr\{W>t\} = \Pr\{\bigcap_{\alpha=1}^{k}(T_\alpha>t)\} = S(t,t,\ldots,t) \ , \tag{2.6}$$

where t is the observed failure time. The hazard rate of the distribution defined in (2.6), which we denote by $a\mu(t)$, and which is the hazard rate due to any (unspecified) cause when all causes are operating, is

$$\begin{aligned} a\mu(t) &= -\frac{dS_W(t)}{dt}\Big/S_W(t) = -\frac{d\log S_W(t)}{dt} \\ &= -\frac{dS(t,\ldots,t)}{dt}\Big/S(t,\ldots,t) = -\frac{d\log S(t,\ldots,t)}{dt} \ . \end{aligned} \tag{2.7}$$

(The prefix a in the symbol $a\mu(t)$ can be regarded as standing for "all".) For a mathematical function $S(t_1,t_2,\ldots,t_k)$ we have

$$dS(t,t,\ldots,t) = \sum_{\alpha=1}^{k} \frac{\partial S(t_1,t_2,\ldots,t_k)}{\partial t_\alpha}\Bigg|_{\{t_j=t\}} dt \ .$$

It follows that

$$-\frac{d\log S(t,t,\ldots,t)}{dt} = \sum_{\alpha=1}^{k} -\frac{\partial \log S(t_1,t_2,\ldots,t_k)}{\partial t_\alpha}\Bigg|_{\{t_j=t\}} . \tag{2.8}$$

Let $a\mu_\alpha(t)$ denote the rate of decrement (i.e. hazard rate) due to cause C_α in *presence* of all causes operating. Clearly, from the mathematical relation (2.8), we have

$$a\mu_\alpha(t) = -\frac{\partial \log S(t_1,t_2,\ldots,t_k)}{\partial t_\alpha}\Bigg|_{\{t_j=t\}} , \tag{2.9}$$

and

$$a\mu(t) = a\mu_1(t) + a\mu_2(t) + \ldots + a\mu_k(t) . \tag{2.9}$$

It should be appreciated that (2.9) is a consequence of the fundamental assumption that each death is due to a single cause; when this assumption does not hold, (2.9) is not neceesarily true.

2.2 Integrating (2.9) over the interval $(0,t)$, we obtain

$$\int_0^t a\mu(u)du = \int_0^t a\mu_1(u)du + \ldots + \int_0^t a\mu_k(u)du . \tag{2.10}$$

Hence

$$\exp\left[-\int_0^t a\mu(u)du\right] = \exp\left[-\int_0^t a\mu_1(u)du - \ldots - \int_0^t a\mu_k(u)du\right]$$

and so

$$S_W(t) = S(t,t,\ldots,t) = G_1(t)\cdot G_2(t)\ldots G_k(t) , \tag{2.11}$$

where

$$G_\alpha(t) = \exp\left[-\int_0^t a\mu_\alpha(u)du\right] \tag{2.12}$$

for $\alpha = 1,2,\ldots,k$. (See also Gail (1975)).

Imagine a 'hypothetical' population in which cause C_α is *acting alone* with force of mortality $a\mu_\alpha(t)$. Let T'_α be the failure time associated with C_α in the "imaginary" population.

We can see from (2.11) that $S_W(t)$ can be calculated as if the event $(W>t)$ were equivalent to the event $[\bigcap_{\alpha=1}^{k}(T'_\alpha>T)]$, where $T'_1, T'_2, \ldots, T'_k$ are mutually independent and T'_α has the survival function $G_\alpha(t)$. Of course, this does not mean that $T_1, T_2, \ldots, T_k$ are necessarily mutually independent, or that T_α necessarily has the survival function $G_\alpha(t)$.

2.3. Suppose that we are interested in the failure time distribution due to all causes operating, *except cause* C_α . Without loss of generality take $\alpha = 1$.

One way of handling this problem, would be *to ignore* deaths due to C_α . This does not mean that there are no such deaths; it only means that we are not interested in them and omit them in the analysis.

We need to modify the notation, which may become rather awkward for some functions.

In fact, the random variable W defined in (2.5) is the minimum of k variates, $T_1, T_2, \ldots, T_k$, and it can be denoted by W_k , i.e.

$$W_k = \min(T_1,T_2,\ldots,T_k) \quad . \tag{2.4a}$$

When only $k-1$ causes are operating, (ignoring C_1), then we may have

$$W_{k-1} = \min(T_2,T_3,\ldots,T_k) \quad . \tag{2.13}$$

From (2.6), we have

$$S_{W_k}(t) = S_{12\ldots k}(t,t,\ldots,t) \quad . \tag{2.6a}$$

When C_1 is ignored, the survival function from the remaining causes is the *marginal* survival function $S_{12...k}(0,t,...,t)$, so that

$$S_{W_{k-1}}(t) = S_{23...k}(t,...,t) = S_{12...k}(0,t,...,t) \tag{2.14}$$

Applying a similar argument to $S_{W_{k-1}}(t)$, we obtain a *new* set of hazard rates due to k-1 causes together, and due to each cause separately which, in general, would not be the same as in the case of the original k causes. Finally,

$$S_{W_{k-1}}(t) = H_2(t) \cdot H_3(t) \cdots H_k(t) , \tag{2.15}$$

and, in general,

$$H_\beta(t) \neq G_\beta(t) . \tag{2.16}$$

(See also Example 4.1).

Ignoring two causes (e.g. C_1, C_2), three causes (e.g. C_1, C_2, C_3), ... etc. (k-1)-causes (e.g. $C_1, C_2, \ldots, C_{k-1}$), we obtain ultimately

$$S_{W_1}(t) = S_{12...k}(0,0,...,t) = S_k(t) , \tag{2.17}$$

where $S_k(t) = \Pr\{T_k > t\}$ is the *marginal* survival function due to cause C_k *alone* when other causes are *ignored*, with hazard rate $\mu_k(t)$ defined in (2.2). Of course, the results are valid for any cause C_α, since the subscript is immaterial.

2.4. As we have mentioned before, $S(t_1,\ldots,t_k)$ cannot be observed nor its form tested, so that the results discussed above have only theoretical meaning. In applications, and in particular, in actuarial work a further assumption is made, namely,

$$a\mu_\alpha(t) \equiv \mu_\alpha(t) \ , \quad \alpha = 1, 2, \ldots, k \tag{2.18}$$

known as the assumption on *identity of forces of mortality*. This does imply

$$S_W(t) = S_{12\ldots k}(t,t,\ldots,t) = S_1(t) \cdot S_2(t) \cdots S_k(t) , \tag{2.19}$$

that is, that the *events* $(T_\alpha > t)$, $\alpha = 1, 2, \ldots, k$, are *independent*. Of course, is does not necessarily mean that the random variables, $T_1, T_2, \ldots, T_k$, are independent (see also Gail (1975)). However, to the best of my knowledge no counter-examples, i.e. such a situation that (2.19) is true, and

$$S_{12\ldots k}(t_1,t_2,\ldots,t_k) = S_1(t_1) \cdot S_2(t_2) \cdots S_k(t_k)$$

does not hold, has been produced.

If assumption (2.18) is true, then we have

$$G_\alpha(t) = S_\alpha(t) \ , \quad \alpha = 1, 2, \ldots, k \ . \tag{2.20}$$

3. SURVIVAL FUNCTIONS FROM DIFFERENT CAUSES WHEN THE HAZARD RATES ARE PROPORTIONAL

3.1. Of special interest is the situation when the ratio of hazard rates for any two causes is *constant* over the whole range $(0,\infty)$. This implies that

$$a\mu_\alpha(t) = c_\alpha \cdot a\mu(t) \ , \quad \alpha = 1, 2, \ldots, k \tag{3.1}$$

with $\sum_\alpha c_\alpha = 1$, since $\sum_\alpha a\mu_\alpha(t) = a\mu(t)$. Hence

$$\begin{aligned} G_\alpha(t) &= \exp\left[-c_\alpha \int_0^t a\mu(u)du\right] \\ &= \exp\left[-\int_0^t a\mu(u)du\right]^{c_\alpha} = [S_W]^{c_\alpha} \end{aligned} \tag{3.2}$$

for $\alpha = 1, 2, \ldots, k$.

Therefore, when the hazard rates, $a\mu_\alpha(t)$, $(\alpha = 1, 2, \ldots, k)$, are *proportional* the evaluation of $G_\alpha(t)$ present no difficulty, provided that the survival function from all causes, $S_W(t)$, is known. It should be appreciated that the assumption of proportionality over the whole range $(0,\infty)$ does note require, in the general case, parametric form of $S_W(t)$ to be known.

3.2. We shall now prove another useful result under the proportionality assumption.

Let $Q_\alpha(t)$ denote the probability of dying from C_α *alone*, but *in presence* of other causes, in the interval $(0,t)$. ($Q_\alpha(t)$ is sometimes called the 'crude' probability). Thus

$$Q_\alpha(t) = \int_0^t a\mu_\alpha(u)\ S_W(u)du \ . \tag{3.3}$$

In particular, the proportion of all deaths which are due to cause C_α is

$$Q_\alpha(\infty) = \int_0^\infty a\mu_\alpha(u)\ S_W(u)du \ . \tag{3.4}$$

Let $C_{(-\alpha)}$ denote all causes excluding C_α , and $F_{\alpha;(-\alpha)}(t)$ denote the (conditional) failure time distribution for deaths due to C_α in the presence of $C_{(-\alpha)}$. Then we have

$$F_{\alpha;(-\alpha)}(t) = Q_\alpha(t)/Q_\alpha(\infty) \ . \tag{3.5}$$

The corresponding survival function is

$$S_{\alpha;(-\alpha)}(t) = 1 - F_{\alpha;(-\alpha)}(t) \ . \tag{3.6}$$

Under the assumption that the hazard rates are proportional, we have

$$a\mu_\alpha(t) = c_\alpha a\mu(t) \quad .$$

It follows that

$$Q_\alpha(t) = c_\alpha \int_0^t a\mu(u)\ S_W(u)du \tag{3.7}$$

and

$$Q_\alpha(\infty) = c_\alpha \quad . \tag{3.7a}$$

Hence

$$\begin{aligned} S_{\alpha;(-\alpha)}(t) &= 1 - Q_\alpha(t)/Q_\alpha(-\infty) \\ &= 1 - \int_0^t a\mu(u)\ S_W(u)du = S_W(t) \quad , \end{aligned} \tag{3.8}$$

since

$$\int_0^t a\mu(u)\ S_W(u)du = \Pr\{W \leqslant t\} = F_W(t) \tag{3.9}$$

is the cumulative failure time distribution from any (unspecified) cause.

We have thus obtained the following result:

If the ratio of the hazard rate due to cause C_α $(\alpha=1,2,\ldots,k)$ *in the presence of the remaining causes* $C_{(-\alpha)}$ *to the hazard rate due to all causes is constant, then the failure time distributions among failures due to* C_α $(\alpha=1,2,\ldots,k)$ *in presence of* $C_{(-\alpha)}$ *are identical with the distribution of the failure time due to all causes.*

Some authors (Sethuraman (1965), David (1970), Moeschberger and David (1971)) have proved this result under the assumption of *independence* of failure times, $T_1, T_2, \ldots, T_k$. As we now can see, this assumption is not necessary to ensure (3.8).

4. EXAMPLES

EXAMPLE 4.1. *Trivariate standard exponential distribution.*

To illustrate some of the results and techniques discussed in Section 2 and 3, we will consider a multivariate generalized Farlie-Gumbel-Morgenstern family of distribtuion. (Butkiewicz(1974), Johnson and Kotz (1975).) For this family, the joint survival function (and, of course, the cumulative distribution function) can be simply expressed in terms of marginal survival functions of the individual variates.

In trivariate case with random variables X, Y, Z (which will be used in the example instead of T_1, T_2, T_3 respectively), having marginal survival functions $S_1(x)$, $S_2(y)$, $S_3(z)$, we have

$$S_{123}(x,y,z) = S_1(x) \cdot S_2(y) \cdot S_3(z)$$
$$\times [1 + \theta_{12} F_1(x) F_2(y) + \theta_{13} F_1(x) F_3(z)$$
$$+ \theta_{23} F_2(y) F_3(z) - \theta_{123} F_1(x) F_2(y) F_3(z)] , \qquad (4.1)$$

with the restriction

$$|\theta_{13} + \theta_{23} \pm \theta_{123}| \leqslant 1 + \theta_{12}$$

and two similar conditions, and

$$|\theta_{123}| \leqslant 1 + \theta_{12} + \theta_{13} + \theta_{23} .$$

(See Johnson and Kotz (1975).)

The joint density for (4.1) is

$$f_{123}(x,y,z) = f_1(x)\cdot f_2(y)\cdot f_3(z)$$

$$\times \Big\{1 + \theta_{12}[1-2F_1(x)][1-2F_2(y)] + \theta_{13}[1-2F_1(x)][1-2F_3(z)]$$

$$+ \theta_{23}[1-2F_2(y)][1-2F_3(z)] + \theta_{123}[1-2F_1(x)][1-2F_2(y)][1-2F_3(z)]\Big\} \quad (4.2)$$

where $F_1(x) = 1 - S_1(x)$ and $f_1(x) = -\frac{dS_1(x)}{dx}$, and similarly for variates Y and Z.

4.1.1. We will consider here the case when the marginal distributions are *standard exponential*, that is

$$S_1(x) = e^{-x}, \quad F_1(x) = 1-e^{-x}, \quad f_1(x) = e^{-x},$$

and similarly for Y and Z. We also take $\theta_{12} = \theta_{13} = \theta_{23} = \theta_{123} = \theta > 0$. (In fact we must then have $0 < \theta < \frac{1}{2}$).

Then the trivariate joint survival function is

$$S_{123}(x,y,z) = S(x,y,z) = e^{-(x+y+z)}[1+\theta(1-e^{-x})(1-e^{-y})+\theta(1-e^{-x})(1-e^{-z})$$

$$+\theta(1-e^{-y})(1-e^{-z})-\theta(1-e^{-x})(1-e^{-y})(1-e^{-z})], \quad (4.3)$$

and the joint density is

$$f_{123}(x,y,z) = f(x,y,z) = e^{-(x+y+z)}[1+\theta(2e^{-x}-1)(2e^{-y}-1)+\theta(2e^{-x}-1)(2e^{-z}-1)$$

$$+\theta(2e^{-y}-1)(2e^{-z}-1)+\theta(2e^{-x}-1)(2e^{-y}-1)(2e^{-z}-1)]. \quad (4.4)$$

The survival function from *all* causes at the observed time t, $S_{W_3}(t)$, is

$$S_{W_3}(t) = S(t,t,t) = e^{-3t}[1+\theta(2+e^{-t})(1-e^{-t})^2]. \quad (4.5)$$

The hazard rate due to all causes, $a\mu(t)$, is

$$a\mu(t) = -\frac{d\log S_{W_3}(t)}{dt} = 3\left[1-\theta\frac{e^{-t}(1-e^{-2t})}{1+\theta(2+e^{-t})(1-e^{-t})^2}\right]. \tag{4.6}$$

The hazard rate for cause C_1 (i.e. associated with failure time X), is

$$a\mu_1(t) = -\left.\frac{\partial\log S(x,y,z)}{\partial x}\right|_{x=y=z=t}$$

$$= 1 - \frac{\theta e^{-t}(1-e^{-2t})}{1+\theta(2+e^{-t})(1-e^{-t})^2} = \frac{1}{3}\,a\mu(t)\ , \tag{4.7}$$

and exactly the same expression holds for $a\mu_2(t)$ and $a\mu_3(t)$.

We then have

$$a\mu_1(t) = a\mu_2(t) = a\mu_3(t) = \frac{1}{3}\,a\mu(t)\ , \tag{4.8}$$

that is, we have a situation in which the hazard rates, $a\mu_\alpha(t)$, are *proportional* over the whole range $(0,\infty)$, with proportionality coefficients $c_1 = c_2 = c_3 = \frac{1}{3}$.

From (3.2), we obtain

$$G_\alpha(t) = [S_{W_3}(t)]^{\frac{1}{3}} = e^{-t}[1+\theta(2+e^{-t})(1-e^{-t})^2]^{\frac{1}{3}} \tag{4.9}$$

for $\alpha = 1, 2, 3$.

From (3.8), the survival function from C_1 *alone* in *presence* of C_2 and C_3 , $S_{1;(-1)}(t)$ is identical with $S_{W_3}(t)$ given by (4.5), that is

$$S_{1;(-1)}(t) = S_{W_3}(t) = e^{-3t}[1-\theta(2+e^{-t})(1-e^{-t})^2]\ , \tag{4.10}$$

which also can be checked by direct calculations. The same expression holds for $S_{2;(-2)}(t)$ and $S_{3;(-3)}(t)$.

4.1.2. Suppose that we now *ignore* cause C_3. Then the joint survival function from C_1 and C_2, $S_{12}(x,y)$, is a bivariate *marginal* of $S_{123}(x,y,z)$, that is

$$S_{12}(x,y) = S_{123}(x,y,0) = e^{-(x+y)}[1+\theta(1-e^{-x})(1-e^{-y})] , \tag{4.11}$$

and the bivariate density function is

$$f_{12}(x,y) = e^{-(x+y)}[1+\theta(2e^{-x}-1)(2e^{-y}-1)] . \tag{4.12}$$

Using the same techniques as in the case of trivariate distribution, we obtain the survival function, $S_{W_2}(t)$, from any of the two causes, C_1 of C_2,

$$S_{W_2}(t) = e^{-2t}[1+\theta(1-e^{-t})^2] . \tag{4.13}$$

Let $a\mu(t;3)$ denote the hazard rate from the remaining causes (C_1 and C_2) together, when C_3 is ignored, and $a\mu_1(t;3)$ the hazard rate from cause C_1 in presence of C_2 when C_3 is ignored, and similarly $a\mu_2(t;3)$.

We have

$$a\mu(t;3) = -\frac{d\log S_{W_2}(t)}{dt} = 2\left[1 - \frac{\theta e^{-t}(1-e^{-t})}{1+\theta(1-e^{-t})^2}\right] , \tag{4.14}$$

and

$$a\mu_1(t;3) = -\left.\frac{\partial S_{12}(x,y)}{\partial x}\right|_{x=y=t} = 1 - \frac{\theta e^{-t}(1-e^{-t})}{1+\theta(1-e^{-t})^2} ,$$

$$= \frac{1}{2}\ a\mu(t;3) , \tag{4.15}$$

and the same expression for $a\mu_2(t;3)$.

Thus

$$a\mu_1(t;3) = a\mu_2(t;3) = \frac{1}{2}\, a\mu(t;3) \ , \qquad (4.16)$$

and $H_\alpha(t)$ - which can be denoted here by $G_\alpha(t;3)$ - is

$$G_\alpha(t;3) = [S_{W_2}(t)]^{\frac{1}{2}} = e^{-t}[1+\theta(1-e^{-t})^2]^{\frac{1}{2}} \ , \qquad (4.17)$$

$\alpha = 1,2$.

Clearly, from (4.9) and (4.17)

$$G_\alpha(t;3) \neq G_\alpha(t) \quad \text{for} \quad \alpha = 1,2 \ .$$

4.1.3. Finally, if we ignore the mortality from C_2 and C_3 , and observe only deaths from C_1 , the survival function, $S_{W_1}(t)$, is the univariate standard exponential, that is $S_{W_1}(t) = S_1(t) = e^{-t}$ with the hazard rate $a\mu_1(t;2,3) = \mu_1(t) = 1$, and the density function $f_1(t) = e^{-t}$.

EXAMPLE 4.2. *Life tables by causes of death.*

Multiple decrement life tables, distinguishing different causes of death are special examples of competing risk analysis. These tables usually include various 'standard' columns. Among these, the column headed al_x represents the number of survivors at the exact age x out of al_0 who started life at age 0 . The al_x column (or more precisely, the proportions al_x/al_0) corresponds to the survival function $S_W(t)$ defined in (2.6) and evaluated at time points $t = x$. The ad_x column gives the total number of deaths between age x and $x + 1$, and different $ad_x^{(\alpha)}$ columns give the corresponding numbers of deaths from different causes.

We may notice that

$$al_x^{(\alpha)} = \sum_{\alpha=x}^{\infty} ad_x^{(\alpha)} \qquad (4.18)$$

for $\alpha = 1, 2, \ldots, k$ gives the number of individuals among al_x now aged x, who eventually die from cuase C_α . (The $al_x^{(\alpha)}$ values are usually not explicitly given in the table). Thus $al_x^{(\alpha)}$ (or more precisely, $al_x^{(\alpha)}/al_0^{(\alpha)}$) corresponds to the survival function for individuals dying from cause C_α in presence of other causes, i.e. to $S_{\alpha;(-\alpha)}(t)$, evaluated at $t = x$.

If the hazard rates due to various causes were proportional, then the ratios al_x/al_0 and $al_x^{(\alpha)}/al_0^{(\alpha)}$ should be approximately the same (in view of the result (3.8)), for each x $(0 < x < \infty)$. This also means that the ratios $al_x^{(\alpha)}/al_x$ should be approximately equal to $al_0^{(\alpha)}/al_0$, for all values of x . In fact, $al_x^{(\alpha)}/al_x$ represents the proportion of individuals of present age x , who eventually will die from cause C_α ; these figures are sometimes given in multiple decrement tables.

Table 4.1 in this paper, is an extract from U.S. Life Tables by Causes of Death: 1959-61, p. 44, (1968). These are abridged tables, with most ages at quinquennial intervals.

Table 4.1 gives the proportions $al_x^{(\alpha)}/al_x$ for: malignant neoplasms (C_1) , and major cardiovascular-renal diseases (C_2) .

We notice that for malignant neoplasms, in the range of age 0-60 this ratio is fairly constant, but it decreases somewhat for older ages. On the other hand, for cardiovascular diseases, this ratio is an increasing function of age so that the proportionality assumption for this cause does not hold.

TABLE 4.1

The ratios $al_x^{(\alpha)}/al_x$ for malignant neoplasms and cardio-vascular-renal diseases. (U.S. Life Table 1959-61, Total population)

Age (x)	Malignant neoplasms	Card.-vasc. -renal diseases	Age (x)	Malignant neoplasms	Card.-vasc. -renal diseases
0	.15154	.61075	45	.15557	.65404
1	.15551	.62688	50	.15332	.66061
5	.15574	.62942	55	.14921	.66917
10	.15573	.63084	60	.14285	.67977
15	.15576	.63208	65	.13362	.69269
20	.15608	.63466	70	.12154	.70830
25	.15659	.63805	75	.10719	.72539
30	.15688	.64127	80	.09113	.74260
35	.15693	.64478	85	.07349	.75728
40	.15658	.64892			

5. IS IGNORING THE DEATHS FROM A GIVEN CAUSE EQUIVALENT TO THE ELIMINATION OF THIS CAUSE?

As was mentioned in Section 3 (see also Gail (1975)), one way of evaluating the survival function when cause C_α is actually eliminated, is to *ignore* the deaths from this cause. In other words, just to consider the *marginal* survival function associated with the remaining causes $C_{(-\alpha)}$.

We now ask: is the effect of C_α actually eliminated by this method? The deaths due to C_α are still occurring but we just do not take notice of them.

It seems more reasonable to think that if we can achieve such a level of health care that a disease associated with C_α, is under control and in practice nobody dies from it, then we effectively *eliminate* C_α as a cause of death. Small pox can serve as an example.

The way in which this comes about is that the competing T_α becomes very large. To present this problem in a formal way, we say that if the limiting (*conditional on* T_α), distribution of surviving from $C_{(-\alpha)}$, given $T_\alpha \to \infty$, exists, then this might be the survival function from other causes, when C_α is *eliminated*. We then should have

$$\lim_{t_\alpha \to \infty} \Pr\{\bigcap_{j=1}^{k-1} (T_j > t) | T_\alpha = t_\alpha\} = S_{(-\alpha)|\infty}(t|\infty) . \qquad (5.1)$$

We may extend this concept to the elimination of two, three,... etc. (k-1) causes, so that finally the survival function from the cause C_β, say, given the effects of the remaining causes was eliminated could be defined as

$$\lim_{t_\alpha \to \infty} \Pr\{T_\beta > t | T_\alpha = t_\alpha \; ; \; \alpha = 1, 2, \ldots ; \alpha \neq \beta\} = S_{\beta|\underset{\sim}{\infty}}(t|\underset{\sim}{\infty}) \qquad (5.2)$$

This is to say, that (5.2) represents the survival function from C_β *alone* when other causes were eliminated (i.e. their failure times were "postponed" to infinity).

Of course, if the assumption of independence of failure times, $T_1, T_2, \ldots, T_k$, is valid then we have

$$\Pr\{T_\beta > t_\beta | T_\alpha = t_\alpha \ ; \ \alpha = 1, 2, \ldots, \alpha \neq \beta\} = S_\beta(t) \ , \tag{5.3}$$

so that the methods described in Section 3 and in this Section produce equivalent results.

We now present two examples, in which the method of conditional vs. marginal distribution, when a cause of death is eliminated, will be discussed.

EXAMPLE 5.1. We consider the *trivariate exponential distribution* from Examples 4.1.

5.1.1. Suppose that cause C_3 is *eliminated*, i.e. $Z \to \infty$. The conditional limiting density function, $f_{12|\infty}(x,y|\infty)$, is

$$f_{12|\infty}(x,y|\infty) = \lim_{z\to\infty} f_{12|3}(x,y,|Z=z) = \lim_{z\to\infty} \frac{f_{123}(x,y,z)}{f_3(z)}$$

$$e^{-(x+y)}[1-\theta(2e^{-x}-1) - \theta(2e^{-y}-1)] \ , \tag{5.4}$$

where $f_{123}(x,yz)$ is given by (4.4) and $f_3(z) = e^{-x}$.

The (conditional) bivariate 'survival function', $S_{12|\infty}(x,y|\infty)$, is

$$S_{12|\infty}(x,y|\infty) = \int_x^\infty\int_y^\infty e^{-(x+y)}[1-\theta(2e^{-x}-1)-\theta(2e^{-y}-1)]dydx$$

$$= e^{-(x+y)}[1+\theta(1-e^{-x}) + \theta(1-e^{-y})] \ . \tag{5.5}$$

and the survival function $S_{(-3)|\infty}(t|\infty)$, is

$$S_{(-3)|\infty}(t|\infty) = S_{12|\infty}(t,t|\infty) = e^{-2t}[1+2\theta(1-e^{-t})] \qquad (5.6)$$

This is different from that given by (4.13).

5.1.2. Suppose that *two* causes, C_2 and C_3, are eliminated, i.e. $y \to \infty$, $x \to \infty$.

The conditional limiting density function, $f_{1|\infty,\infty}(x|\infty,\infty)$, is

$$f_{1|\infty,\infty}(x,\infty,\infty) = \lim_{\substack{x\to\infty \\ y\to\infty}} \frac{f_{123}(x,y,z)}{f_{12}(x,y)} = e^{-x}[1 - \frac{\theta}{1+\theta}(2e^{-x}-1)] , \qquad (5.7)$$

where $f_{123}(x,y,z)$ is given by (4.4) and $f_{12}(x,y)$ by (4.12).

Thus the survival function, $S_{1|\infty\ \infty}(t|\infty,\infty)$ from cause C_1 *alone*, when C_2 and C_3 are eliminated, is

$$S_{1|\infty,\infty}(t|\infty,\infty) = \int_t^{\infty} e^{-x}[1 - \frac{\theta}{1+\theta}(2e^{-x}-1)]\,dx$$

$$= e^{-t}[1 + \frac{\theta}{1+\theta}(1-e^{-t})] , \qquad (5.8)$$

which is different from the marginal e^{-t}.

EXAMPLE 5.2. *Bivariate Gompertz distribution.*

We wish to consider here some modification of bivariate extreme-value Gumbel distribution. Gumbel (1965) (see also Johnson and Kotz (1971), p. 251) suggested a general bivariate form

$$F_{12}(x,y) = F_1(x)\cdot F_2(y)\ \exp\left\{-\theta\left[\frac{1}{\log F_1(x)} + \frac{1}{\log F_2(y)}\right]\right\} , \qquad (5.9)$$

$$\text{for}\quad 0 < \theta < 1 ,$$

where $F_1(x)$, $F_2(y)$ are univariate extreme-value distributions. We suggest here some modification of (5.9). First, instead of $F_{12}(x,y)$, $F_1(x)$ and $F_2(x)$, we use $S_{12}(x,y)$, $S_1(x)$ and $S_2(y)$ respectively. (Note that this produces a different distribution that (5.9).)

Second, we shall use in this example, univariate Gompertz distributions, each being a truncated (from below at time $x = 0$, or $y = 0$) extreme-value Type 1 distribution.

The algebra is rather lengthy, and we confine ourselves to presenting only the final results.

The hazard rates and survival functions for the two marginal Gompertz distributions are:

$$\mu_1(x) = R_1 e^{a_1 x} , \quad S_1(x) = \exp\left[\frac{R_1}{a_1}\left(1-e^{a_1 x}\right)\right] ; \tag{5.10}$$

$$\mu_2(y) = R_2 e^{a_2 y} ; \quad S_2(y) = \exp\left[\frac{R_2}{a_2}\left(1-e^{a_2 y}\right)\right] . \tag{5.10a}$$

The joint survival function, $S_{12}(x,y)$ is,

$$S_{12}(x,y) = \exp\left\{\frac{R_1}{a_1}\left(1-e^{a_1 x}\right) + \frac{R_2}{a_2}\left(1-e^{a_2 y}\right) - \theta \frac{\frac{R_1}{a_1}\cdot\frac{R_2}{a_2}\left(1-e^{a_1 x}\right)\left(1-e^{a_2 y}\right)}{\frac{R_1}{a_1}\left(1-e^{a_1 x}\right) + \frac{R_2}{a_2}\left(1-e^{a_2 y}\right)}\right\} . \tag{5.11}$$

Putting $x = y = t$, we obtain the 'observable' survival function $S_{W_2}(t) = S_{12}(t,t)$.

After rather lengthy algebra, we find the limiting survival function, $S_{1|\infty}(t|\infty)$, that is

$$S_{2|\infty}(t|\infty) = \exp[(1-\theta)R_2 e^{a_2 t}] \tag{5.12}$$

with the hazard rate, $\mu_{2|\infty}(t)$, say,

$$\mu_{2|\infty}(t) = (1-\theta)R_2 e^{a_2 t} , \tag{5.13}$$

which is again a Gompertz distribution with parameters a_2 and $R_2' = (1-\theta)R_2$.

6. CONCLUDING REMARKS

The results presented in this paper are intended as contributions to both theoretical and practical aspects of competing risk analysis.

(i) Various kinds of survival functions from single or combined causes, in presence or absence of other causes were introduced and defined using formal distribution theory. Example 4.2 illustrates their relationship to the multiple decrement life table functions.

(ii) Clearly, such random variables as "times due to die" from different causes cannot be observed directly, and so the survival functions, $S(t_1,t_2,\ldots,t_k)$, $S_\alpha(t)$ or $S_{\alpha|\infty}(t|\infty)$, $\alpha = 1, 2, \ldots, k$, cannot be estimated without additional assumptions. Nevertheless the analysis enhances the investigator's awareness of the possible consequences of relationships among survival functions under special assumptions (for example, the assumption $a\mu_\alpha(t) = \mu_\alpha(t)$).

(iii) In constructing multiple decrement life tables, various assumptions on the behavior of the al_x function in small intervals are made. The most common are: uniform distribution of deaths, constant hazard rate, and proportional hazard rates from different causes (e.g. Chiang (1961)). It would be probably possible to design some Monte Carlo studies to assess the effects of these assumptions. Especially some parametric models such as exponential of Gompertz (see Examples 4.1, 5.1 and 5.2) would be useful to consider.

(iv) Finally it seems feasible to conduct some experiments on laboratory animals in which diseases are genetically controlled, or on mechanical devices in which time to failure of various parts depend on the strength of material of which they are made. It would then be possible to estimate directly, under some circumstances, the survival function, $S_{\alpha|\infty}(t|\infty)$ and possibly $S(t_1,...,t_k)$ for small number of causes (e.g. $k = 2$ or $k = 3$) .

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