

<sup>1</sup> This research was supported by the Air Force Office of Scientific Research under Contract #AFOSR-75-2796.

**AN IMPROVED STATEMENT OF OPTIMALITY  
FOR SEQUENTIAL PROBABILITY RATIO TESTS**

by

Gordon D. Simons<sup>1</sup>

*Department of Statistics  
University of North Carolina at Chapel Hill*

Institute of Statistics Mimeo Series #1037

November, 1975

ABSTRACT

An improved version of the optimality property of sequential probability ratio tests is stated.

---

AMS 1970 *subject classifications*. Primary 62L10

*Key words and phrases*: Optimality, sequential tests, sequential probability ratio tests, Kullback-Leibler information number.

An Improved Statement of Optimality  
for Sequential Probability Ratio Tests

by Gordon Simons<sup>1</sup>  
*Univeristy of North Carolina at Chapel Hill*

Wald and Wolfowitz (1948) first proved that sequential probability ratio tests (SPRT's) possess a strong optimality property. Improvements on their statement of optimality have been given by Burkholder and Wijsman (1960 and 1963) and by J. K. Ghosh (1961). The intent of this paper is to discuss another improvement.

We shall assume the usual iid model (See Lehmann (1959), page 97.) with respect to two probability measures  $P_0$  and  $P_1$ . In particular, every observation has the density  $p_0$  under  $P_0$ , or  $p_1$  under  $P_1$ . Let  $T$  be an SPRT  $S(A_0, A_1)$ ,  $0 < A_0 < 1 < A_1 < \infty$ , with error probabilities  $\alpha_0$  and  $\alpha_1$ , and with expected sample sizes  $E_0N$  and  $E_1N$ . Further, let  $T'$  be a competing test with error probabilities  $\alpha'_0$  and  $\alpha'_1$ , and with expected sample sizes  $E_0N'$  and  $E_1N'$ . The optimality property stated by Wald and Wolfowitz says that if

$$(1) \quad \alpha'_0 \leq \alpha_0 \quad \text{and} \quad \alpha'_1 \leq \alpha_1,$$

and if

---

<sup>1</sup>*This research was supported by the Air Force Office of Scientific Research under Contract # AFOSR-75-2796.*

$$(2) \quad E_0 N' < \infty \quad \text{and} \quad E_1 N' < \infty,$$

then

$$(3) \quad E_0 N \leq E_0 N' \quad \text{and} \quad E_1 N \leq E_1 N' .$$

We shall show that (1) can be replaced by the weaker assumption

$$(4) \quad \beta'_0 \leq \beta_0 \quad \text{and} \quad \beta'_1 \leq \beta_1 ,$$

where  $\beta_0 = \alpha_0 / (1 - \alpha_1)$ ,  $\beta_1 = \alpha_1 / (1 - \alpha_0)$  and the quantities  $\beta'_0$  and  $\beta'_1$  are defined analogously. The conditions in (4) imply that  $\alpha'_0 + \alpha'_1 \leq \alpha_0 + \alpha_1$  but permit one of the inequalities in (1) to be violated.

As an aside, it appears to us that these ratios (the  $\beta$ 's) may be more useful measures of error than are error probabilities. For instance, they appear in the fundamental result

$$(5) \quad \beta_1 \leq A_0 \quad \text{and} \quad \beta_0 \leq A_1^{-1} .$$

(See Lehmann (1959), page 99.) Moreover, they appear in various places in the literature on sequential tests. We shall refer to them as *normalized error probabilities*. (Observe that  $\beta_0 = \frac{P_0(\text{rejecting } H_0)}{P_1(\text{rejecting } H_0)}$  if the test always terminates. Thus, the error probability in the numerator is being normalized by a probability of non-error in the denominator.) Besides the result we have mentioned, we shall discuss another well known result which can be strengthened by replacing assumptions about error probabilities with assumptions about normalized error probabilities. We suspect that many other examples could be found.

Proof that (4) and (2) imply (3) For the sake of brevity, we shall draw rather freely from the notation and results appearing on pages 105 and 106 of Lehmann (1959).

The conditions in (1) are used by Wald and Wolfowitz only to conclude that

$$(6) \quad \pi w_0 \alpha_0' + (1-\pi)w_1 \alpha_1' \leq \pi w_0 \alpha_0 + (1-\pi)w_1 \alpha_1 ,$$

where  $\pi$ ,  $0 < \pi < 1$ , is the prior probability that  $p_0$  is the correct density, and where  $w_0 > 0$  and  $w_1 > 0$  are losses due to type I and type II errors, respectively. Thus, it suffices to show that (4) implies (6).

According to Figure 3 (page 106 of Lehmann),

$$\pi' \leq \frac{w_1}{w_0 + w_1} \leq \pi'',$$

where  $\pi'$  and  $\pi''$  are values in the open interval (0,1) which satisfy

$$A_0 = \frac{\pi}{1-\pi} \frac{1-\pi''}{\pi''} \quad \text{and} \quad A_1 = \frac{\pi}{1-\pi} \frac{1-\pi'}{\pi'} .$$

Thus,

$$A_0 \leq \frac{\pi w_0}{(1-\pi)w_1} \leq A_1 ,$$

which, in view of (5), implies

$$(7) \quad \beta_1 \leq \frac{\pi w_0}{(1-\pi)w_1} \leq \beta_0^{-1} .$$

Finally, (6) follows from (4) and (7). This is most easily seen by first showing that the conditions in (4) are algebraically equivalent to

$$\beta_0(\alpha_1' - \alpha_1) \leq \alpha_0 - \alpha_0' \quad \text{and} \quad \alpha_1' - \alpha_1 \leq \beta_1(\alpha_0 - \alpha_0');$$

and (6) is algebraically equivalent to

$$(\alpha_1' - \alpha_1) \leq \frac{\pi w_0}{(1-\pi)w_1} (\alpha_0 - \alpha_0'). \quad \square$$

We shall now discuss another result which can be strengthened by the introduction of normalized error probabilities. Let  $I(p_0, p_1) = E_0 \log(p_0(X)/p_1(X))$  (a Kullback-Liebler information number), where  $X$  represents an observation appearing in the random sample. Suppose  $\alpha_0^*$  and  $\alpha_1^*$  are fixed positive numbers whose sum  $\alpha_0^* + \alpha_1^* < 1$ . Then, for any test whose stopping variable is  $N$  and whose error probabilities are  $\alpha_0 \leq \alpha_0^*$  and  $\alpha_1 \leq \alpha_1^*$ ,

$$(8) \quad E_0 N \geq \frac{\alpha_0^* \log \frac{\alpha_0^*}{1-\alpha_1^*} + (1-\alpha_0^*) \log \frac{1-\alpha_0^*}{\alpha_1^*}}{I(p_0, p_1)} .$$

This follows from the well known result that

$$E_0 N \geq \frac{\alpha_0 \log \frac{\alpha_0}{1-\alpha_1} + (1-\alpha_0) \log \frac{1-\alpha_0}{\alpha_1}}{I(p_0, p_1)} ,$$

and from the fact that the function

$$g(x,y) = x \log \frac{x}{1-y} + (1-x) \log \frac{1-x}{y}, \quad 0 < x, y < 1,$$

is convex in each of its arguments. It is not difficult to show (8) holds when the conditions  $\alpha_0 \leq \alpha_0^*$  and  $\alpha_1 \leq \alpha_1^*$  are replaced by the weaker conditions

$$(9) \quad \beta_0 \leq \beta_0^* \quad \text{and} \quad \beta_1 \leq \beta_1^*,$$

where  $\beta_0^* = \alpha_0^*/(1-\alpha_1^*)$  and  $\beta_1^* = \alpha_1^*/(1-\alpha_0^*)$ . One must show that (9) implies  $g(\alpha_0, \alpha_1) \geq g(\alpha_0^*, \alpha_1^*)$ .

#### References

- [1] Burkholder, D.L. and Wijsman, R.A. (1960). Optimum properties and admissibility of sequential tests. *Ann. Math. Statist.* 31, 232 (abstract).
- [2] Burkholder, D.L. and Wijsman, R.A. (1963). "Optimum properties and admissibility of sequential tests. *Ann. Math. Statist.* 34, 1-17.
- [3] Ghosh, J.K. (1961). On the optimality of probability ratio tests in sequential and multiple sampling. *Calcutta Statist. Assoc. Bull.* 10, 73-92.
- [4] Lehmann, E.L. (1959). *Testing Statistical Hypotheses*. Wiley, New York.
- [5] Wald, A. and Wolfowitz, J. (1948). Optimum character of the sequential probability ratio test. *Ann. Math. Statist.* 19, 326-339.