ON THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR MARTINGALES

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ABSTRACT: Necessary and sufficient conditions for the functional central limit theorem for a double array of random variables are sought. It is argued that this is a martingale problem only if the variables truncated at some fixed point c are asymptotically a martingale difference array. Under this hypothesis, necessary and sufficient conditions for convergence in distribution to a Brownian motion are obtained when the normalization is given (i) by the sums of squares of the variables, (ii) by the conditional variances and (iii) by the variances. The results are proved by comparing the various normalizations with a "natural" normalization.

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1. INTRODUCTION

Since the final solution of the classical central limit problem through the works of Lévy, Lindeberg, and Feller, research on the convergence in distribution of sums of random variables has developed in two directions. Firstly the classical results have been extended to the random processes obtained by interpolating the partial sums. Secondly, results have been obtained also for dependent summands. In this context perhaps the most important dependence property is that of a martingale.

The purpose of the present paper is to find conditions that are necessary and sufficient for the functional central limit theorem for martingales. Unfortunately it is not possible to give as clearcut a solution to this problem as in the classical situation with independent summands. One reason for this is that a number of different normalizations (or time-scales) all seem reasonable, but that they lead to different results. Another reason is that any array of random variables with finite means can be made a martingale difference array by adding random variables that are zero except on a set of asymptotically negligible probability. This alteration would then not change the convergence or non-convergence of the distribution of the summation processes based on the array, so any such array could be regarded as a martingale difference array, which would make the problem of getting conditions for convergence to a Brownian motion rather meaningless. Hence one has to introduce some restriction to make the problem not only superficially a martingale problem. An appropriate restriction is given as Condition (1) of Section 2. Once this condition is introduced, our approach is very simple. We show that there is a "natural" time-scale that makes the summation process converge in distribution, and then get conditions for convergence when using other time-scales by comparing these time-scales with the natural one.

We have not attempted to find necessary conditions for the convergence to normality of the one-dimensional distribution of a normalized martingale, and in fact it seems to be difficult to find non-trivial conditions for this (cf. Dvoretsky (1972), Section 6). However, from a pedogogical point of view it may be preferable to get necessary conditions for the functional central limit theorem, since this avoids the extra assumption that the summands are asymptotically small.

Successively weaker sufficient conditions for the central limit theorem for martingales have been obtained by a number of authors. Early important results were proved by P. Lévy (see e.g. his 1937 book; ref. [9]) and much of the subsequent work has relied on methods developed by him. Billingsley (1961) and, independently Ibragimov (1962), proved convergence to normality when the martingale differences are stationary, ergodic and with finite variance. Further weakening of the conditions were made by Dvoretsky (1972), Brown (1971), and Scott (1973). Drogin (1972) considered random time-scales and also got results on necessity. Our point of view is similar to that of Drogin. The most recent results known to the present author are those of McLeish (1974) and from the sequel it can be seen that his sufficient conditions are rather close to the necessary ones.

The plan of this paper is as follows; in Section 2 the necessary notation is developed and the natural time-scale is found. In Section 3 normalization by means of sums of squares and by means of conditional variances are considered, while in Section 4 the normalization is given by variances. Finally, Section 5 contains a comment on a remaining problem.

2. THE NATURAL TIME-SCALE.

For $n = 1,2,\ldots$ let $\{X_{n,i}\}_{i=1}^{\infty}$ be a sequence of random variables on a probability space $(\Omega_n, \mathcal{B}_n, P_n)$, let $\mathcal{B}_{n,i}$ be the sub-sigmaalgebra of $\mathcal{B}_n$ that is generated by $X_{n,1},\ldots,X_{n,i}$ and put $S_n(k) = \sum_{i=1}^{k} X_{n,i}$. Furthermore let $\tau_n(t);\ t\in[0,1]$ be stopping times of $\{X_{n,i}\}_{i=1}^{\infty}$ that are increasing and right continuous in t a.s. In the sequel we will without further comment assume that

$$\tau_n(1) < \infty \quad \text{a.s.}, \quad n\geq 1.$$

Then $\{S_n \circ \tau_n(t):\ t\in[0,1]\}_{n=1}^{\infty}$ is a sequence of random variables in $D(0,1)$, the space of functions on $[0,1]$ which are right-continuous and have left-hand limits. We let $D(0,1)$ be endowed with the Skorokhod topology and let B be a Brownian motion on $D(0,1)$. For brevity we abuse notation slightly and write $S\circ\tau_n$ for $S_n\circ\tau_n$ and $E_i(\cdot)$ for $E(\cdot\,\|\,\mathcal{B}_{n,i})$ when the expectation is taken of variables in the n'th row $\big(E_0(\cdot)=E(\cdot)\big)$. The object of this paper is to find conditions for $S\circ\tau_n \xrightarrow{d} B$ when $\{X_{n,i}\}$ is a martingale difference array (m.d.a.) i.e. when $E_{i-1}(X_{n,i})=0$, $i\geq 2$, $n\geq 1$, so that the partial sums in each row form a martingale.

However, as was noted in the introduction, if the $X_{n,i}$'s have finite means then any array $\{X_{n,i}\}$ can be made a m.d.a. by adding variables which take large values, but with low probabilities, in such a way that the asymptotic distribution of $S\circ\tau_n$ is not changed. For $i>0$ put $X'_{n,i} = X_{n,i} I(|X_{n,i}|\leq c)$ and $X''_{n,i} = X_{n,i} - X'_{n,i}$. Convergence of the distribution of $S\circ\tau_n$ to a Brownian motion entails that the maximum of the summands tends to zero in probability and hence, with a probability tending to one, all the $X''_{n,i}$, $1\le i\le\tau_n(1)$ are zero. Thus we are essentially concerned only with the distribution of the array $\{X'_{n,i}\}$, and if the problem is to be a martingale one $\{X'_{n,i}\}$ has to be, at least asymptotically, a m.d.a. Formally this can be written as

$$(1)\qquad \max_{1\le k\le\tau_n(1)} \left|\sum_{i=1}^{k} E_{i-1}(X'_{n,i})\right| \xrightarrow{P} 0 \quad \text{as} \quad n\to\infty.$$

However, once this condition is assumed to hold there is no need to require that the original array, $\{X_{n,i}\}$, is a m.d.a. and this will not be done unless explicitly stated. Moreover, it is no restriction to assume that e.g. $c=1$ so that

$$X'_{n,i} = X_{n,i}I(|X_{n,i}|\le 1) \quad \text{and}$$

$$X''_{n,i} = X_{n,i}I(|X_{n,i}|>1).$$

Furthermore introduce

$$\xi_{n,i} = X'_{n,i} - E_{i-1}(X'_{n,i}).$$

Then $|\xi_{n,i}| \le 2$ a.s. and $\{\xi_{n,i}\}$ is a m.d.a. Our first result is that the sum of squares of $\{\xi_{n,i}\}$ gives a natural time-scale for the summation process. To state the result we need the further notation $M_n = \max_{1\le i\le\tau_n(1)} X_{n,i}$.

<u>THEOREM 1.</u> Let $\tau_n(t) = \inf\{k;\ \sum_{i=1}^{k}\xi^2_{n,i}>t\}$, $t\in[0,1]$, and assume that (1) is satisfied. Then $S\circ\tau_n \xrightarrow{d} B$ if and only if $M_n \xrightarrow{P} 0$.

<u>PROOF.</u> From $S\circ\tau_n \xrightarrow{d} B$ it follows immediately that $M_n \xrightarrow{P} 0$, so only the reverse implication remains to be proved. Assuming that $M_n \xrightarrow{P} 0$ it follows

from (1) that $\max_{1 \le k \le \tau_n(1)} |\sum_{i=1}^{k} X_{n,i} - \sum_{i=1}^{k} \xi_{n,i}| \xrightarrow{P} 0$ and thus, putting $S'_n(k) = \sum_{i=1}^{k} \xi_{n,i}$, it is enough to prove that $S'_n \circ \tau_n \xrightarrow{d} B$ and since $|\xi_{n,i}| \le 2$ a.s. it follows from e.g. Theorem 1 of Drogin (1972) that $S'_n \circ \tau_n \xrightarrow{d} B$. (We might as well have used the results of [3], [11], or [10], the proof in McLeish (1974) perhaps being the easiest one. Furthermore, that proof can be somewhat simplified in the present case.) □

REMARK 2. It is easy to see that an equivalent time-scale is given by $\tau_n(t) = \inf\{k; \sum_{i=1}^{k} E_{i-1}(\xi_{n,i}^2) > t\}$ (c.f. Theorem 5 below). Furthermore it should be noted that for these time-scales $\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{1} t$ as $n \to \infty$, for $t \in [0,1]$.

The main tool for the rest of the paper is Lemma 3 below which shows that $\tau_n(t) = \inf\{k; \sum_{i=1}^{k} \xi_{n,i}^2 > t\}$ is not only a natural time-scale, but that it is also in a certain sense minimal.

LEMMA 3. Let $\tau_n(t)$ be stopping times of $\{X_{n,i}\}$ that are increasing and right continuous in t a.s., and assume that (1) is satisfied. Then $S \circ \tau_n \xrightarrow{d} B$ if $M_n \xrightarrow{P} 0$ and if furthermore

(2) $$\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} t \quad \text{as} \quad n \to \infty, \quad t \in [0,1].$$

Conversely, if $S \circ \tau_n \xrightarrow{d} B$ and if furthermore

(3) $$\limsup_{n \to \infty} E\left(\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2\right) \le t, \quad t \in [0,1],*$$

then $M_n \xrightarrow{P} 0$ and (2) holds, also if convergence in probability is replaced by

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* Actually it is enough that (3) holds for $t=1$.

convergence in the mean.

PROOF. For the first part it is as in the previous proof sufficient to show that $S'_n \circ \tau_n \xrightarrow{d} B$, where $S'_n(k) = \sum_{i=1}^{k} \xi_{n,i}$. This is easy to do by comparing the time-scale τ_n of this lemma with the natural time-scale of Theorem 1 above. However, since $S'_n \circ \tau_n \xrightarrow{d} B$ is also implied by Theorem 3.2 of McLeish (1974) we omit the details.

For the converse part we first note that $M_n \xrightarrow{P} 0$ again follows immediately, and that we then also have that $S'_n \circ \tau_n \xrightarrow{d} B$. Next it has to be proved that $\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} t$, $t \in (0,1]$ (that (2) holds for $t=0$ follows from (3), and by dividing both members by t it is seen that it suffices to show that

$$\sum_{i=1}^{\tau_n(1)} \xi_{n,i}^2 \xrightarrow{P} 1 \quad \text{as} \quad n \to \infty. \tag{4}$$

Now the functional $x(\cdot) \to \sum_{i=1}^{k} \{x(\frac{i}{k}) - x(\frac{i-1}{k})\}^2$ is a.s. B-continuous and hence

$$\sum_{i=1}^{k} \{S'_n \circ \tau_n(\tfrac{i}{k}) - S'_n \circ \tau_n(\tfrac{i-1}{k})\}^2 \xrightarrow{d} \sum_{i=1}^{k} \{B(\tfrac{i}{k}) - B(\tfrac{i-1}{k})\}^2$$

as $n \to \infty$. Furthermore the latter sum has mean 1 and variance $2/k$ so $\sum_{i=1}^{k} \{B(\frac{i}{k}) - B(\frac{i-1}{k})\}^2 \xrightarrow{P} 1$ as $k \to \infty$. It follows that it is possible to find a sequence $n' = n'(n)$ of integers with $n' \to \infty$ as $n \to \infty$ such that

$$\sum_{i=1}^{n'} \{S'_n \circ \tau_n(\tfrac{i}{n'}) - S'_n \circ \tau_n(\tfrac{i-1}{n'})\}^2 \xrightarrow{P} 1 \quad \text{as} \quad n \to \infty. \tag{5}$$

To simplify notation we will for the rest of the proof write $\tau(i)$ for $\tau_n(\frac{i}{n'})$ when dealing with variables from the n'th row, $1 \le i \le n'$ $(\tau(0)=0)$. Again suppressing the row index, let $\tau'(i)$ be the minimum of $\tau(i)$ and of $\inf\{k > \tau(i-1);\ |\sum_{i=\tau(i-1)+1}^{k} \xi_{n,i}| > 1\}$ and denote the event that

$\tau(i) = \tau'(i)$, $1 \le i \le n'$, by A_n. Let $Y_n = \max_{1\le i\le n'} \sup_{\frac{i-1}{n'}<t\le\frac{i}{n'}} |S_n'\circ\tau_n(t) - S_n'\circ\tau_n(\frac{i-1}{n'})|$ so that $A_n^c \subset \{Y_n > 1\}$ and introduce the $D(0,1)$ modulus of continuity $\omega_n(\delta) = \inf_{\{t_i\}} \max_{0<i\le r} \sup_{t_{i-1}\le t,s\le t_i} |S_n'\circ\tau_n(t) - S_n'\circ\tau_n(s)|$, where the infinum is taken over $\{t_i\}$ satisfying $0 = t_0 < t_1 < \ldots < t_r = 1$, $t_i - t_{i-1} > \delta$, $i = 1,\ldots,r$. Then $Y_n \le 2\omega_n(\frac{1}{n'}) + \max_{1\le i\le\tau_n(1)} |\xi_{n,i}|$ and since $n' \to \infty$ it follows from Theorem 15.2 of Billingsley (1968) that $\omega_n(\frac{1}{n'}) \xrightarrow{P} 0$. As also $\max_{1\le i\le\tau_n(1)} |\xi_{n,i}| \xrightarrow{P} 0$ we have that

$$(6) \qquad P(A_n^c) \le P(\{Y_n > 1\}) \to 0 \quad \text{as} \quad n\to\infty.$$

Hence, putting $\zeta_{n,i} = \sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}$, it follows from (5) and (6) that

$$(7) \qquad \sum_{i=1}^{n'} \zeta_{n,i}^2 \xrightarrow{P} 1 \quad \text{as} \quad n\to\infty.$$

Since $\max_{1\le i\le n'} |\zeta_{n,i}| \le Y_n$ we have by the reasoning above that $\max_{1\le i\le n'} |\zeta_{n,i}| \xrightarrow{P} 0$ as $n\to\infty$. Moreover, as $|\xi_{n,i}| \le 2$ for all n and i, it follows from the definition of $\tau'(i)$ that $\max_{1\le i\le n'} |\zeta_{n,i}| \le 3$.

For n fixed $\{S_n'(k)\}_{k=1}^{\infty}$ is a martingale, and as $\tau'(i)$ is a stopping time also, $\{S_n'(k\wedge\tau'(i))\}_{k=1}^{\infty}$ is a martingale. Since $\sum_{k=2}^{\infty} \left(S_n'(k\wedge\tau'(i)) - S_n'(\{k-1\}\wedge\tau'(i))\right)^2 \le \sum_{k=1}^{\tau_n(1)} \xi_{n,k}^2$ and since $E(\sum_{k=1}^{\tau_n(1)} \xi_{n,k}^2) < \infty$ for n large enough by (3), the martingale $\{S_n'(k\wedge\tau'(i))\}_{k=1}^{\infty}$ is square integrable for large n. Then, by the optional stopping theorem $\left\{S_n'(\{\tau(i-1)+k\}\wedge\tau'(i))\right\}_{k=0}^{\infty}$ is a square integrable martingale and hence mean square convergent. Since $\lim_{k\to\infty} S_n'(\{\tau(i-1)+k\}\wedge\tau'(i)) - S_n'(\tau(i-1)\wedge\tau'(i)) = \zeta_{n,i}$ a.s. by definition, it follows that

$$(8) \qquad E(\zeta_{n,i}^2 \| \mathcal{B}_{n,\tau(i-1)}) = E\left(\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2 \| \mathcal{B}_{n,\tau(i-1)}\right).$$

In particular we have from (3) that $E\left(\sum_{i=1}^{n'} \zeta_{n,i}^2\right) \le 1 + o(1)$. Together with (7) this shows that $\{\sum_{i=1}^{n'} \zeta_{n,i}^2\}_{n=1}^{\infty}$ is uniformly integrable (see e.g. Chung (1968) Theorem 4.5.4).

Let $d_n = \sum_{k=1}^{\tau_n(1)} \xi_{n,k}^2 - \sum_{i=1}^{n'} \zeta_{n,i}^2$ and let $d_n' = \sum_{i=1}^{n'} \sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2 - \sum_{i=1}^{n'} \zeta_{n,i}^2$ for $n = 1,2,\ldots$. Equation (8) and the inequality $(a^2+b^2) \le 2a^2+2b^2$ imply that

$$Ed_n'^2 \le 2 \sum_{i=1}^{n'} E\left\{\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2 - E\left(\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2 \| \mathcal{B}_{n,\tau(i-1)}\right)\right\}^2$$

$$+ 2 \sum_{i=1}^{n'} E\{\zeta_{n,i}^2 - E(\zeta_{n,i}^2 \| \mathcal{B}_{n,\tau(i-1)})\}^2 = S_1 + S_2 \text{ , say.}$$

Here $E\{\zeta_{n,i}^2 - E(\zeta_{n,i}^2 \| \mathcal{B}_{n,\tau(i-1)})\}^2 \le E\zeta_{n,i}^4$ so $S_1 \le E \sum_{i=1}^{n'} \zeta_{n,i}^4$, and similarly $E\left\{\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2 - E\left(\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2 \| \mathcal{B}_{n,\tau(i-1)}\right)\right\}^2 \le E\left\{\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2\right\}^2$

By a wellknown martingale inequality (see Burkholder (1974), Theorem 3.2) there is a universal constant c such that

$$E\left\{\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}^2\right\}^2 \le c\, E\left\{\sum_{k=\tau(i-1)+1}^{\tau'(i)} \xi_{n,k}\right\}^4$$

$$= c\, E\, \zeta_{n,i}^4 \; .$$

(Remember that $\sum_{k=\tau(i-1)+1}^{\tau'(i)\wedge N} \xi_{n,k}$ converges almost surely as $N\to\infty$ and thus, since $\left|\sum_{k=\tau(i-1)+1}^{\tau'(i)\wedge N} \xi_{n,k}\right| \le 3$, also converges in the 4'th mean.) Hence

$$E\, d_n'^2 \le 2(1+c)E \sum_{i=1}^{n'} \zeta_{n,i}^4 \le$$

$$\le 2(1+c)E\{ \max_{1\le i\le n'} \zeta_{n,i}^2 \sum_{i=1}^{n'} \zeta_{n,i}^2\} \to 0$$

as $n\to\infty$, since $\{\sum_{i=1}^{n'} \zeta_{n,i}^2\}_{n=1}^{\infty}$ is uniformly integrable and since $\max_{1\le i\le n'} \zeta_{n,i}^2 \le 9$ and $\max_{1<i<n'} \zeta_{n,i}^2 \xrightarrow{P} 0$. Thus $d_n' \xrightarrow{P} 0$, and combining this with (6) we have that for arbitrary $\varepsilon>0$

$$P(\{|d_n|>\varepsilon\}) \le P(\{|d_n|>\varepsilon\}\cap A_n) + P(A_n^c) \le$$

$$\le P(\{|d_n'|>\varepsilon\}) + P(A_n^c) \to 0$$

as $n\to\infty$. By (7) this proves that (4) holds and thus that (2) is satisfied. It now only remains to be observed that (2) and (3) together give that $\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{1} t$, also. □

It should perhaps be noted that in the converse part of the lemma, the condition (3) cannot be deleted entirely, not even if $\{\tau_n(t)\}$ is non-random.

3. RANDOM TIME-SCALES.

In Section 2 above it is shown that a natural time-scale is given by the sums of the squares of the truncated and recentered summands. In this section some ways of expressing this time-scale more directly in terms of the original array $\{X_{n,i}\}$ will be investigated. The first result shows when it is possible to normalize directly by sums of squares of the $X_{n,i}$'s.

THEOREM 4. Let $\tau_n(t) = \inf\{k;\ \sum_{i=1}^{k} X_{n,i}^2>t\}$, $t\in[0,1]$, and assume that (1) is satisfied. Then $S\circ\tau_n \xrightarrow{d} B$ if and only if both $M_n \xrightarrow{P} 0$ and

$$\text{(9)} \qquad \sum_{i=1}^{\tau_n(1)} E_{i-1}^2 \, (X'_{n,i}) \xrightarrow{P} 0 \quad \text{as} \quad n\to\infty.$$

PROOF. By definition

$$X_{n,i}^2 = \xi_{n,i}^2 + E_{i-1}^2(X'_{n,i}) + X''^2_{n,i} + 2E_{i-1}(X'_{n,i})\big(X'_{n,i}-E_{i-1}(X'_{n,i})\big).$$

Either by assumption, or else since $S\circ\tau_n \xrightarrow{d} B$ we have that $M_n \xrightarrow{P} 0$ and thus $\sum_{i=1}^{\tau_n(t)} X''_{n,i} \xrightarrow{P} 0$. Furthermore, by considering $\tau_n(t)\wedge N$ and letting $N\to\infty$, Fatou's lemma gives that

$$E\left\{\sum_{i=1}^{\tau_n(t)} \big(X'_{n,i}-E_{i-1}(X'_{n,i})\big)E_{i-1}(X'_{n,i})\right\}^2 \le E \sum_{i=1}^{\tau_n(1)} \big(X'_{n,i}-E_{i-1}(X'_{n,i})\big)^2 E_{i-1}^2(X'_{n,i}) \le$$

$$\le E \sum_{i=1}^{\tau_n(1)} X'^2_{n,i}\, E_{i-1}^2(X'_{n,i}) \le$$

$$\le E \sum_{i=1}^{\tau_n(1)} X'^4_{n,i} \le$$

$$\le 2\, E \max_{1\le i\le\tau_n(1)} X'^2_{n,i} \to 0 \quad \text{as} \quad n\to\infty,$$

where the last inequality holds since $\sum_{i=1}^{\tau_n(1)-1} X'^2_{n,i} < 1$ and $|X'_{n,\tau_n(1)}| \le 1$. Hence for $t\in[0,1]$

$$\sum_{i=1}^{\tau_n(t)} X_{n,i}^2 = \sum_{i=1}^{\tau_n(t)} \big(\xi_{n,i}^2 + E_{i-1}^2(X'_{n,i})\big) + r_n ,$$

where $r_n \xrightarrow{P} 0$ as $n\to\infty$. Thus

$$\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 = \sum_{i=1}^{\tau_n(t)} \big(X_{n,i}^2 - E_{i-1}^2(X'_{n,i})\big) - r_n =$$

$$= t + \theta\, X^2_{n,\tau_n(t)} + \sum_{i=1}^{\tau_n(t)} E_{i-1}^2(X'_{n,i}) - r_n ,$$

for some θ; $0\le\theta\le 1$. Since $X^2_{n,\tau_n(t)} \le M_n^2 \xrightarrow{P} 0$ we have that $\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} t$

for all $t \in [0,1]$ if and only if (9) holds. Now

$$E \sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \le E \sum_{i=1}^{\tau_n(t)} X'^2_{n,i}$$

$$\le t + o(1).$$

so Lemma 3 is applicable and the theorem follows. □

As seen above, under weak conditions the sums of squares of the $X_{n,i}$'s give the natural time scale. This normalization has the further advantage that it is readily available and that it does not depend on the underlying probability measure. However, in analogy with the case of independent summands it might also be interesting to normalize by the conditional variances (cf. Remark 2). This possibility was introduced by Lévy, and is the random normalization that has been most investigated.

THEOREM 5. Let $\tau_n(t) = \inf\{k; \sum_{i=1}^{k} E_{i-1}(X_{n,i}^2) > t\}$ and assume that (1) is satisfied. Then $S \circ \tau_n \xrightarrow{d} B$ if and only if both $M_n \xrightarrow{P} 0$ and

(10) $$\sum_{i=1}^{\tau_n(1)-1} \{E_{i-1}^2(X'_{n,i}) + E_{i-1}(X''^2_{n,i})\} + r_n \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty,$$

where $r_n = 1 - \sum_{i=1}^{\tau_n(1)-1} E_{i-1}(X_{n,i}^2)$.

REMARK 6. A simpler condition that together with $M_n \xrightarrow{P} 0$ implies (10) is $\sum_{i=1}^{\tau_n(1)} \{E_{i-1}^2(X'_{n,i}) + E_{i-1}(X''^2_{n,i})\} \xrightarrow{P} 0$. If already the original array $\{X_{n,i}\}$ is a m.d.a. then $E_{i-1}^2(X'_{n,i}) = E_{i-1}^2(X''_{n,i}) \le E_{i-1}(X''^2_{n,i})$ so in that case (10) is equivalent to

(10') $$\sum_{i=1}^{\tau_n(t)-1} E_{i-1}(X''^2_{n,i}) + r_n \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

Moreover, together with $M_n \xrightarrow{P} 0$ this also implies (1).

PROOF. We first prove that (1), (10), and $M_n \xrightarrow{P} 0$ are sufficient to ensure $S\circ\tau_n \xrightarrow{d} B$. By definition

$$E_{i-1}(X_{n,i}^2) = E_{i-1}(\xi_{n,i}^2) + E_{i-1}^2(X'_{n,i}) + E_{i-1}(X''^2_{n,i})$$

and hence for $t\in[0,1]$

$$(11)\quad \sum_{i=1}^{\tau_n(t)-1} E_{i-1}(\xi_{n,i}^2) = t - \Big(t - \sum_{i=1}^{\tau_n(t)-1} E_{i-1}(X_{n,i}^2)\Big) - \sum_{i=1}^{\tau_n(t)-1} \{E_{i-1}^2(X'_{n,i}) + E_{i-1}(X''^2_{n,i})\}.$$

It follows that if $M_n \xrightarrow{P} 0$ and (10) hold then $\sum_{i=1}^{\tau_n(t)-1} E_{i-1}(\xi_{n,i}^2) \xrightarrow{P} t$ and thus, since (1) and $M_n \xrightarrow{P} 0$ together imply that $\max_{1\le i\le\tau_n(1)} |\xi_{n,i}| \xrightarrow{P} 0$, also

$$(12)\quad \sum_{i=1}^{\tau_n(t)} E_{i-1}(\xi_{n,i}^2) \xrightarrow{P} t \quad \text{as} \quad n\to\infty.$$

Let the natural time-scale be $\tau'_n(t) = \inf\{k: \sum_{i=1}^{k} \xi_{n,i}^2 > t\} \wedge \tau_n(1)$, $t\in[0,1]$ and observe that $\sum_{i=1}^{\tau'_n(t)} \xi_{n,i}^2 \le t + \max_{1\le i\le\tau_n(1)} \xi_{n,i}^2 \xrightarrow{1} t$. Now

$$(13)\quad E\Big\{\sum_{i=1}^{\tau'_n(t)} \big(\xi_{n,i}^2 - E_{i-1}(\xi_{n,i}^2)\big)\Big\}^2 \le E\sum_{i=1}^{\tau'_n(t)} \xi_{n,i}^4 \le$$

$$\le E\Big\{\max_{1\le i\le\tau_n(1)} \xi_{n,i}^2 \sum_{i=1}^{\tau'_n(t)} \xi_{n,i}^2\Big\} \to 0 \quad \text{as} \quad n\to\infty,$$

Since $4 \ge \max_{1\le i\le\tau_n(1)} \xi_{n,i}^2 \xrightarrow{P} 0$, so also $P\Big(\sum_{i=1}^{\tau'_n(t)} E_{i-1}(\xi_{n,i}^2) \le t + \delta\Big) \to 1$ for any $\delta>0$. Hence, combining this with (12) we have, for $\varepsilon>0$, that

$$P\big(\tau_n(t) \le \tau_n'(t-\varepsilon)\big) \le P\Big(\sum_{i=1}^{\tau_n(t)} E_{i-1}(\xi_{n,i}^2) \le \sum_{i=1}^{\tau_n'(t-\varepsilon)} E_{i-1}(\xi_{n,i}^2)\Big) \to 0 \quad \text{as} \quad n\to\infty.$$

In particular $P\big(\tau_n'(t-\varepsilon) < \tau_n(1)\big) \to 1$ and thus $\sum_{i=1}^{\tau_n'(t-\varepsilon)} \xi_{n,i}^2 \xrightarrow{P} t-\varepsilon$ so

$$(14) \quad P\Big(\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 < t - 2\varepsilon\Big) \le P\Big(\sum_{i=1}^{\tau_n'(t-\varepsilon)} \xi_{n,i}^2 < t - 2\varepsilon\Big) + o(1) \to 0$$

as $n\to\infty$. Furthermore

$$(15) \qquad E \sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 = E \sum_{i=1}^{\tau_n(t)-1} E_{i-1}(\xi_{n,i}^2) + 0(1) \le t + o(1),$$

and since $\varepsilon>0$ is arbitrary, it follows from (14) and (15) that $\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} t$ for $t\in[0,1]$ (cf. e.g. Lemma 2.11 of McLeish (1974)), which by Lemma 3 proves that $S\circ\tau_n \xrightarrow{d} B$.

Conversely, suppose $S\circ\tau_n \xrightarrow{d} B$ as $n\to\infty$. Then $M_n \xrightarrow{P} 0$ and since (15) holds Lemma 3 is again applicable and thus, in particular, $\sum_{i=1}^{\tau_n(1)} \xi_{n,i}^2 \xrightarrow{1} 1$. Moreover, (13) then holds also if $\tau_n'(t)$ is replaced by $\tau_n(1)$ and it follows that $\sum_{i=1}^{\tau_n(1)} E_{i-1}(\xi_{n,i}^2) \xrightarrow{P} 1$ as $n\to\infty$, which by (12) proves that (10) holds. □

4. NON-RANDOM TIME-SCALES.

If the summands are independent it is sufficient to consider deterministic time scales, the most obvious one being given by the variances. In this section we will treat non-random time-scales. First, in Theorem 6 below, we will assume that the original array $\{X_{n,i}\}$ is a m.d.a., and then a more general case is treated in Theorem 8.

THEOREM 6. Let $\{X_{n,i}\}$ be a m.d.a., put $\tau_n(t) = \inf\{k;\ E\sum_{i=1}^{k} X_{n,i}^2 > t\}$ and assume that (1) is satisfied. Then $S\circ\tau_n \xrightarrow{d} B$ if and only if both $M_n \xrightarrow{P} 0$ and

$$\sum_{i=1}^{\tau_n(t)} X_{n,i}^2 \xrightarrow{P} t \quad \text{as } n\to\infty, \quad t\in[0,1]. \tag{16}$$

REMARK 7. If $\{X_{n,i}\}$ is a m.d.a. and $\tau_n(t)$ is as above, then $M_n \xrightarrow{P} 0$ and (16) together imply (1), so for sufficiency it is not necessary to assume that (1) holds. A number of equivalent sufficient conditions are given by Scott (1973).

PROOF. It is easy to show that if $M_n \xrightarrow{P} 0$ and (16) holds then the conditions of the first part of Lemma 3 are satisfied and hence $S\circ\tau_n \xrightarrow{d} B$. However, since the result is wellknown we omit the details.

For the reverse implication, assume that $S\circ\tau_n \xrightarrow{d} B$. Then $M_n \xrightarrow{P} 0$ and moreover

$$E\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \le E\sum_{i=1}^{\tau_n(t)-1} \xi_{n,i}^2 + o(1) \le$$
$$\le E\sum_{i=1}^{\tau_n(t)-1} X_{n,i}^2 + o(1) \le$$
$$\le t + o(1).$$

Hence, from the converse part of Lemma 3, $\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{1} t$ as $n\to\infty$, $t\in[0,1]$. Now,

$$t \ge E\sum_{i=1}^{\tau_n(t)-1} X_{n,i}^2 = E\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 + o(1) + E\sum_{i=1}^{\tau_n(t)-1} \{E_{i-1}^2(X'_{n,i}) + E_{i-1}(X''^2_{n,i})\},$$

so $E\sum_{i=1}^{\tau_n(t)-1} E_{i-1}^2(X'_{n,i}) \to 0$ as $n\to\infty$.

Proceeding as in the proof of Theorem 4, it is enough to note that hence also $E\sum_{i=1}^{\tau_n(1)} \xi_{n,i}^2 E_{i-1}^2(X'_{n,i}) \le E(\max_{1\le i\le\tau_n(1)} E_{i-1}^2(X'_{n,i}) \sum_{i=1}^{\tau_n(1)} \xi_{n,i}^2) \to 0$ as $n\to\infty$ to conclude that $\sum_{i=1}^{\tau_n(t)} X_{n,i}^2 - \sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} 0$, $t\in[0,1]$, and thus that (16) holds. □

<u>THEOREM 8.</u> Let $\tau_n(t) = \inf\{k;\ E\sum_{i=1}^{k} \xi_{n,i}^2 > t\}$ and assume that (1) holds. Then $S\circ\tau_n \xrightarrow{d} B$ if and only if $M_n \xrightarrow{P} 0$ and furthermore

$$\sum_{i=1}^{\tau_n(t)} \xi_{n,i}^2 \xrightarrow{P} t \quad \text{as } n\to\infty,\quad t\in[0,1].$$

<u>PROOF.</u> Theorem 8 is just a special case of Lemma 3. □

5. A CONCLUDING REMARK.

We have argued that the condition (1) is needed to make the martingale property come into play. However, the necessary and sufficient conditions of Theorem 6 and, if it is assumed that $\{X_{n,i}\}$ is a m.d.a., of Theorem 5 imply that (1) holds. Hence one might hope to obtain the results of those theorems without assuming (1). Sufficiency is of course obvious, but as regards necessity the best the present author has thus far been able to do is to replace (1) by the requirement that $\{M_n^2\}_{n=1}^{\infty}$ is uniformly integrable.

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