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AN INVARIANCE PRINCIPLE FOR LINEAR COMBINATIONS OF ORDER STATISTICS*

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ABSTRACT

Based on an (almost sure) reverse-martingale representation for linear combinations of order statistics (with smooth weight functions), a backward invariance principle (relating to the tail sequence) is established and the underlying regularity conditions are critically examined.

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Key Words and Phrases: Almost sure convergence, Bahadur-representation, linear functions of order statistics, invariance principles, reverse martingale, smooth weight function, weak convergence and Wiener process.

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1. Introduction. Let $\{X_i , i \geq 1\}$ be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with a distribution function (df) F, defined on $(-\infty, \infty)$. For every $n(\geq 1)$, the order statistics corresponding to $X_1,\ldots,X_n$ are denoted by $X_{n,1} \leq \ldots \leq X_{n,n}$ and let

$$T_n = \sum_{i=1}^{n} c_{n,i} X_{n,i} , \quad n \geq 1 , \tag{1.1}$$

where the $c_{n,i}$, $1 \leq i \leq n$, $n \geq 1$ are known constants. Asymptotic normality of (the standardized form of) T_n has been studied under diverse regularity conditions and by diverse techniques by a host of research workers; we may refer to Stigler (1969) and Shorack (1969) where other references are cited. The present investigation concerns with the tail-sequence $\{T_k , k \geq n\}$ and provides suitable invariance principles under appropriate regularity conditions. Instead of generalizing the Pyke-Shorack (1968) approach to the tail-sequence of empirical processes and incorporating the same to derive the invariance principles, we have formulated our results through some reverse-martingale characterizations of $\{T_n\}$; viz., Theorems 1 and 2. Even if $\{T_n\}$ may not exactly be a reverse martingale, it may be approximated by a suitable reverse martingale sequence, so that under suitable regularity conditions pertaining to this order of approximation, the invariance principle continues to hold (see Theorem 3). The current approach also yields a different proof of the asymptotic normality of T_n under somewhat different regularity conditions.

Along with the preliminary notions, the main theorems are introduced in Section 2. The proofs of the theorems are sketched in Section 3.

Section 4 deals with the almost sure (a.s.) convergence of certain functions of order statistics having relevance to the main theorems. The last section is devoted to some concluding remarks as well as to applications of the main theorems.

2. The main theorems. Let $\phi_n = \{\phi_n(t): 0 < t \leq 1\}$ be defined by

(2.1) $$c_{n,i} = n^{-1}\phi_n(t) \quad \text{for} \quad (i-1)/n < t \leq i/n\ ,\quad i = 1,\ldots,n\ ,$$

and, conventionally, we let

(2.2) $$c_{n,0} = c_{n,n+1} = 0 \quad \text{for} \quad n \geq 1\ .$$

Also, let $u(t) = 1$ or 0 according as t is $\geq$ or < 0 and let

(2.3) $$F_n(x) = n^{-1}\sum_{i=1}^{n} u(x-x_i)\ ,\quad -\infty < x < \infty$$

be the empirical d.f. Then, T_n may be rewritten as

(2.4) $$T_n = \int_{-\infty}^{\infty} \phi_n(F_n(x))x dF_n(x)\ ,\quad n \geq 1\ .$$

We are interested in the case of *smooth weights* where

(2.5) $$\lim_{n\to\infty} \phi_n(u) = \phi(u) \quad \text{exists for every} \quad 0 < u < 1\ ,$$

and, we define

(2.6) $$\mu = \int_{-\infty}^{\infty} \phi(F(x))x dF(x)\ ,$$

(2.7) $$\sigma^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [F(x\wedge y) - F(x)F(y)]\phi(F(x))\phi(F(y))dxdy\ ,$$

where $a \wedge b = \min(a,b)$ and we assume that both μ and σ^2 are finite.

For every $n(\geq 1)$, consider a stochastic process $W_n = \{W_n(t), t \in I\}$ $(I = [0,1])$ by introducing a sequence of integer-valued, non-increasing and left-continuous functions $\{k_n(t), t \in I\}$ where $k_n(t) = \min\{k: n/k \leq t\}$, $t \in I$ and then letting

$$W_n(t) = n^{\frac{1}{2}}(T_{k_n(t)} - \mu)/\sigma, \quad t \in I. \tag{2.8}$$

In order that W_n is properly defined in its lower boundary, we assume that

$$T_n \to \mu \text{ a.s., as } n \to \infty \quad (\text{i.e., } W_n(0) = 0 \text{ with probability } 1). \tag{2.9}$$

Then W_n belongs to the $D[0,1]$ space and we associate with it the Skorkhod J_1-topology. Also, let $W = \{W(t), t \in I\}$ be a standard Wiener process on I. Our primary concern is to show that under suitable regularity conditions, as $n \to \infty$,

$$W_n \xrightarrow{\mathcal{D}} W, \text{ in the } J_1\text{-topology on } D[0,1]. \tag{2.10}$$

For this purpose, we define for every $n \geq 1$

$$c^*_{n+1,i} = \frac{i-1}{n+1} c_{n,i-1} + \frac{n-i+1}{n+1} c_{n,i}, \tag{2.11}$$

$$d_{n+1,i} = c_{n+1,i} - c^*_{n+1,i}, \quad i \leq i \leq n+1,$$

$$U_n = n(T_n - T_{n+1}) \text{ and } T^*_{n+1} = \sum_{i=1}^{n+1} d_{n+1,i} X_{n+1,i}. \tag{2.12}$$

<u>Theorem 1</u>. *If* $d_{n,i} = 0$ *for* $1 \leq i \leq n$ *and all* $n \geq n_0(\geq 1)$ *and if*

$$0 < \lim_{n\to\infty} n \operatorname{Var}(T_n) = \sigma^2 < \infty, \tag{2.13}$$

then the convergence of the finite-dimensional distributions (f.d.d.) of $\{W_n\}$ *(to those of* W*) insures* (2.10).

We shall see in Section 3 that (2.9) also holds under the hypothesis of the theorem. Stigler (1969) has shown that (2.13) holds under suitable regularity conditions and further, T_n can then be written as $S_n + R_n$, where S_n is an average of independent random variables and $R_n = o_p(n^{-\frac{1}{2}})$. As such, for finitely many T_{nj}'s, his decomposition and an application of the central limit theorem (for a double sequence of independent r.v.'s) on the S_{nj}'s yield the convergence of the f.d.d.'s of $\{W_n\}$ to those of W under the regularity conditions of Section 4 of Stigler (1969). Hence, under the additional assumption that $d_{n,i} = 0$, $\forall\ 1 \le i \le n$, $n \ge n_0$, his asymptotic normality result extends to the invariance principle in (2.10).

Let $I(A)$ be the indicator function of a set A and also let

$$\sigma_n^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{n,i}c_{n,j}(n+1)^{-2}[(i\wedge j)(n+1)-ij](X_{n+1,i+1}-X_{n+1,i})(X_{n+1,j+1}-X_{n+1,j}) \,. \tag{2.14}$$

Then, the following theorem provides a set of different regularity conditions pertaining to (2.10) where $\{F_n,\ n \ge 1\}$ is defined in the beginning of Section 3.

<u>Theorem 2</u>. *If* $d_{n,i} = 0$ *for* $1 \le i \le n$ *and* $n \ge n_0(\ge 1)$ *and if*

$$n^2\sigma_n^2 \to \sigma^2 \text{ a.s., as } n \to \infty \,, \tag{2.15}$$

$$E[U_n^2 I(|U_n| > \varepsilon n^{\frac{1}{2}}) | F_{n+1}] \to 0 \text{ a.s., as } n \to \infty,\ \forall\ \varepsilon > 0 \,, \tag{2.16}$$

then (2.10) *holds.*

We may note that for both Theorems 1 and 2, the assumption that the $d_{n,i}$ are all equal to 0 is very crucial. As we shall see later on that in many cases, the $d_{n,i}$ may not be exactly equal to 0, though they are very closely so. Towards this, we present the following.

Theorem 3. *If* $N^{3/2}T_N^* \to 0$ *a.s., as* $N \to \infty$, *then* (2.10) *holds under the rest of assumptions of Theorem* 1 *or Theorem* 2.

The proofs of the theorems are outlined in Section 3. Section 4 is devoted to a.s. convergence and uniform integrability of linear functions of order statistics having direct impact on (2.15), (2.16) and the first condition of Theorem 3.

3. Proofs of the main Theorems. Let $\mathcal{F}_n = \mathcal{F}(X_{n,1},\dots,X_{n,n},X_{n+j}, j \geq 1)$ be the σ-field generated by $(X_{n,1},\dots,X_{n,n})$ and X_{n+j} , $j \geq 1$ for $n \geq 1$. Then $\mathcal{F}_n$ is non-increasing in n .

Lemma 3.1. *If for some* $n_0(\geq 1)$, $d_{n,i} = 0$ *for every* $1 \geq i \geq n$, $n \geq n_0$, *then* $\{T_n, \mathcal{F}_n ; n \geq n_0\}$ *is a reverse martingale.*

Proof. Given $\mathcal{F}_{n+1}$, X_{n+1} can assume the values $X_{n+1,1},\dots,X_{n+1,n+1}$ with equal conditional probability $(n+1)^{-1}$. Thus, the possible realizations of $(X_{n,1},\dots,X_{n,n})$, given $\mathcal{F}_{n+1}$, are $(X_{n+1,1},\dots,X_{n+1,k-1},X_{n+1,k+1},\dots,X_{n+1,n+1})$, $k = 1,\dots,n+1$ and these are conditionally equally likely; here for $k = 1$ (or $n+1$) the vector starts (ends) with $X_{n+1,2}$ $(X_{n+1,n+1})$. Hence for $n \geq 1$,

$$
\begin{aligned}
(3.1)\qquad E(T_n|\mathcal{F}_{n+1}) &= \textstyle\sum_{i=1}^{n} c_{n,i}\, E[X_{n,i}|\mathcal{F}_{n+1}] \\
&= \textstyle\sum_{i=1}^{n} c_{n,i} \left\{\frac{n-i+1}{n+1} X_{n+1,i} + \frac{i}{n+1} X_{n+1,i+1}\right\} \\
&= \textstyle\sum_{i=1}^{n+1} \left\{\frac{i-1}{n+1} c_{n,i-1} + \frac{n-i+1}{n+1} c_{n,i}\right\} X_{n+1,i} \quad (c_{n,0} = c_{n,n+1} = 0)
\end{aligned}
$$

$$= \sum_{i=1}^{n+1} c^*_{n+1,i} X_{n+1,i}$$

$$= T_{n+1} - \sum_{i=1}^{n+1} d_{n+1,i} X_{n+1,i} .$$

Since $d_{n,i}$ are all equal to 0, the lemma follows from (3.1). Q.E.D.

As an illustration, we consider the following.

Example 3.1. Let k be a non-negative integer, $n_0 = 2k+1$ and

(3.2) $$T_n = \left\{\sum_{i=1}^{n} \binom{i-1}{k}\binom{n-i}{k} X_{n,i}\right\} \Big/ \binom{n}{2k+1} \quad \text{for } n \geq n_0 .$$

Here $(n+1)^{-1}\{(i-1)c_{n,i-1} + (n-i+1)c_{n,i}\} = (n+1)^{-1}\left\{(i-1)\binom{i-2}{k}\binom{n-i+1}{k} + (n-i+1)\binom{i-1}{k}\binom{n-1}{k}\right\}\binom{n}{2k+1}^{-1} = \binom{n+1}{2k+1}^{-1}\binom{i-1}{k}\binom{n+1-i}{k} = c_{n+1,i}$, $1 \leq i \leq n+1$. Thus, Lemma 3.1 applies here. For $k=0$, T_n is the sample mean while for $k \geq 1$, it is a robust competitor of the sample mean; see Sen (1964).

Let us now consider the proof of Theorem 1. Here, (2.9) is insured by Lemma 3.1 and (2.13). Also, note that for every $0 \leq s < s+\delta \leq 1$,

(3.3) $$\sup_{s\leq t\leq s+\delta} |W_n(t) - W_n(s)| = \max_{k_1\leq q\leq k_2} n^{\frac{1}{2}}\sigma^{-1}|T_q - T_{k_2}| ,$$

where

(3.4) $$k_1^{-1}n \leq s+\delta < (k_1-1)^{-1}n \quad \text{and} \quad k_2^{-1}n \leq s < (k_2-1)^{-1}n .$$

Since $\{|T_q - T_{k_2})|, \mathcal{F}_q ;\ q \leq k_2\}$ has the reverse sub-martingale property, by reversing the order of the index set and using Lemma 4 of Brown (1971), we obtain that for every $\varepsilon > 0$,

(3.5) $$P\{\max_{k_1\leq q\leq k_2} |T_q - T_{k_2}| > \sigma\varepsilon n^{-\frac{1}{2}}\}$$

$$\leq (2/\varepsilon\sigma)(E\{n[T_{k_1} - T_{k_2}]^2\}P\{n^{\frac{1}{2}}|T_{k_1} - T_{k_2}| > \varepsilon\sigma/2\})^{\frac{1}{2}},$$

where by (2.13) and (3.4), as $n \to \infty$, $E\{n[T_{k_2} - T_{k_1}]^2\} \to \delta\sigma^2$ and also the convergence of f.d.d.'s of $\{W_n\}$ to those of W implies that as $n \to \infty$

$$n^{\frac{1}{2}}(T_{k_1} - T_{k_2})/\sigma \xrightarrow{L} N(0,\delta) \ . \tag{3.6}$$

Finally, if $\Phi(x)$ be the standard normal d.f., then for every $x \geq 1$, $2\{1-\Phi(x)\} \leq (2/\pi)^{\frac{1}{2}} x^{-1} \exp(-\frac{1}{2}x^2)$, and hence, for every $\varepsilon > 0$, we can choose $\delta(>0)$, so small that the right hand side of (3.5) is less than $\eta\delta$, for some arbitrary $\eta > 0$. Hence, $\{W_n\}$ is tight. Q.E.D.

Consider next the proof of Theorem 2. Here also, by virtue of Lemma 3.1, we are in a position to apply the invariance principles for reverse martingales for our purpose. Towards this, note that

$$\begin{aligned} &\operatorname{Var}(X_{n,i} | F_{n+1}) \\ &= \frac{n-i+1}{n+1} X^2_{n+1,i} + \frac{i}{n+1} X^2_{n+1,i+1} - \left\{\frac{n-i+1}{n+1} X_{n+1,i} + \frac{i}{n+1} X_{n+1,i+1}\right\}^2 \\ &= \frac{i(n+1-i)}{(n+1)^2} [X_{n+1,i+1} - X_{n+1,i}]^2 , \quad 1 \leq i \leq n \end{aligned} \tag{3.7}$$

and, similarly, for $1 \leq i < j \leq n$,

$$\begin{aligned} &\operatorname{Cov}(X_{n,i}, X_{n,j} | F_{n+1}) \\ &= \frac{i(n+1-j)}{(n+1)^2} [X_{n+1,i+1} - X_{n+1,i}][X_{n+1,j+1} - X_{n+1,j}] \ . \end{aligned} \tag{3.8}$$

Thus, by (1.1), (3.7) and (3.8),

$$E\{(T_n - T_{n+1})^2 | F_{n+1}\} = \sigma^2_n \ , \quad \text{defined by} \quad (2.14) \ . \tag{3.9}$$

Hence, by (2.15) and (3.9),

$$(n/\sigma^2) \sum_{N \geq n} E\{(T_N - T_{N+1})^2 | F_{N+1}\} \xrightarrow{p} 1 \quad \text{as} \quad n \to \infty \ . \tag{3.10}$$

Also, for every $\varepsilon > 0$, as $n \to \infty$,

$$(3.11) \qquad (n/\sigma^2) \sum_{N\geq n} E\{(T_N - T_{N+1})^2 I(|T_N - T_{N+1}| > \varepsilon\sigma\sqrt{n}) | F_{N+1}\}$$

$$= (n/\sigma^2) \sum_{N\geq n} N^{-2} E\{U_N^2 I(|U_N| > \varepsilon\sigma \sqrt{n}\ | F_{N+1}\}$$

$$\xrightarrow{a.s.} 0 , \text{ by } (2.16) .$$

Hence, by an updated version of a basic theorem of Loynes (1970) we conclude from (3.10) and (3.11) that (2.10) holds. Q.E.D.

Finally, consider the proof of Theorem 3. Let

$$(3.12) \qquad \tilde{U}_k = T_k - E(T_k|F_{k+1}) = (T_k - T_{k+1}) + T^*_{k+1} , \text{ by } (3.1) .$$

Therefore, for every $q \geq n$, by the first condition of Theorem 3,

$$(3.13) \qquad n^{\frac{1}{2}}(T_q - \sum_{N\geq q}\tilde{U}_N) = n^{\frac{1}{2}}(\sum_{n\geq q+1}T^*_N) \to 0 \text{ a.s, as } n \to \infty .$$

Since $E[\tilde{U}_k|F_{k+1}] = 0$ for every $k \geq n_0$, $\{U_n, F_n ; n \geq n_0\}$ is a reverse martingale — difference sequence. Further, $E\{\tilde{U}_k^2\} = E\{E[\tilde{U}_k^2|F_{k+1}]\} \leq$ $E\{E[(T_k - T_{k+1})^2|F_{k+1}]\} = E(T_k - T_{k+1})^2$, $\forall\ k \geq n_0$, and hence,

$$(3.14) \qquad E(\sum_{k_1\leq k\leq k_2}\tilde{U}_k)^2 \leq E(T_{k_2} - T_{k_1})^2 , \ \forall\ n_0 \leq k_1 < k_2 < \infty .$$

Further (3.13) and the convergence of the f.d.d.'s of $\{W_n\}$ to those of W insures the same for the process where T_N is replaced by $\sum_{k\geq N}\tilde{U}_k$, $N \geq n$. Hence, the proof of Theorem 1 readily extends. For Theorem 2, we note that (3.14) and the first condition of Theorem 3 implies that

$$(3.15) \qquad \sigma_N^2 - 0(N^{-3}) = E\{\tilde{U}_N^2|\ _{N+1}\} \leq \sigma_N^2 , \ \forall\ N \geq n_0 ,$$

Thus, by (2.15) and (3.15), here also (3.10) holds with $T_N - T_{N+1}$ being replaced by $\tilde{U}_N$, $N \ge n$. Also, for every $N \ge n$,

$$(3.16)\quad E[\tilde{U}_N^2 I(|\tilde{U}_N| > \varepsilon\sigma n^{-\frac{1}{2}}) | F_{N+1}]$$

$$= E[(T_N - T_{N+1})^2 I(|\tilde{U}_N| > \varepsilon\sigma n^{-\frac{1}{2}}) | F_{n+1}] - E\{T_{N+1}^{*2} I(|\tilde{U}_N| > \varepsilon\sigma n^{-\frac{1}{2}})\}$$

$$\le E[(T_N - T_{N+1})^2 I(|T_N - T_{N+1} + T_{N+1}^*| > \varepsilon\sigma n^{-\frac{1}{2}}) | F_{N+1}]$$

$$\le E\{(T_N - T_{N+1})^2 I(|T_N - T_{N+1}| > \tfrac{1}{2}\varepsilon\sigma n^{-\frac{1}{2}}) | F_{n+1}\} +$$

$$\sigma_N^2 E\{I(|T_{N+1}^*| > \tfrac{1}{2}\varepsilon\sigma n^{-\frac{1}{2}}) \; | F_{n+1}\}$$

$$= E\{(T_N - T_{N+1})^2 I(|T_N - T_{N+1}| > \tfrac{1}{2}\varepsilon\sigma n^{-\frac{1}{2}}) | F_{n+1}\} + 0(N^{-2}) \quad \text{a.s.},$$

by (2.15) and the fact that $N^{-3/2} T_N^* \to 0$ a.s., as $N \to \infty$. Hence, (3.11) holds with $T_N - T_{N+1}$ being replaced by $\tilde{U}_N$, $N \ge n$. Therefore, the proof of Theorem 2 readily extends to that of Theorem 3. Q.E.D.

4. <u>A.S. convergence and uniform integrability of functions of order statistics</u>.

In this section, we shall elaborate (2.15), (2.16) and the first condition of Theorem 3, and formulate suitable regularity conditions under which these hold.

In this context, we consider first the a.s. convergence of an arbitrary linear compound of order statistics. Let

$$(4.1)\qquad V_n = \textstyle\sum_{i=1}^n h_{n,i} X_{n,i} = \displaystyle\int_{-\infty}^{\infty} \psi_n(F_n(x)) x \; dF_n(x),$$

where $h_{n,i} = n^{-1}\psi_n(t)$ for $(i-1)/n < t \le i/n$, $1 \le i \le n$ and let

$$(4.2)\qquad \nu = \int_{-\infty}^{\infty} \psi(F(x)) x \; dF(x),$$

where we assume that $\psi_n(u) \to \psi(u)$ as $n \to \infty$ for $\forall\ 0 < u < 1$ and the integral in (4.2) exists. Our concern is to study suitable regularity conditions under which

$$V_n \to \nu \text{ a.s., as } n \to \infty . \tag{4.3}$$

Note that whenever $\{V_n , \mathcal{F}_n ; n \geq n_0\}$ is a reverse martingale, we may use the reverse martingale convergence theorem to prove (4.3). Hence, in the sequel, we only consider the case where the above reverse martingale property may not hold. We assume that

$$\psi(t) = \psi_1(t) - \psi_2(t) , \ 0 < t < 1 \text{ where } \psi_j(t) \text{ is } \nearrow \text{ in } t \text{ and is continuous inside } I , \ j = 1,2 . \tag{4.4}$$

Then, the following theorems relate to (4.3) under different sets of regularity conditions.

<u>Theorem 4.1</u>. *Let* r *and* s *be two positive integers satisfying* $r^{-1} + s^{-1} = 1$ *and such that*

$$E|X|^r < \infty \text{ and } \limsup_n n^{-1}\textstyle\sum_{i=1}^n |\psi_n(\frac{i}{n})|^s < \infty . \tag{4.5}$$

Then, under (4.4), (4.3) *holds.*

<u>Proof</u>. We may rewrite $V_n - \nu$ as

$$V_n - \nu = \int_{-\infty}^{\infty} [\psi_n(F_n(x)) - \psi(F(x))] x dF_n(x) + \int_{-\infty}^{\infty} x\psi(F(x)) d[F_n(x) - F(x)] . \tag{4.6}$$

By the Kintchine strong low of large numbers, the second term on the r.h.s.

of (4.6) a.s. converges to 0 as $n \to \infty$. We write first term as

(4.7) $$\int_{-\infty}^{-C} + \int_{-C}^{C} + \int_{C}^{\infty} [\psi_n(F_n(x)) - \psi(F(x))] x dF_n(x)$$

$$= I_{n1} + I_{n2} + I_{n3}\ , \quad \text{say}\ ,$$

where $C(0 < C < \infty)$ is so chosen that for given $\varepsilon > 0$ and $\eta > 0$,

(4.8) $$\int_{-\infty}^{-C} + \int_{C}^{\infty} |x\psi(F(x))| dF(x) < \tfrac{1}{2}\varepsilon \quad \text{and} \quad \int_{-\infty}^{-C} + \int_{C}^{\infty} |x|^r dF(x) < \eta\ .$$

For I_{n2} , $|x|$ is bounded, so that by (4.4) and the Glivenko-Cantelli theorem, $\sup\{x|\psi_n(F_n(x)) - \psi(F(x))| : |x| \le C\} \to 0$ a.s., as $n \to \infty$, and hence, $I_{n2} \to 0$ a.s. For $I_{n1} + I_{n3}$, note that $\left\{\int_{-\infty}^{-C} + \int_{C}^{\infty} \psi(F(x)) x dF_n(x)\ ,\ F_n\ ;\ n \ge 1\right\}$ is a reverse martingale (uniformly integrable), so that by (4.8), $\left|\int_{-\infty}^{-C} + \int_{C}^{\infty} x\psi(F(x)) dF_n(x)\right| < \tfrac{1}{2}\varepsilon$ a.s., as $n \to \infty$. Further, by the Hölder-inequality

(4.9) $$\left|\int_{-\infty}^{-C} + \int^{\infty} \psi_n(F_n(x)) x dF_n(x)\right|$$

$$\le \left[\int_{-\infty}^{-C} + \int_{C}^{\infty} |x|^r dF_n(x)\right]^{r^{-1}} \left[\int_{-\infty}^{-C} + \int_{C}^{\infty} |\psi_n(F_n(x))|^s dF_n(x)\right]^{s^{-1}}\ ,$$

where $\left\{\int_{-\infty}^{-C} + \int_{C}^{\infty} |x|^r dF_n(x)\ ,\ F_n\ ;\ n \ge 1\right\}$ is a reverse martingale (uniformly integrable), so that by (4.8), $\int_{-\infty}^{-C} + \int_{C}^{\infty} |x|^r dF_n(x) < \eta$ a.s., as $n \to \infty$. Finally, by (4.5)

(4.10) $$\int_{-\infty}^{-C} + \int_{C}^{\infty} |\psi_n(F_n(x))|^s dF_n(x) \le n^{-1} \sum_{i=1}^{n} |\psi_n(i/(n+1)|^s < \infty\ .$$

Thus, by proper choice of $\eta(>0)$, the r.h.s. of (4.9) can be made $\le \tfrac{1}{2}\varepsilon$ a.s., as $n \to \infty$. Thus, $I_{n1} + I_{n3} \to 0$ a.s., as $n \to \infty$. Q.E.D.

In the above theorem, we need that $E|X|^r < \infty$ for some $r > 1$. This may be relaxed (at the cost of additional restrictions on ψ_n) as follows.

Theorem 4.2. *Suppose that* (i) $\psi_n(i/n) = \psi(i/(n+1))$, $1 \le i \le n$ *or* $\psi_n(u) \to \psi(u)$, $0 < u < 1$ *as* $n \to \infty$ *and* $\int_{-\infty}^{\infty}\psi_n(F(x))x dF_n(x) = n^{-1}\sum_{i=1}^{n} \psi_n(F(X_i))X_i \to \nu$ *a.s., as* $n \to \infty$, (ii) ψ_n *(and hence,* ψ*) satisfies the Lipschitz condition:* $|\psi_n(u) - \psi_n(v)| \le k|u-v|$ *for all* $u, v \in I$, *and* (iii) $E|X|^r < \infty$ *for some* $r > \frac{1}{2}$. *Then* (4.3) *holds.*

Proof. By (4.6)-(4.7), here also, we need to show only that $I_{n1} + I_{n3} \to 0$ a.s., as $n \to \infty$, and the crucial step is to show that as $n \to \infty$,

$$\left|\int_{-\infty}^{-C} + \int_{C}^{\infty}\{\psi_n(F_n(x))-\psi_n(F(x))\}x dF_n(x)\right| \to 0 \quad \text{a.s.} \tag{4.11}$$

In this context, we use the following result, due to Ghosh (1972):

for every $\eta > 0$, there exists a $K_1 (= K_\eta < \infty)$, such that as $n \to \infty$,

$$\sup_{-\infty<x<\infty} n^{\frac{1}{2}}(\log n)^{-1}\{F(x)[1-F(x)]\}^{-\frac{1}{2}+\eta}|F_n(x)-F(x)| \le K_1 \quad \text{a.s.} \tag{4.12}$$

Hence, by (4.12) and the Lipschitz condition,

$$\begin{aligned} |x\{\psi_n(F_n(x))-\psi_n(F(x))\}| &\le K|x||F_n(x)-F(x)| \\ &\le KK_1 n^{-\frac{1}{2}}(\log n)|x|\{F(x)[1-F(x)]\}^{\frac{1}{2}-\eta} \quad \text{a.s.} \\ &= (KK_1)n^{-\frac{1}{2}}(\log n)|x|^{1-\frac{3}{2}r+r\eta}\{|x|^r F(x)[1-F(x)]\}^{\frac{1}{2}-\eta}|x|^r . \end{aligned} \tag{4.13}$$

Now, $E|X|^r < \infty \Rightarrow |x|^r F(x)[1-F(x)]$ is bounded for all x and it converges to 0 as $x \to \pm\infty$. Also, $dF_n(x) = 0$ unless $x = X_{n,i}$, $1 \le i \le n$, $E|X|^r < \infty \Rightarrow \max_{1\le i\le n}|X_{n,i}|^r = o(n)$ a.s., as $n \to \infty$. Hence, by (4.13), as $n \to \infty$, (4.14) $$\left|\left\{\int_{-\infty}^{-C} + \int_{C}^{\infty}\right\} x[\psi_n(F_n(x)) - \psi_n(F(x))]dF_n(x)\right|$$

$$\leq \left[(KK_1)n^{-\frac{1}{2}}(\log n)\left\{\max_{1\leq r\leq n}|X_{n,i}|\right\}^{1-\frac{3}{2}r+r\eta}\right.$$

$$\left.\left\{\sup_{-\infty<x<\infty}|x|^r F(x)[1-F(x)]\right\}^{\frac{1}{2}-\eta}\left\{\int_{-\infty}^{-C} + \int_{C}^{\infty}|x|^r dF_n(x)\right\}\right] , \quad \text{a.s.}$$

$$= [O(n^{-\frac{1}{2}}\log n)][\{O(n^{\frac{1}{r}})\}^{1-\frac{3}{2}r+r\eta)}][O(1)][O(1)] \quad \text{a.s.}$$

$$= o(1) \quad \text{a.s., as} \quad r > \tfrac{1}{2} . \qquad \text{Q.E.D.}$$

Remark. In the context of the asymptotic normality of $n^{\frac{1}{2}}(T_n - \mu)$, Stigler (1969) and Shorack (1969), both, for technical reasons, required that $c_{n,i} = 0$ for all $i \leq k_n$ and $i \geq n-k_n + 1$ where $k_n \uparrow$ in n but $n^{-1}k_n \downarrow 0$ as $n \to \infty$, and they assume that for some $r > 0$, $E|X|^r < \infty$. This is also possible in our case. Note that $E|X|^r < \infty \Rightarrow \max_{k_n\leq i\leq n-k_n+1}|X_{n,i}|^r = o(n/k_n)$ and hence, in (4.14), we need that $k_n \geq n^{\alpha}$ for some $\alpha > 2(1-2r)/(2-3r)$ if $0 < r \leq \frac{1}{2}$.

Let us now examine the regularity conditions pertaining to (2.15). For simplicity, we let $c_{n,i} = n^{-1}\phi(i/(n+1))$, $1 \leq i \leq n$ and assume that ϕ satisfy (4.4). Then, we may rewrite σ_n^2 in (2.14) as

(4.15) $$n^2\sigma_n^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[F_{n+1}(x\wedge y) - F_{n+1}(x)F_{n+1}(y)]\phi(F_{n+1}(x))\phi(F_{n+1}(y))dxdy .$$

We assume that

(4.16) $$L = \int_{-\infty}^{\infty}|\phi(F(x))|\{F(x)[1-F(x)]\}^{\frac{1}{2}}dx < \infty ,$$

which insures that $\sigma^2 \leq L^2 < \infty$. Also, let $E = \{(x,y): -\infty<x, y<\infty\}$ and for $C > 0$, $E_C = \{(x,y): |x| < C, |y| < C\}$, $E_C^* = E - E_C$. Further, we choose $C(<\infty)$ such that for some given $\eta > 0$, (4.17) $$\int_{-\infty}^{-C} + \int_{C}^{\infty} |\phi(F(x))| \{F(x)[1-F(x)]\}^{\frac{1}{2}} dx < \eta/4L .$$

Then, from (2.7) and (4.17), we have

(4.18) $$\left|\sigma^2 - \iint_{E_C} [F(x\wedge y)-F(x)F(y)]\phi(F(x))\phi(F(y))dxdy\right| < \eta/2 .$$

By (4.4) (for $\phi = \psi$) and the Glivenko-Cantelli theorem, as $n \to \infty$,

(4.19) $$\iint_{E_C} \{F_{n+1}(x\wedge y)-F_{n+1}(x)F_{n+1}(y)\}\phi(F_{n+1}(x))\phi(F_{n+1}(y))dxdy$$
$$\to \iint_{E_C} [F(x\wedge y)-F(x)F(y)]\phi(F(x))\phi(F(y))dxdy \quad \text{a.s.}$$

Finally,

(4.20) $$\left|\iint_{E_C^*} [F_{n+1}(x\ y)-F_{n+1}(x)F_{n+1}(y)]\phi(F_{n+1}(x))\phi(F_{n+1}(y))dxdy\right|$$
$$\le 2\left(\int_{-\infty}^{\infty} \{F_{n+1}(x)[1-F_{n+1}(x)]\}^{\frac{1}{2}} |\phi(F_{n+1}(x))| dx\right) \cdot$$
$$\left(\int_{-\infty}^{-C} + \int_{C}^{\infty} \{F_{n+1}(y)[1-F_{n+1}(y)\}^{\frac{1}{2}} |\phi(F_{n+1}(y))| dy\right).$$

As such, if we let

(4.21) $$\psi_n^*\left(\frac{i}{n+1}\right) = \{[i(n+1-i)]^{\frac{1}{2}}/(n+1)\}\left|\phi\left(\frac{i}{n+1}\right)\right| , \quad 1 \le i \le n ;$$
$$\psi_n^*(0) = \psi_n^*(1) = 0 ,$$

and note that by (4.17), $\psi_n^*(\frac{i}{n+1})$ is bounded for every $1 \le i \le n$ and it tends to 0 as $i/(n+1) \to 0$ or 1 , we obtain on letting

(4.22) $$\psi_n\left(\frac{i}{n+1}\right) = n\left[\psi_n^*\left(\frac{i-1}{n+1}\right) - \psi_n^*\left(\frac{i}{n+1}\right)\right] , \quad 1 \le i \le n ,$$

and integrating by parts

$$\text{(4.23)} \qquad \int_{-\infty}^{\infty} \{F_{n+1}(x)[1-F_{n+1}(x)]\}^{\frac{1}{2}} |\phi(F_{n+1}(x))| dx$$

$$= \int_{-\infty}^{\infty} \psi_n^*(F_{n+1}(x))dx = -\int_{-\infty}^{\infty} x d\psi_n^*(F_{n+1}(x))$$

$$= \int_{-\infty}^{\infty} \psi_n(F_{n+1}(x)) x dF_{n+1}(x) \; .$$

Thus, if ψ_n, defined by (4.22), satisfies the hypothesis of Theorem 4.1 or 4.2 (or the remark there after), the r.h.s. of (4.23) converges a.s. to L, and similarly, the second factor on the r.h.s. of (4.20) can be made (a.s.) $\le \frac{1}{2}\eta$, as $n \to \infty$. This leads us to the following.

Theorem 4.3. *If* (4.16) *holds, for* $c_{n,i} = n^{-1}\phi(i/(n+1))$, $1 \le i \le n$, ϕ *satisfies* (4.4) *and* ψ_n, *defined by* (4.22), *satifies the hypothesis of Theorem* 4.1 *or* 4.2, *then* (2.15) *holds*.

We proceed on to (2.16). Note that if $X_{n+1} = X_{n+1,k} (1 \le k \le n+1)$, then under the condition that $d_{n,i} = 0$, $\forall\ 1 \le i \le n$, $n \ge n_0$, by (2.12),

$$U_n = n\left\{\sum_{i=1}^{k-1} c_{n,i}X_{n+1,i} + \sum_{i=k+1}^{n+1} c_{n,i-1}X_{n+1,i} - \sum_{i=1}^{n+1} c_{n+1,i}X_{n+1,i}\right\}$$

$$= n\Big\{\sum_{i=1}^{k-1}\Big[\frac{i}{n+1} c_{n,i} - \frac{i-1}{n+1} c_{n,i-1}\Big]X_{n+1,i} - c_{n,k}X_{n+1,k}$$

$$\text{(4.24)} \qquad -\sum_{i=k+1}^{n+1}\Big[\frac{n-i+1}{n+1} c_{n,i} - \frac{n-i+2}{n+1} c_{n,i-1}\Big]X_{n+1,k}\Big\}$$

$$= \int_{-\infty}^{\infty} [I(x \ge X_{n+1,k}) - F_{n+1}(x)]\phi(F_{n+1}(x))dx$$

$$= U_{n+1,k}, \text{ say, for } k = 1,\ldots,n+1 \; .$$

Hence, for every $\lambda > 0$,

$$\text{(4.25)} \qquad E[U_n^2 I(|U_n| > \lambda)|\mathcal{F}_{n+1}] = \frac{1}{n+1} \sum_{k=1}^{n+1} U_{n+1,k}^2 I(|U_{n+1,k}| > \lambda) \; .$$

Thus, it suffices to show that as $n \to \infty$,

(4.26) $$n^{-\frac{1}{2}}\left|\int_{-\infty}^{\infty}[I(x \geq X_{n+1}) - F_{n+1}(x)]\phi(F_{n+1}(x)dx\right| \to 0 \quad \text{a.s.}$$

For this purpose, we define

(4.27) $$Z_n = \int_{-\infty}^{\infty}[I(x \geq X_n) - F(x)]\phi(F(x))dx\ , \quad n \geq 1\ ,$$

(4.28) $$Z_n^* = \int_{-\infty}^{\infty}[I(x \geq X_n) - F_n(x)]\phi(F_n(x)dx\ , \quad n \geq 1\ ,$$

and decomposing the range $(-\infty,\infty)$ as $(-\infty,-C)$, $[-C,C]$, (C,∞) , $C > 0$, we denote the corresponding components of Z_n (or Z_n^*) as $Z_n^{(1)}(C)$, $Z_n^{(2)}(C)$ and $Z_n^{(3)}(C)$ (or $Z_n^{*(1)}(C)$, $Z_n^{*(2)}(C)$ and $Z_n^{*(3)}(C)$) , respectively. Since $\{Z_n\ ,\ n \geq 1\}$ are i.i.d.r.v. with $EZ_n = 0$ and $EZ_n^2 = \sigma^2$, defined by (2.7), it follows that $n^{-\frac{1}{2}}|Z_n| \to 0$ a.s., as $n \to \infty$. In a similar manner it follows that

(4.29) $$\max_{1\leq\ell\leq 3} n^{-\frac{1}{2}}|Z_n^{(\ell)}| \to 0 \quad \text{a.s., as} \quad n \to \infty\ .$$

Also, if ϕ satisfies (4.4), then for every $C > 0$,

(4.30) $$\begin{aligned} Z_n^{*(2)}(C) - Z_n^{(2)}(C) &= \int_{-C}^{C}[F(x)-F_n(x)]\phi(F_n(x))dx \\ &+ \int_{-C}^{C}[I(x \geq X_n) - F(x)][\phi(F_n(x)) - \phi(F(x))]dx \\ &\to 0 \quad \text{a.s., as} \quad n \to \infty\ . \end{aligned}$$

Further, for every $C > 0$,

(4.31) $$\begin{aligned} n^{-\frac{1}{2}}|Z_n^{*(1)}(C) + Z_n^{*(3)}(C)| &\leq \int_{-\infty}^{-C} + \int_{C}^{\infty} n^{-\frac{1}{2}}|I(x\geq X_n)-F_n(x)|\phi(F_n(x))|dx \\ &\leq \int_{-\infty}^{-C} + \int_{C}^{\infty}\{F_n(x)[1-F_n(x)]\}^{\frac{1}{2}}|\phi(F_n(x))|dx\ . \end{aligned}$$

As such, as in (4.20), by choosing C adequately large, the r.h.s. of (4.31) can be made $\le \frac{1}{2}\eta$ a.s., as $n \to \infty$. Hence, we obtain from (4.29)-(4.31), the following:

Theorem 4.4. *Under the assumptions of Theorem* 4.3, (2.16) *holds.*

Finally, let us examine the first condition of Theorem 3. Here also, we take

$$(4.32) \qquad c_{n,k} = n^{-1}\phi\left(\frac{i}{n+1}\right), \quad 1 \le i \le n, \quad c_{n,0} = c_{n,n+1} = 0 .$$

Then, by (2.11) and (4.32), we have

$$(4.33) \quad d_{n+1,i} = (n+1)^{-1}\left\{\phi\left(\frac{i}{n+2}\right) - \frac{i-1}{n}\phi\left(\frac{i-1}{n+1}\right) - \frac{n-i+1}{n}\,\phi\left(\frac{i}{n+1}\right)\right\}, \quad 1\le i\le n+1 ,$$

where conventionally $(k/n)\phi(k/n) = 0$ if $k=0$ and $((n-k)/n)\phi(k/n) = 0$ if $k=n$. Thus, on defining

$$(4.34) \quad \psi_n\left(\frac{i}{n+1}\right) = (n+1)^{3/2}\left[\phi\left(\frac{i}{n+2}\right) - \frac{i-1}{i}\phi\left(\frac{i-1}{n+1}\right) - \frac{n-i+1}{n}\phi\left(\frac{i}{n+1}\right)\right], \quad 1\le i\le n+1 ,$$

it remains to verify the conditions of Theorems 4.1 or 4.2 and to show that ν defined by (4.2) is equal to 0. Let ϕ' and ϕ'' be the first and second derivative, of ϕ. First, consider the simplest case, where ϕ'' is bounded inside I. Then, by (4.34) and simple expansion, we get

$$(4.35) \quad \left|\psi_n\left(\frac{i}{n+1}\right)\right| \le O((n+1)^{-1/2})\left\{\left|\frac{n-2i+2}{n+2}\,\phi'\left(\frac{i}{n+2}\right)\right| + \frac{i(n+1-i)}{(n+1)^2}\left|\phi''\left(\frac{i}{n+1}\right)\right|\right\} ,$$

so that by the same technique as in Theorem 4.2, it follows that

$$(4.36) \qquad [E|X|^r < \infty \text{ for some } r > \tfrac{1}{2}] \Rightarrow N^{3/2}T_N^* \to 0 \text{ a.s., as } N \to \infty .$$

Consider next the case where for some $\delta > 0$ and $0 < K < \infty$, (4.37) $|\phi'(u)| \le K[u(1-u)]^{-\frac{3}{2}+\delta}$ and $|\phi''(u)| \le K[u(1-u)]^{-\frac{5}{2}+\delta}$, $0<u<1$.

Note then $\psi_n(1/(n+1)) = -[(n+1)^{\frac{1}{2}}/n(n+2)]\phi'(\theta/(n+1) + (1-\theta)/(n+2))$, $0<\theta<1$, $\psi_n((n+1)/(n+1)) = [(n+1)^{\frac{1}{2}}/n(n+2)]\phi'(1-\theta/(n+1)-(1-\theta)/(n+2))$, $0<\theta<1$ and for $2 \le i \le n$, on noting that $(n+1)^{-\frac{1}{2}} \le \{i(n+1-i)/(n+1)^2\}^{\frac{1}{2}}$,

$$(4.38) \qquad \left|\psi_n\left(\frac{i}{n+1}\right)\right| \le K_1\left\{\left|\frac{n-2i+2}{n+2}\right|\frac{\{i(n+1-i)\}^{\frac{1}{2}}}{(n+1)}\left|\phi'\left(\frac{i}{n+2}\right)\right| + \frac{\{i(n+1-i)\}^{3/2}}{(n+1)^3}\left|\phi''\left(\frac{i}{n+1}\right)\right|\right\}$$

$$\le K^*\{i(n+1-i)/(n+1)^2\}^{-1+\delta}, \quad 2 \le i \le n .$$

Also, by (4.35) and (4.37), $\psi_n(u) \to 0$ as $n \to \infty$, for every $0<u<1$. Thus, if for some η: $0<\eta<\delta<1$,

$$(4.39) \qquad r = (\delta-\eta)^{-1} \quad\text{and}\quad s = (1-\delta+\eta)^{-1} ,$$

then $E|X|^r < \infty$, insures that (4.5) holds, and hence, by Theorem 4.1, $N^{3/2}T_N^* \to 0$ a.s., as $N \to \infty$. For $\delta \le 1$, here we require $E|X|^r < \infty$ for some $r > 1$. On the other hand, if in (4.37), $\delta \ge 1$, all we need is the convergence of $\int_{-\infty}^{\infty} |x|\{F(x)[1-F(x)\}^{\delta-1}dF(x)$, which holds when $E|X|^r < \infty$ for some $r \ge 1/\delta$. In this set up, for unbounded ϕ' (or ϕ'') , we need $r > 2/3$. However, if we let $c_{n,i} = 0$, for $i \le k_n$ and $i \ge n - k_n + 1$ where we let $k_n = n^\alpha$, $0<\alpha<1$, then on noting that

$$(4.40) \qquad [u(1-u)]^\gamma \le K^*[n^{-1}k_n]^\gamma = 0(n^{-\gamma(1-\alpha)}), \forall n^{-1}k_n<u<1-n^{-1}k_n, \gamma>0 ,$$

and proceeding as in (4.38)-(4.39), we obtain that under (4.37) for $N^{3/2}T_N^* \to 0$ a.s., all we need is that

(4.41) $\gamma(1-\alpha) = \frac{1}{2}$ and replace δ by $\delta + \gamma - \frac{1}{2}$ in (4.39) .

Thus, for every $\delta > 0$, γ can be made adequately large by choosing α accordingly, and hence, in the truncated case, $E|X|^r < \infty$ for some $r > 0$ is sufficient.

5. Some general remarks. In (1.1) and elsewhere, we have considered a linear combination of the $X_{n,i}$. More generally, one may also consider

$$T_n^g = \sum_{i=1}^n c_{n,i} g(X_{n,i}) \ , \quad \text{where} \quad g = \{g(t)\ ,\ -\infty < t < \infty\}\ , \tag{5.1}$$

and under suitable conditions on g (such as the absolute continuity etc.), obtain parallel results. Note that Lemma 3.1 holds for X being replaced by $g(X)$, so also (3.7) and (3.8). In Section 4, we need to replace $E|X|^r$ by $E|g(X)|^r$. Secondly, one can also consider

$$\tilde{T}_n = \sum_{i=1}^n c_{n,i} X_{n,i} + \sum_{i=1}^s c_{n,i}^0 X_{n,[np_i]+1}\ , \tag{5.2}$$

where the constants $c_{n,i}^0$ satisfy $|c_{n,i}^0 - c_i^0| = o(n^{-\frac{1}{2}})$, $0 < |c_i^0| < \infty$, $i = 1,\dots,s$, $0 < p_1 < \dots < p_s < 1$ and $[s]$ denotes the largest integer contained in s . Let ξ_i be defined (uniquely) by $F(\xi_i) = p_i$, $1 \le i \le s$ and

$$\mu^0 = \sum_{i=1}^s c_i^0 \xi_i\ . \tag{5.3}$$

From the results of Bahadur (1966), it follows that if $f(\xi_{p_i}) > 0$ $1 \le i \le s$ and in some neighborhood of ξ_{p_i} , $f'(x)$ is bounded, then as $n \to \infty$,

$$(X_{n,[np_i]+1} - \xi_{p_i}) + (F_n(\xi_{p_i}) - p_i)/f(\xi_{p_i}) = 0(n^{-3/4}\log n) \text{ a.s.}\ , \tag{5.4}$$

for $i = 1,\dots,s$. Further, it is well-known that

$$\{(F_n(\xi_{p_i}) - p_i\ ,\ 1 \le i \le s)\ ,\ \mathcal{F}_n;\ n \ge 1\} \text{ is a reverse martingale.} \tag{5.5}$$

Thus, $n^{\frac{1}{2}}(\tilde{T}_n - \mu - \mu^0)$ is a.s. (as $n \to \infty$) equivalent to

$$n^{\frac{1}{2}}(T_n-\mu) + n^{\frac{1}{2}}\sum_{i=1}^{s} c_i^0[F_n(\xi_{p_i})-p_i]/f(\xi_{p_i}) \quad (5.6)$$

As such, Theorems 1, 2 and 3 readily extends to $\{\tilde{T}_N , N \geq n\}$ as well; σ^2 has to be adjusted only.

Ghosh (1972) considered an almost sure representation of T_n in terms of i.i.d.r.v.'s and required the conditions that (i) $E|X|^r < \infty$ for some $r > 2$ and (ii) $\phi''(u)$ is bounded inside I. In so far as the Theorems of Section 2 are concerned, we do not need the boundedness of ϕ'' or that $E|X|^r < \infty$ for some $r > 2$. Also, Ghosh and Sen (1976) have also proved (2.15) (in the context of sequential tests based on linear functions of order statistics) under the assumption that $E|X|^2 < \infty$ and ϕ'' bounded. Our Theorem 4.3 improves this result under weaker conditions. This result and the basic Theorems of Section 2 enable us to study the convergence of the OC functions of sequential tests based on $\{T_n\}$ under weaker regularity conditions.

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