A UNIFORM CENTRAL LIMIT THEOREM
USEFUL IN NONLINEAR TIME SERIES REGRESSION

by

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The purpose of this report is to prove in detail a central limit theorem useful in nonlinear time series regression. The main ideas of the proof are due to Goebel (1974). The contribution here is to obtain a stronger conclusion than his Theorem 3 and correct some minor errors in his proof.

Lemma 1. Let

$$S_n(\theta) = Z_{kn}(\theta) + X_{kn}(\theta) \qquad (k = 1, 2, \ldots, \quad n = 1, 2, \ldots)$$

Assume that for every $\delta > 0$

$$\lim_{k \to \infty} P[\,|X_{kn}(\theta)| > \delta] = 0$$

uniformly in n and θ. Assume

$$\lim_{n \to \infty} P[Z_{kn}(\theta) \le z] = N(z;\, 0,\, \sigma_k^2(\theta))$$

and

$$\lim_{k \to \infty} \sigma_k^2(\theta) = \tau^2(\theta)$$

uniformly in θ where $0 < \ell \le \tau^2(\theta) \le \mu < \infty$ for all θ. Then

$$\lim_{n \to \infty} P[S_n(\theta) \le z] = N(z;\, 0,\, \tau^2(\theta))$$

uniformly in θ.

Proof: Given $\varepsilon > 0$ there is a $\delta > 0$ depending on ε but not θ or n such that

$$N(z + \delta;\, 0,\, \tau^2(\theta)) < N(z;\, 0,\, \tau^2(\theta)) + \varepsilon$$

because $\tau^2(\theta)$ is suitably bounded from above and below. There is a

k which depends on δ but not on θ or n such that

$$N(z + \delta;\, 0,\, \sigma_k^2(\theta)) < N(z + \delta;\, 0,\, \tau^2(\theta)) + \epsilon$$

$$P[\,|X_{kn}(\theta)| > \delta] < \epsilon$$

by hypothesis and the uniform convergence of $\sigma_k^2(\theta)$ to $\tau^2(\theta)$. Then there is an n^* depending on k and δ but not on θ such that for all $n > n^*$

$$P[Z_{kn}(\theta) \le z + \delta] < N(z + \delta;\, 0,\, \sigma_k^2(\theta)) + \epsilon \ .$$

Consequently, given $\epsilon > 0$ there is an n^* which does not depend on θ such that

$$\begin{aligned} P[S_n(\theta) \le z] &= P[Z_{kn}(\theta) + X_{kn}(\theta) \le z] \\ &\le P[Z_{kn}(\theta) \le z + \delta,\, |X_{kn}(\theta)| \le \delta] + P[\,|X_{kn}(\theta)| > \delta] \\ &< P[Z_{kn}(\theta) \le z + \delta] + \epsilon \\ &< N(z + \delta;\, 0,\, \sigma_k^2(\theta)) + 2\epsilon \\ &< N(z + \delta;\, 0,\, \tau^2(\theta)) + 3\epsilon \\ &< N(z;\, 0,\, \tau^2(\theta)) + 4\epsilon \end{aligned}$$

for all $n > n^*$. Similar arguments can be used to show that

$$P[S_n(\theta) \le z] > N(z;\, 0,\, \tau^2(\theta)) - 4\epsilon$$

for $n > n'$ where n' does not depend on θ . $\Box$

Lemma 2. If $\sup_\theta c_t^2(\theta) < \infty$ for each t and

$$\lim_{n \to \infty} n^{-1} \Sigma_{t=1}^{n} c_t^2(\theta) = \bar{c}(\theta)$$

uniformly in θ where $0 < \bar{\ell} \leq \bar{c}(\theta) \leq \bar{\mu} < \infty$ for all θ then

$$\lim_{n \to \infty} n^{-1} \sup_{1 \leq t \leq n} \sup_\theta c_t^2(\theta) = 0 .$$

Proof: There is a sequence m_n such that

$$\sup_{1 \leq t \leq n} \sup_\theta c_t^2(\theta) = \sup_\theta c_{m_n}^2(\theta) .$$

The lemma would be true trivially if m_n were bounded for all n so we assume the contrary. Given $\epsilon > 0$ there is an n^* which does not depend on θ such that for $n > n^*$ we have

$$n^{-1} \sup_{1 \leq t \leq n} \sup_\theta c_t^2(\theta)$$

$$= \sup_\theta \left[\left(\frac{m_n}{n}\right) \frac{1}{m_n} \Sigma_{t=1}^{m_n} c_t^2(\theta) - \left(\frac{m_n - 1}{n}\right) \frac{1}{m_n - 1} \Sigma_{t=1}^{m_n - 1} c_t^2(\theta) \right]$$

$$< \sup_\theta \left[\left(\frac{m_n}{n}\right)(\bar{c}(\theta) + \epsilon) - \left(\frac{m_n - 1}{n}\right)(\bar{c}(\theta) - \epsilon) \right]$$

$$= \sup_\theta [\bar{c}(\theta)/n + 2m_n \epsilon/n + \epsilon/n]$$

$$\leq \bar{\mu}/n + 2\epsilon - \epsilon/n . \quad \square$$

Theorem. Let $\{Z_t\}$ be the generalized linear process

$$Z_t = \Sigma_{j = -\infty}^{\infty} a_j e_{t-j} \qquad (t = 0, \pm 1, \ldots)$$

where $\Sigma_{j=-\infty}^{\infty} |a_j| < \infty$ and the e_t are independently and identically distributed with mean zero and finite variance $\sigma^2 > 0$. Let $\{c_t(\theta)\}$ be a sequence for which $\sup_\theta c_t^2(\theta) < \infty$ for each t and for which

$$\lim_{n \to \infty} n^{-1} \Sigma_{t=1}^{n-|h|} c_t(\theta) c_{t+|h|}(\theta) = \bar{c}(h,\theta)$$

uniformly in θ where $0 < \bar{\ell} \leq \Sigma_{i=-k}^{k} \Sigma_{j=-k}^{k} a_i a_j \bar{c}(i-j, \theta) \leq \bar{\mu} < \infty$ for all θ and all k. Then

$$S_n(\theta) = (1/\sqrt{n}) \Sigma_{t=1}^{n} c_t(\theta) Z_t$$

converges in distribution to the normal distribution with mean zero and variance

$$\tau^2(\theta) = \sigma^2 \Sigma_{i=-\infty}^{\infty} \Sigma_{j=-\infty}^{\infty} a_i a_j \bar{c}(i-j, \theta) = \Sigma_{k=-\infty}^{\infty} \bar{c}(k,\theta)\gamma(k)$$

uniformly in θ.

Proof: The proof consists of verifying the assumptions of Lemma 1. We split $S_n(\theta)$ as

$$S_n = (1/\sqrt{n}) \Sigma_{t=1}^{n} c_t \Sigma_{j=-\infty}^{\infty} a_j e_{t-j}$$

$$= \Sigma_{t=1}^{n} \Sigma_{j=-k}^{k} (c_t a_j/\sqrt{n}) e_{t-j} + \Sigma_{t=1}^{n} \Sigma_{|j|>k} (c_t a_j/\sqrt{n}) e_{t-j}$$

$$= Z_{kn} + X_{kn}$$

where we have suppressed the argument θ for simplicity.

Let $\delta > 0$ be given. Since $P[|X_{kn}| > \delta] < Var(X_{kn})/\delta^2$ by Chebysheff's inequality, the first assumption of Lemma 1 may be verified by showing $\lim_{k \to \infty} \mathrm{Var}(X_{kn}) = 0$ uniformly in n and θ. For given $\epsilon > 0$ there is a k_o depending only on ϵ such that $(\Sigma_{i > k} |a_i|)^2 + (\Sigma_{i < -k} |a_i|)^2 < \epsilon$ for all $k > k_o$. Then

$$\tfrac{1}{2} \mathrm{Var}(X_{kn}) \le \mathrm{Var}[\Sigma_t \Sigma_{j > k}(c_t a_j/\sqrt{n}) e_{t-j}] + \mathrm{Var}[\Sigma_t \Sigma_{j < -k}(c_t a_j/\sqrt{n}) e_{t-j}]$$

$$= (\sigma^2/n)\Sigma_{i > k} \Sigma_{j > k} a_i a_j \Sigma_{t \in T} c_t c_{t-j+i}$$

$$+ (\sigma^2/n) \Sigma_{i < -k} \Sigma_{j < -k} a_i a_j \Sigma_{t \in T} c_t c_{t-j+i}$$

where

$$T = \{t: \ 1 \le t \le n, \ 1 \le t - j + i \le n\}$$

$$\le (\sigma^2/n)\Sigma_{i > k} \Sigma_{j > k}|a_i a_j|(\Sigma_{t \in T} c_t^2)^{\frac{1}{2}}(\Sigma_{t \in T} c_{t-j+i}^2)^{\frac{1}{2}}$$

$$+ (\sigma^2/n)\Sigma_{i < -k} \Sigma_{j < -k} |a_i a_j|(\Sigma_{t \in T} c_t^2)^{\frac{1}{2}}(\Sigma_{t \in T} c_{t-j+i}^2)^{\frac{1}{2}}$$

$$\le \sigma^2(\Sigma_{i > k} |a_i|)^2(n^{-1}\Sigma_{t=1}^n c_t^2) + \sigma^2(\Sigma_{i < -k} |a_i|)^2(n^{-1}\Sigma_{t=1}^n c_t^2)$$

$$\le \sigma^2 \bar{c}(0, \theta)[(\Sigma_{i > k} |a_i|)^2 + (\Sigma_{i < -k} |a_i|)^2]$$

$$< \sigma^2 \bar{\mu} \epsilon/a_o^2 .$$

This establishes the first condition of Lemma 1.

For n larger than $2k + 1$ we may split $Z_{kn}(\theta)$ as (see Figure 1)

Figure 1.

e_{16}													a_{-3}
e_{15}												a_{-3}	a_{-2}
e_{14}											a_{-3}	a_{-2}	a_{-1}
e_{13}										a_{-3}	a_{-2}	a_{-1}	a_0
e_{12}									a_{-3}	a_{-2}	a_{-1}	a_0	a_1
e_{11}								a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2
e_{10}							a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3
e_9						a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3	
e_8					a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3		
e_7				a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3			
e_6			a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3				
e_5		a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3					
e_4	a_{-3}	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3						
e_3	a_{-2}	a_{-1}	a_0	a_1	a_2	a_3							
e_2	a_{-1}	a_0	a_1	a_2	a_3								
e_1	a_0	a_1	a_2	a_3									
e_0	a_1	a_2	a_3										
e_{-1}	a_2	a_3											
e_{-2}	a_3												
	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}

The sum $\sum_{t=1}^{13}\sum_{j=-3}^{3} c_t a_j e_{t-j}$ is obtained by multiplying each a_j by the corresponding terms on the horizontal and vertical axes and summing. The split is accomplished by segregating terms associated with those a_j between the horizontal lines.

$$Z_{kn} = (1/\sqrt{n}) \Sigma_{p=k+1}^{n-k} e_p \Sigma_{i=-k}^{k} a_i c_{p+i}$$

$$+ (1/\sqrt{n})(\Sigma_{t=1}^{2k} \Sigma_{j=-k}^{k-t} c_t a_j e_{t-j} + \Sigma_{t=n-k+1}^{n} \Sigma_{j=n-k+1-t}^{k} c_t a_j e_{t-j})$$

$$= U_{kn} + V_{kn} .$$

The variance of V_{kn} is bounded by

$$n^{-1} \sup_{1 \leq t \leq n} \sup_\theta c_t^2(\theta)[\mathcal{E}(\Sigma_{t=1}^{2k} \Sigma_{j=-k}^{k} |a_j| \, |e_{t-j}|)^2$$

$$+ \mathcal{E}(\Sigma_{t=n-k+1}^{n} \Sigma_{j=-k}^{k} |a_j| \, |e_{t-j}|)^2] .$$

The term in square brackets does not vary with n so we have

$$\mathrm{Var}(V_{kn}) \leq [n^{-1} \sup_{1 \leq t \leq n} \sup_\theta c_t^2(\theta)] \cdot B$$

which converges to zero as n tends to infinity uniformly in θ by Lemma 2. Consequently, if we show

$$\lim_{n \to \infty} P[U_{kn} \leq z] = N(z;\ 0,\ \sigma_k^2(\theta))$$

uniformly in θ it follows that

$$\lim_{n \to \infty} P[Z_{kn} \leq z] = N(z;\ 0,\ \sigma_k^2(\theta))$$

uniformly in θ .

Set $d_t = \Sigma_{i=-k}^{k} a_i c_{t+i}$. By Theorem 1 of Hertz (1969)

$$\sup_z |P[\sqrt{n}\, U_{kn}/s_n \leq z] - N(z;\ 0,\ 1)| \leq \Delta_{kn}(\theta)$$

where

$$\Delta_{kn}(\theta) = K\, s_n^{-3} \int_0^{s_n} \Psi_n(u)du$$

$$s_n^2 = \sigma^2 \Sigma_{t=k+1}^{n-k} d_t^2 ,$$

$$\Psi_n(c) = \Sigma_{t=k+1}^{n-k} d_t^2 \int_{|d_t e| > c} e^2 d\, F(e) ,$$

and K is a finite constant. Now

$$\Delta_{kn}(\theta) = s_n^{-2} \int_0^1 \Psi(s_n v)dv$$

$$= s_n^{-2} \Sigma_{t=k+1}^{n-k} d_t^2 \int_0^1 \int_{|d_t e| > v\, s_n} e^2 d\, F(e)dv$$

$$\leq s_n^{-2} \Sigma_{t=k+1}^{n-k} d_t^2 \int_o^1 \int_{e^2 > v^2 \inf_{1 \leq t \leq n} \inf_\theta (s_n^2/d_t^2)} e^2 d\, F(e)dv$$

$$= \int_0^1 \int_{e^2 > v^2 \inf_{1 \leq t \leq n} \inf_\theta (s_n^2/d_t^2)} e^2 d\, F(e)dv \quad .$$

Thus, if we show that

$$\lim_{n \to \infty} \inf_{1 \leq t \leq n} \inf_\theta (s_n^2/d_t^2) = \infty$$

we will have $\lim_{n \to \infty} \Delta_{kn}(\theta) = 0$ uniformly in θ by the dominated convergence theorem and the fact that $\int_{-\infty}^{\infty} e^2 d\, F(e) = \sigma^2 < \infty$. Now

$$\lim_{n \to \infty} n^{-1} s_n^2 = \lim_{n \to \infty} \mathrm{Var}(U_{kn})$$

$$= \sigma^2 \Sigma_{i=-k}^{k} \Sigma_{j=-k}^{k} a_i a_j \lim_{n \to \infty} n^{-1} \Sigma_{p=k+1}^{n-k} c_{p+i}(\theta) c_{p+j}(\theta)$$

$$= \sigma^2 \Sigma_{i=-k}^{k} \Sigma_{j=-k}^{k} a_i a_j\, \bar{c}(i-j, \theta)$$

$$= \sigma_k^2(\theta)$$

uniformly in θ. Moreover, $\sigma_k^2(\theta)$ is bounded from below by $\bar{\ell}$ uniformly in θ whence, for ϵ with $0 < \epsilon < \bar{u}$, there is an n_o independent of θ, such that for $n > n_o$

$$\inf_{1 \le t \le n} \inf_\theta (s_n^2/d_t^2) > (\bar{u}-\epsilon)/\sup_{1 \le t \le n} \sup_\theta n^{-1} d_t^2$$

$$\ge (\bar{u}-\epsilon)/\sup_{1 \le t \le n} \sup_\theta n^{-1} (\Sigma_{i=-k}^k a_i^2)(\Sigma_{i=-k}^k c_{t+i}^2)$$

$$= (\bar{u}-\epsilon)/(2k+1)(\Sigma_{i=-k}^k a_i^2) \sup_{1 \le t \le n} \sup_\theta n^{-1} c_t^2$$

$$\to \infty .$$

Thus,

$$|P(U_{kn} \le z) - N(z;\, 0,\, \sigma_k^2(\theta))|$$

$$\le |P[\sqrt{n}\, U_{kn}/s_n \le \sqrt{n}\, z/s_n] - N(\sqrt{n}\, z/s_n;\, 0,\, 1)|$$

$$+ |N(\sqrt{n}\, z/s_n;\, 0,\, 1) - N(z/\sigma_k(\theta);\, 0,\, 1)|$$

$$\le \Delta_{kn}(\theta) + |N(\sqrt{n}\, z/s_n;\, 0,\, 1) - N(z/\sigma_k(\theta);\, 0,\, 1)|$$

$$\to 0$$

uniformly in θ as n tends to infinity.

Lastly, we verify that $\lim_{k \to \infty} \sigma_k^2(\theta) = \tau^2(\theta)$ uniformly in θ.

$$|\tau^2(\theta) - \sigma_k^2(\theta)| = |2\sigma^2 \Sigma_{i > |k|} \Sigma_{j=-\infty}^\infty a_i a_j \bar{c}(i-j,\, \theta)|$$

$$\le 2\sigma^2 \Sigma_{i > |k|} \Sigma_{j=-\infty}^\infty |a_i a_j \bar{c}(i-j,\, \theta)|$$

$$\le 2\sigma^2 \Sigma_{i > |k|} \Sigma_{j=-\infty}^\infty |a_i a_j| \bar{c}(0,\, \theta)$$

because

$$n^{-1}|\Sigma_{t=1}^{n-|k|} c_t(\theta)c_{t+|k|}(\theta)| \leq n^{-1} \Sigma_{t=1}^{n} c_t^2(\theta)$$

$$\leq 2\sigma^2(\bar{u}/a_o^2)(\Sigma_{j=-\infty}^{\infty} |a_j|)(\Sigma_{i>|k|} |a_i|) .$$

The last term on the right does not depend on θ and may be made arbitrarily small by increasing k . □

The result which finds application in nonlinear time series regression is the following corollary.

Corollary. Let $\{Z_t\}_{t=-\infty}^{\infty}$ be the generalized linear process

$$Z_t = \Sigma_{j=-\infty}^{\infty} a_j e_{t-j} \qquad (t = 0, \pm 1, \ldots)$$

where $\Sigma_{j=-\infty}^{\infty} |a_j| < \infty$ and the e_t are independently distributed with mean zero and finite variance $\sigma^2 > 0$. Let $\{c_t\}_{t=1}^{\infty}$ be a sequence of p-vectors for which the limit

$$\bar{c}(h) = \lim_{n\to\infty} \Sigma_{t=1}^{n-|h|} c_t c'_{t+|h|}$$

exists for all $h = 0, \pm 1, \ldots$. Assume that for each non-zero p-vector λ there are finite constants $\bar{\ell}$ and $\bar{u}$ which do not depend on k such that

$$0 < \bar{\ell} \leq \Sigma_{i=-k}^{k} \Sigma_{j=-k}^{k} a_i a_j \lambda'\bar{c}(i-j)\lambda \leq \bar{u} .$$

Then

$$S_n = (1/\sqrt{n}) \Sigma_{t=1}^{n} c_t Z_t$$

converges in distribution to the p-variate normal with mean vector zero and variance-covariance matrix

$$V = (\sigma^2/2) \Sigma_{i=-\infty}^{\infty} \Sigma_{j=-\infty}^{\infty} a_i a_j [\bar{c}(i-j) + \bar{c}'(i-j)]$$

$$= \tfrac{1}{2} \Sigma_{h=-\infty}^{\infty} \gamma(h)[\bar{c}(h) + \bar{c}'(h)] .$$

Proof: We apply 2c.4 of Rao (1965, p. 103). Let $X \sim N_p(0, V)$. By the Theorem, $\lambda' S_n$ converges in distribution to a normal with mean zero and variance

$$\tau = \Sigma_{h=-\infty}^{\infty} \gamma(h) \lim_{n\to\infty} n^{-1} \Sigma_{t=1}^{n-|h|} \lambda' c_t c'_{t+|h|} \lambda$$

$$= \Sigma_{h=-\infty}^{\infty} \gamma(h) \lim_{n\to\infty} n^{-1} \Sigma_{t=1}^{n-|h|} \tfrac{1}{2} \lambda' [c_t c'_{t+|h|} + c_{t+|h|} c'_t] \lambda$$

$$= \Sigma_{h=-\infty}^{\infty} \gamma(h) \tfrac{1}{2} \lambda' [\bar{c}(h) + \bar{c}'(h)] \lambda$$

$$= \lambda' V \lambda .$$

Thus $\lambda' S_n$ converges in distribution to $\lambda' X$ for every non-zero λ. (The matrix $\Sigma_{h=-\infty}^{\infty} \gamma(h) \bar{c}(h)$ is positive definite by assumption but it is not symmetric. This is the reason for the term $\frac{1}{2}[\bar{c}(h) + \bar{c}'(h)]$ in V.) ▯

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