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ON THE ASYMPTOTIC DISTRIBUTIONS OF THE ELEMENTARY SYMMETRIC FUNCTIONS CONSIDERED AS TEST STATISTICS IN TWO MULTIVARIATE TESTS

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1. INTRODUCTION

Tomsky (1974) applied Roy's union-intersection principle to various multivariate tests by considering the index set as a set consisting of matrices of order $k \times p$, $1 \leq k \leq p$. This index set from which the component hypotheses are established is usually considered as a set of vectors, i.e. $k = 1$ (see Roy (1957), Morrison (1967) p. 118). By representing these component test-statistics in terms of the elementary symmetric functions of the characteristic roots of the matrix appearing in the likelihood ratio component test-statistic, he derived a new class of multivariate test-statistics for various tests.

We shall consider two tests here, namely the tests of equality of mean vectors from samples from multivariate normal populations and the test of independence between subvectors that are normally distributed. The asymptotic distributions of the elementary symmetric functions proposed as test-statistics, will be derived here.

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2. TESTING EQUALITY OF SEVERAL MEAN VECTORS

The test of equality of mean vectors of several normal populations can be framed into the following situation: Let $X(p \times q)$ be distributed $N\left(M(p \times q), \Sigma \otimes I_q\right)$ and $A(p \times p)$ be distributed independently as $W(\Sigma,n)$. The random matrix of interest in testing $M = 0$ against $M \neq 0$ is either

2.1 $$F = X'A^{-1}X \quad \text{for} \quad q < p$$

or

2.2 $$V = B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}} \quad \text{for} \quad q \geq p$$

where $B = XX'$ and $B^{\frac{1}{2}}$ the symmetric square root of B distributed according to $W(\Sigma,n,\Omega)$, $\Omega = \Sigma^{-1}MM'$. Both $F(q \times q)$ or $V(p \times p)$ are distributed noncentral multivariate beta with $(p, n + q - p)$ and (q,n) degrees of freedom respectively. In the case of F the noncentrality parameter is taken as $\Omega = M'\Sigma^{-1}M$.

Tomsky (1974) obtained the following class of test statistics for this problem:

2.2 $$T_{jk} = f_j\left(1 + \lambda_1(V), 1 + \lambda_2(V), \ldots, 1 + \lambda_k(V)\right), \quad k \leq p,$$

where $\lambda_1(V) \geq \lambda_2(V) \geq \ldots \geq \lambda_p(V)$ are the characteristic roots of V and $f(\circ)$ is the j-th esf of the characteristic roots $\lambda_i(V)$, $i = 1, \ldots, k$. If $k = p$, $T_{jp} = tr_j(I + V)$, and this is the class of statistics we are considering here. It is interesting to note that $T_{1p} = |I + V|$, the likelihood ratio statistic and $T_{11} = 1 + \lambda_1(V)$, Roy's "maximum root" statistic.

We shall now consider the asymptotic distribution of $tr_j(I_q + F)$.

Lemma 2.1. (de Waal (1974)) Let $B(q \times q)$ be p.d.s., $\partial = \left(\frac{1}{2}(1 + \delta_{rs})\frac{\partial}{\partial\sigma_{rs}}\right)$, $r,s = 1, \ldots ,q$, and $R(q \times q)$ any fixed p.d.s. matrix, then

$$2.4 \quad etr(B\partial)\exp\left(\nu tr_j(I_q + \nu^{-1}\Sigma)\right)\Big|_{\Sigma=\nu R^{-1}} = etr(B\Gamma_j)\exp\left(\nu tr_j(I + R^{-1})\right) + 0(\nu^{-1})$$

where $\Gamma_j = (-1)^{j-1}\sum_{i=1}^{j}(-1)^{j-i}R^{1-i}tr_{j-i}R^{-1}$.

Lemma 2.2. (de Waal (1974)) If $F(q \times q)$ is defined as in 2.1 then for any fixed $R(q \times q)$ p.d.s. and $\nu = int$

$$2.5 \quad E\ etr(\nu trRF) = etr(-\tfrac{1}{2}\Omega)etr(\tfrac{1}{2}\Omega R^{-1})(1 - 2it)^{-\frac{1}{2}pq}etr\left(\frac{it}{1-2it}\,\Omega R^{-1}\right) + 0(n^{-1}) ,$$

where $\Omega = M'\Sigma^{-1}M$.

Remark: If however F is replaced by V defined in 2.2, then 2.5 stays the same except that R is then of order p and Ω is defined as $\Sigma^{-\frac{1}{2}}MM'\Sigma^{-\frac{1}{2}}$.

Theorem 2.1: If F is defined as in 2.1, $\Omega = M'\Sigma^{-1}M$ and

$$2.6 \qquad \xi_1 = \frac{n}{\tau(j)}\left\{tr_j(I_q + F) + trR^{-1}\Gamma_j - tr_j(I + R^{-1})\right\} ,$$

then

$$P(\xi_1 < x) = P\left(\chi_f^2(\delta^2) < x\right) + 0(n^{-1})$$

where $f = pq$, $\delta^2 = tr(\frac{1}{2}\Omega)$, $\tau(j) = \dfrac{tr\Omega}{tr\Omega\Gamma_j^{-1}}$ and R^{-1} any fixed p.d.s.

matrix. Γ_j is given in lemma 2.1.

Proof: Let $\nu = \text{int}$ and τ any constant, then the characteristic function of

$$n \, \text{tr}_j \left(\frac{1}{\tau}(I + F)\right)$$

after expanding it into a Taylor series at

$$\frac{\nu F}{\tau} = \frac{\nu R^{-1}}{\tau} \tag{2.7}$$

and applying lemma 2.1, can be written as

$$\begin{aligned} E \exp\left(\nu \text{tr}_j \left(\frac{1}{\tau}(I + F)\right)\right) &= E \, \text{etr}\left(\frac{\nu}{\tau}(F - R^{-1})\partial\right) \exp\left(\nu \text{tr}_j \left(\frac{1}{\tau} I + \nu^{-1}\Sigma\right)\right)\Big|_{\Sigma=\nu R^{-1}} \\ &= E \, \text{etr}\left(\frac{\nu}{\tau^j}(F - R^{-1})\Gamma_j\right) \exp\left(\nu \text{tr}_j \left(\frac{1}{\tau}(I + R^{-1})\right)\right) + 0(n^{-1}) \\ &= \exp\left(\frac{\nu}{\tau^j}\left(\text{tr}_j (I + R^{-1}) - \text{tr} R^{-1}\Gamma_j\right)\right) E \, \text{etr}\left(\frac{\nu}{\tau^j} F\Gamma_j\right) + 0(n^{-1}) \; . \end{aligned}$$

Hence using lemma 2.2 the characteristic function of

$$n\tau^{-j}\{\text{tr}_j(I + F) + \text{tr} R^{-1}\Gamma_j - \text{tr}_j(I + R^{-1})\}$$

can be written as

$$\text{etr}(-\tfrac{1}{2}\Omega)\,\text{etr}(\tfrac{1}{2}\tau^j \Omega\Gamma_j^{-1})(1 - 2it)^{-\frac{1}{2}pq}\,\text{etr}\left(\frac{it\tau^j}{1-2it}\Omega\Gamma_j^{-1}\right) + 0(n^{-1}) \; .$$

If we let

$$\tau^j = \frac{\text{tr}\Omega}{\text{tr}\Omega\Gamma_j^{-1}} = \tau(j) \quad \text{say ,}$$

the theorem is proved.

<u>Corollary 2.1.</u> If $V(p \times p)$ is defined as in 2.2, $\Omega = \Sigma^{-\frac{1}{2}} MM' \Sigma^{-\frac{1}{2}}$ and

2.8 $$\xi_2 = \frac{n}{\tau(j)} \{tr_j(I_p + V) + trR^{-1}\Gamma_j - tr_j(I_p + R^{-1})\}$$

then

2.9 $$P(\xi_2 < x) = P\left(\chi_f^2(\delta^2) < x\right) + O(n^{-1})$$

where f , δ^2 and $\tau(j)$ are defined in theorem 2.1 and for $R^{-1}(p \times p)$ p.d.s. Γ_j stays the same as in lemma 2.1.

<u>Proof:</u> This corollary follows directly from the remark on lemma 2.2 and the theorem.

Since

$$\operatorname*{plim}_{n\to\infty} nF = \Omega$$

we would like to choose R^{-1} given in 2.7 as close to the nil matrix as possible. So let N^* be a fixed large value and let

2.10 $$R^{-1} = \frac{\Omega}{N^*} .$$

Then ξ_1 defined in 2.6 becomes

2.11 $$\xi_1 = \frac{nN^{*(j-1)} tr\Omega\Omega^{-1}(j)}{tr\Omega} \left\{tr_j(I + F) + \frac{1}{N^{*j}} tr\Omega\Omega(j) - tr_j(I + \frac{1}{N^*}\Omega)\right\}$$

where

2.12 $$\Omega(j) = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} \Omega^{i-1} tr_{j-i}\Omega .$$

We note that

$$\operatorname*{plim}_{n\to\infty} \frac{\xi_1}{n} = \frac{N^{*(j-1)}\operatorname{tr}\Omega\Omega^{-1}(j)}{\operatorname{tr}\Omega}\left\{\binom{q}{j} + \frac{1}{N^{*j}}\operatorname{tr}\Omega\Omega(j) - \operatorname{tr}_j(I + \frac{1}{N^*}\Omega)\right\} .$$

3. TEST FOR INDEPENDENCE

Let $A(p \times p)$ be distributed $W(\Sigma,n)$ and A and Σ be partitioned as

$$A = \begin{pmatrix} A_{11}(q\times q) & A_{12} \\ A_{21} & A_{22} \end{pmatrix} , \quad \Sigma = \begin{pmatrix} \Sigma_{11}(q\times q) & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} ;$$

then (Troskie (1969))

3.1 $$A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21} \sim W(\Sigma_{11.2}, n - p + q)$$

and conditional on A_{22} independent of $A_{11.2}$

3.2 $$G|A_{22} \sim W(\Sigma_{11.2}, p - q, \Omega)$$

where $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, $G = A_{12}A_{22}^{-1}A_{21}$, $\Omega = \Sigma_{11.2}^{-\frac{1}{2}}\beta A_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}$ and $\beta = \Sigma_{12}\Sigma_{22}^{-1}$.

Let

3.3 $$R = A_{11}^{-\frac{1}{2}}A_{12}A_{22}^{-1}A_{21}A_{11}^{-\frac{1}{2}} ,$$

then R is called the generalized correlation matrix.

The following esf's are among the class of test-statistics obtained by Tomsky (1974) for testing independence, i.e. $\Sigma_{12} = 0$:

3.4 $$\operatorname{tr}_j(I - R)^{-1} = \operatorname{tr}_j(I + A_{11.2}^{-1}G) .$$

The following theorem can now be proved:

<u>Theorem 3.1.</u> Let $R = A_{11}^{-\frac{1}{2}}A_{12}A_{22}^{-1}A_{21}A_{11}^{-\frac{1}{2}}$, $\Omega = \Sigma_{11.2}^{-\frac{1}{2}}\beta A_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}$, $\beta = \Sigma_{12}\Sigma_{22}^{-1}$, then conditional on A_{22} the characteristic function of

$$3.5 \qquad ntr_j\left(\tfrac{1}{\tau}(I - R)^{-1}\right)$$

can be written as

$$3.6 \quad E\exp\left[itntr_j\left(\tfrac{1}{\tau}(I - R)^{-1}\right)\right] = \exp\left[\frac{itn}{\tau^j}\left(tr_j(I + Q) - trQ\Gamma_j\right)\right]etr(-\tfrac{1}{2}\Omega)$$
$$etr\left(\tfrac{1}{2}\tau^j\Omega\Gamma_j^{-1}\right)(1 - 2it)^{-\frac{1}{2}q(p-q)}\,etr\left(\frac{it\tau^j}{1-2it}\Omega\Gamma_j^{-1}\right) + $$
$$+ O(n^{-1}),$$

where Γ_j is given in 3.8.

<u>Proof:</u> By expanding the characteristic function of 3.5 as a Taylor series at

$$\frac{itnA_{11.2}^{-1}G}{\tau} = \frac{itnQ}{\tau}$$

for Q p.d.s. fixed, it can be written conditional on A_{22} as

$$3.7 \quad E\exp\left[itntr_j\left(\tfrac{1}{\tau}(I - R)^{-1}\right)\right] = \exp\left[\frac{itn}{\tau^j}\left(tr_j(I + Q) - trQ\Gamma_j\right)\right]E\,etr\left(\frac{itn}{\tau^j}A_{11.2}^{-1}G\Gamma_j\right) +$$
$$+ O(n^{-1})$$

$$= \exp\left[\frac{itn}{\tau^j}\left(tr_j(I + Q) - trQ\Gamma_j\right)etr(-\tfrac{1}{2}\Omega)\right.$$
$$etr\left(\tfrac{1}{2}\tau^j\Omega\Gamma_j^{-1}\right)(1 - 2it)^{-\frac{1}{2}q(p-q)}\,etr\left(\frac{it\tau^j}{1-2it}\right)\Omega\Gamma_j^{-1}) +$$
$$+ O(n^{-1})$$

where

3.8 $$\Gamma_j = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} Q^{i-1} \mathrm{tr}_{j-i} Q \ .$$

This is true since the characteristic roots of $A_{11.2}^{-1}G$ are also the characteristic roots of $G^{\frac{1}{2}}A_{11.2}^{-1}G^{\frac{1}{2}}$ which is distributed conditional on A_{22} as a noncentral multivariate beta distribution with $p - q$ and $n - p + q$ degrees of freedom and noncentrality parameter $\Omega = \Sigma_{11.2}^{-\frac{1}{2}}\beta A_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}$. Hence the theorem follows by applying lemma 2.2 remembering that F is distributed as a noncentral multivariate beta with p and $n + q - p$ degrees of freedom and noncentrality parameter $\Omega = M'\Sigma^{-1}M$. Q.E.D.

But since $A_{22}(p - q \times p - q)$ is distributed $W(\Sigma_{22},n)$, we can find the unconditional characteristic function of $n\mathrm{tr}_j\left(\frac{1}{\tau}(I - R)^{-1}\right)$ and hence the unconditional distribution function.

<u>Theorem 3.2.</u> Let $R = A_{11}^{-\frac{1}{2}}A_{12}A_{22}^{-1}A_{11}^{-\frac{1}{2}}$, $\Sigma_{11}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-\frac{1}{2}} = \frac{1}{n}\Theta$ and

3.9 $$\xi_3 = \frac{n\mathrm{tr}\Theta\Theta^{-1}(j)}{\mathrm{tr}\Theta} \{\mathrm{tr}_j(I - R)^{-1} + \mathrm{tr}\Theta\Theta(j) - \mathrm{tr}_j(I + \Theta)\}$$

then

$$P(\xi_3 < x) = P\left(\chi_f^2(\delta^2) < x\right) + O(n^{-1})$$

where

$$f = q(p - q) \ , \quad \delta^2 = \mathrm{tr}(\tfrac{1}{2}\Theta) \quad \text{and}$$

$$\Theta(j) = (-1)^{j-1}\sum_{i=1}^{j}(-1)^{j-i}\Theta^{i-1}\mathrm{tr}_{j-i}\Theta \ .$$

<u>Proof:</u> From 3.6 the unconditional characteristic function of $ntr_j\left(\frac{1}{\tau}(I - R)^{-1}\right)$ is given by

3.10 $$E \exp\left[itntr_j\left(\frac{1}{\tau}(I - R)^{-1}\right)\right] = \exp\left[\frac{itn}{\tau^j}\left(tr_j(I + Q) - trQ\Gamma_j\right)\right](1 - 2it)^{-\frac{1}{2}q(p-q)}$$

$$E_{A_{22}} etr(WA_{22}) + 0(n^{-1})$$

where $W = -\frac{1}{2}\beta'\Sigma_{11.2}^{-1}\beta + \frac{1}{2}\tau^j\beta'\Sigma_{11.2}^{-\frac{1}{2}}\Gamma_j^{-1}\Sigma_{11.2}^{-\frac{1}{2}}\beta + \frac{it\tau^j}{1-2it}\beta'\Sigma_{11.2}^{-\frac{1}{2}}\Gamma_j^{-1}\Sigma_{11.2}^{-\frac{1}{2}}\beta$

and the expectation is taken w.r.t. the density of A_{22} . But

3.11 $$E_{A_{22}} etr(WA_{22}) = |\Sigma_{22}^{-1} - 2W|^{-\frac{1}{2}n} |\Sigma_{22}|^{-\frac{1}{2}n}$$
$$= |I - 2\Sigma_{22}^{\frac{1}{2}}W\Sigma_{22}^{\frac{1}{2}}|^{-\frac{1}{2}n} .$$

If we define the population generalized correlation matrix as

3.12 $$P = \Sigma_{11}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-\frac{1}{2}}$$

and assume that

3.13 $$P = \frac{1}{n}\Theta \quad \text{(see Sugiura (1969))}$$

then

$$\Sigma_{11.2}^{-1} = \Sigma_{11}^{-1} + 0(\tfrac{1}{n}) .$$

Let $P_1 = \Sigma_{22}^{\frac{1}{2}}\beta'\Sigma_{11}^{-\frac{1}{2}} = \frac{1}{\sqrt{n}}\Theta_1$ say, where $\Theta_1'\Theta_1 = \Theta$, then

3.14 $$\Sigma_{22}^{\frac{1}{2}}W\Sigma_{22}^{\frac{1}{2}} = -\frac{1}{2}\,\frac{1}{n}\Theta + \frac{1}{2}\tau^j\,\frac{1}{n}\Theta_1\Gamma_j^{-1}\Theta_1' + \frac{it\tau^j}{(1-2it)n}\Theta_1\Gamma_j^{-1}\Theta_1' + 0(n^{-2}) .$$

Using the relation

$$|I - \frac{1}{n}\xi|^{-nx} = \text{etr } x\xi + O(n^{-1})$$

3.11 can therefore be written as

3.15 $$E_{A_{22}}\text{etr}(WA_{22}) = \text{etr}\left(-\frac{1}{2}\Theta + \frac{1}{2}\tau^j\Theta\Gamma_j^{-1} + \frac{it\tau^j}{1-2it}\Theta\Gamma_j^{-1}\right) + O(n^{-1})$$
$$= \text{etr}\left(\frac{it\tau(j)\Theta\Gamma_j^{-1}}{1-2it}\right) + O(n^{-1})$$

where τ^j is chosen such that

$$\text{tr}(\tfrac{1}{2}\Theta) = \tfrac{1}{2}\tau^j\text{tr}\Theta\Gamma_j^{-1}$$

i.e. $$\tau^j = \frac{\text{tr}\Theta}{\text{tr}\Theta\Gamma_j^{-1}} = \tau(j)$$ say.

Substitute 3.15 in 3.10 then it follows that the characteristic function of

3.16 $$\frac{n}{\tau(j)}\left\{\text{tr}_j(I - R)^{-1} + \text{tr}Q\Gamma_j - \text{tr}_j(I + Q)\right\} =$$
$$= (1 - 2it)^{-\frac{1}{2}q(p-q)}\text{etr}\left(\frac{it\tau(j)\Theta\Gamma_j^{-1}}{1-2it}\right) + O(n^{-1}) .$$

Since we assumed $P = \frac{1}{n}\Theta$ in 3.13, it follows that

$$\Sigma_{11.2}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11.2}^{-\frac{1}{2}} = \frac{1}{n}\Theta + O(n^{-2})$$

i.e.

$$\operatorname*{plim}_{n\to\infty} nA_{11.2}^{-\frac{1}{2}}GA_{11.2}^{-\frac{1}{2}} = \Theta .$$

Therefore, let $Q = \Theta$ in 3.16 and

3.17 $$\Theta(j) = \Gamma_j\Big|_{\Theta=\Theta}$$
$$= (-1)^{j-1}\Sigma_{i=1}^{j}(-1)^{j-i}\Theta^{i-1}\text{tr}_{j-i}\Theta ,$$

and the theorem follows.

We notice that

$$\underset{n\to\infty}{\text{plim}}\ \frac{1}{n}\xi_3 = \frac{\text{tr}\Theta\Theta^{-1}(j)}{\text{tr}\Theta}\left\{\text{tr}_j I + \text{tr}\Theta\Theta(j) - \text{tr}_j(I + \Theta)\right\} .$$

REFERENCES

[1] de Waal, D. J. (1974): An asymptotic distribution for the j-th esf of the generalised Hotelling's beta matrix. *Inst. of Statist. Mimeo Series no.* 966, University of North Carolina.

[2] Morrison, D. F. (1967): *Multivariate Statistical Methods*, McGraw-Hill, New York.

[3] Roy, S. N. (1957): *Some Aspects of Multivariate Analysis*, Wiley, New York.

[4] Sugiura, N. (1969): Asymptotic non-null distributions of the likelihood ratio criteria for covariance matrix under local alternatives, *Inst. of Statist. Mimeo Series no.* 609, University of North Carolina.

[5] Tomsky, J. L. (1974): A new class of multivariate tests based on the union-intersection principle. Tech report no. 83, Department of Statist., Stanford, Univ.

[6] Troskie, C. G. (1969): The generalised multiple correlation matrix. *S.A. Statist. J.* 3, 109-121.