A SYMMETRY PROPERTY OF THE GENOCCHI NUMBERS

by

Dominique Dumont

Dêpartement de Mathêmtique
Universitê de Strasbourg

Dominique Foata

Department of Statistics
University of North Carolina at Chapel Hill

Institute of Statsstics Mimeo Series No. 981

January, 1975

A SYMMETRY PROPERTY OF THE GENOCCHI NUMBERS

by

Dominique Dumont* and Dominique Foata**

ABSTRACT

A refinement of the Genocchi numbers into a sum of symmetric coefficients is obtained, together with a combinatorial interpretation for this refinement.

*Université de Strasbourg, France

**University of North Carolina, Chapel Hill,U.S.A.

1. INTRODUCTION.

The *Genocchi numbers* G_{2n} $(n > 0)$ are positive integers, which may be defined by means of their exponential generating function (or rather that of the coefficients $(-1)^n G_{2n}$)

(1) $$2u/(e^u + 1) = u + \sum_{n\geq 1} (u^{2n}/(2n)!)\ (-1)^n G_{2n} .$$

They are the closest integers to the *Bernoulli numbers* B_{2n} $(n > 0)$ in the following sense. Let the latter numbers be defined by

(2) $$u/(e^u - 1) = 1 - u/2 + \sum_{n\geq 1} (u^{2n}/(2n)!)\ (-1)^{n+1} B_{2n} .$$

Then, the following identity is a simple consequence of von Staudt-Clausen theorem (see e.g. Carlitz [2], or Hardy and Wright [6] p. 91)

(3) $$a(a^{2n} - 1)\ B_{2n} \equiv 0 \mod 1$$

for any integer a . The least non-trivial integer to be considered is $a = 2$ and this gives precisely the Genocchi numbers

(4) $$G_{2n} = 2(2^{2n} - 1)\ B_{2n} \qquad (n > 0) .$$

The first values of these two sequences of numbers are shown in table 1.

TABLE 1

n	1	2	3	4	5	6	7
G_{2n}	1	1	3	17	155	2073	38227
B_{2n}	1/6	1/30	1/42	1/30	5/66	691/2730	7/6

The purpose of this paper is to prove the following two theorems 1a and 2 .

1a. Theorem. *Let* $(F_n(x,y,x))_{n>0}$ *be the sequence of polynomials in three variables* x, y, z *defined by the recurrence relation*

$$F_1(x,y,z) = 1 \tag{5}$$

and for $n \geq 2$

$$F_n(x,y,z) = (x+z)\ (y+z)\ F_{n-1}(x,y,z+1) - z^2\ F_{n-1}(x,y,z)\ . \tag{6}$$

Then, for any $n > 0$ *the polynomial* $F_n(x,y,z)$ *is symmetric with respect to the set of the three variables* x, y, z . *Moreover*

$$F_n(1,1,1) = G_{2n+2} \qquad (n > 0)\ . \tag{7}$$

Clearly, the coefficients of the polynomials $F_n(x,y,z)$ are non-negative integers. Theorem 1 then gives a *refinement* of the Genocchi numbers into symmetric coefficients. More precisely, let

$$F_n(x,y,z) = \sum_{1\leq i,j,k} a_{n,i,j,k}\ x^{i-1}\ y^{j-1}\ z^{k-1} \qquad (n > 0)\ . \tag{8}$$

A straightforward algebraic argument shows that (5) and (6) are equivalent to the recurrence relation on the $a_{n,i,j,k}$'s

$$a_{1,1,1,1} = 1\ ,\quad a_{1,i,j,k} = 0 \ \text{ if } \ (i,j,k) \neq (1,1,1) \tag{9}$$

and for $n \geq 2$, $1 \leq i,\ j,\ k \leq n$

(10) $$a_{n,i,j,k} = \sum_{k-1\le\ell\le n-1} \binom{\ell-1}{k-1} a_{n-1,i-1,j-1,\ell} + \binom{\ell-1}{k-2} (a_{n-1,i-1,j,\ell} + a_{n-1,i,j-1,\ell}) + \binom{\ell-1}{k-3} a_{n-1,i,j,\ell} .$$

We then have another way of stating theorem 1

1b. <u>Theorem</u>. *Let* $a_{n,i,j,k}$ $(n \ge 1 , \quad 1 \le i, j, k \le n)$ *be the sequence of integers defined by* (9) *and* (10) . *Then, for any sequence of integers* (i, j, k) *and any permutation* (i', j', k') *of* (i, j, k)

(11) $$a_{n,i',j',k'} = a_{n,i,j,k} .$$

Moreover

(12) $$G_{2n+2} = \sum_{1\le i,j,k\le n} a_{n,i,j,k} .$$

A table of the first values of the coefficients $a_{n,i,j,k}$ appears in table 2.

Theorem 1a is proved in section 2. It is a consequence of an earlier result due to Carlitz [2], Riordan and Stein [7].

The main result of our paper is stated in theorem 2, which provides with a *combinatorial interpretation* for the sequence $(F_n(x,y,z))_{n>0}$. A map f of the interval $[2n] = \{1, 2, \ldots, 2n\}$ into itself is said to be *exceedant*, if the inequality $x \le f(x)$ holds for every x in [2n] . For each $n > 0$ let A_n denote the set of all exceedant surjections of [2n] onto the subset $\{2, 4, 6, \ldots, 2n\}$ of the even integers. Let f belong to A_n and x to [2n] . The ordered pair (x, f(x)) is said to be an *upper record* of f if either $x = 1$, or $1 < x \le 2n$ and $f(y) < f(x)$ for all y with $1 \le y < x$, a *fixed point* of f if $f(x) = x$ and a *maximum occurrence* of f if $1 \le x \le 2n-1$ and $f(x) = 2n$. The number of upper records (resp. fixed points, maximum occurrences) of f is denoted by $I(f)$ (resp. $J(f)$, $K(f)$).

2. Theorem. *For each* $n > 0$ *the trivariate generating function*

$$\sum\{x^{I(f)}\, y^{J(f)}\, z^{K(f)} \;:\; f \in A_n\} \tag{13}$$

of the vector (I, J, K) *over* A_n *is equal to* $xyz\, F_n(x,y,z)$.

It follows from theorems 1 and 2 that

$$G_{2n+2} = \text{card } A_n \qquad (n > 0)\,. \tag{14}$$

In fact, this result was already established by the first author [4], when he gave the first combinatorial interpretations for the Genocchi numbers. It also follows from theorem 1 that the distribution of the vector (I, J, K) is *symmetric*. This property of symmetry can be proved directly, as shown in sections 3 and 4. Let $a_{n,i,j,k}$ be the number of elements f in A_n having $I(f) = i$ upper records, $J(f) = j$ fixed points, and $K(f) = k$ maximum occurrences $(1 \le i,j,k,n)$. In section 3 it is proved that $a_{n,i,j,k}$ satisfies recurrence relation (10). To prove the symmetry it suffices to establish (10) after permuting the role of i and k. More precisely, it suffices to prove that

(15) $$a_{n,i,j,k} = \sum_{i-1\le\ell\le n-1} \binom{\ell-1}{i-1} a_{n-1,\ell,j-1,k-1} + \binom{\ell-1}{i-2} (a_{n-1,\ell,j,k-1} + a_{n-1,\ell,j-1,k}) + \binom{\ell-1}{i-3} a_{n-1,\ell,j,k} .$$

This is done in section 4. As the distribution of the vector (I, J, K) is symmetric, there exist *involutions* $f \to f'$ and $f \to f''$ of the set A_n with the following properties

$$I(f') = J(f) , \quad J(f') = I(f) , \quad K(f') = K(f)$$

on the one hand, and

$$I(f'') = I(f) , \quad J(f'') = K(f) , \quad K(f'') = J(f)$$

on the other hand. Such involutions are described in section 5. They are based upon the combinatorial constructions given in section 3 and 4. Numerical tables are added at the end of the paper.

The authors thank Professor Carlitz for his interest in the present results, and also his giving them an interesting note [3] with explicit formulas for $F_n(x,y,z)$.

2. THE GANDHI CONJECTURE.

Gandhi [5] proposed the following conjecture about Genocchi numbers. Let $(P_n(z))_{n\ge 0}$ be the sequence of polynomials in one variable z defined by the recurrence relation

(16)
$$P_0(z) = 1$$
$$P_n(z) = z^2 P_{n-1}(z+1) - (z-1)^2 P_{n-1}(z) \quad \text{for} \quad n \geq 1 .$$

Then, Gandhi conjectured that

(17)
$$G_{2n+2} = P_n(1)$$

held for any $n \geq 0$. The conjecture was immediately proved to be correct by Carlitz [2] and Riordan and Stein [7] . The change of variables

(18)
$$Q_n(z) = P_{n-1}(z+1) \qquad (n \geq 1)$$

yields a sequence of polynomials $(Q_n(z))_{n>0}$ defined by the recurrence relation

(19)
$$Q_1(z) = 1$$
$$Q_n(z) = (z+1)^2 Q_{n-1}(z+1) - z^2 Q_n(z) \qquad (n > 1) .$$

As $P_n(1) = P_{n-1}(2) = Q_n(1)$, relation (17) implies

(20)
$$G_{2n+2} = Q_n(1) \qquad (n > 0) .$$

When comparing (6) with (19) it is readily seen that

(21)
$$Q_n(z) = F_n(1,1,z) \qquad (n > 0) .$$

Hence (7) is a consequence of (20) and (21) . The proof of theorem 1 will be completed if the symmetry of the polynomials $F_n(x,y,z)$ $(n > 0)$ is established. This can be done as follows. Relation (6) is clearly symmetric with respect to the pair $\{x,y\}$. It then suffices to show that (6) also holds when x and z are permuted. In other words, it suffices to establish the following recurrence relation

$$(22)\quad F_n(x,y,z) = (x+y)(x+z)\ F_{n-1}(x+1,y,z) - x^2\ F_{n-1}(x,y,z) \qquad (n > 1)\ .$$

First $F_1(x,y,z) = 1$ and (6) imply that $F_2(x,y,z) = xy + yz + zx$. Now let $m > 2$ and assume that (22) has been established for any $n < m$. Then

$$\begin{aligned} F_m(x,y,z) &= (x+z)(y+z)\ F_{m-1}(x,y,z+1) - z^2\ F_{m-1}(x,y,z) \\ &= (x+z)(y+z)((x+y)(x+z)\ F_{m-2}(x+1,y,z+1) - x^2\ F_{m-2}(x,y,z+1)) \\ &\quad - z^2((x+y)(x+z)\ F_{m-2}(x+1,y,z) - x^2\ F_{m-2}(x,y,z)) \end{aligned}$$

by induction on m. By associating the first with the third term, and the second with the fourth one, and using (6) we clearly obtain (22) (with m replacing n).

3. EXCEEDANT SURJECTIONS.

For any $n, i, j, k > 0$ denote by $A_{n,i,j,k}$ the set of all exceedant surjections f of $[2n]$ onto $\{2, 4, 6, \ldots, 2n\}$ which have $I(f) = i$ upper records, $J(f) = j$ fixed points, and $K(f) = k$ maximum occurrences. Also let

$$A_{n,\cdot,\cdot,k} = \bigcup_{i,j\geq 1} A_{n,i,j,k} \quad\text{and}\quad A_{n,i,\cdot,\cdot} = \bigcup_{j,k\geq 1} A_{n,i,j,k}\ .$$

The purpose of this section is to prove that $a_{n,i,j,k} = \operatorname{card} A_{n,i,j,k}$

satisfies recurrence relation (10) . As was noticed in the introduction, this is equivalent to proving theorem 2.

Let

(23) $a_{n,.,.,k} = \text{card } A_{n,.,.,k}$ and $a_{n,i,.,.} = \text{card } A_{n,i,.,.}$.

If $(x, f(x))$ is an upper record (resp. fixed point, maximum occurrence) of an exceedant surjection f, it will be convenient to say that x is the *index* and $f(x)$ the *value* of the ordered pair $(x, f(x))$. For any $k, \ell > 0$ denote by $\begin{pmatrix}[\ell+1]\\ k-1\end{pmatrix}$ the set of all the $\begin{pmatrix}\ell+1\\ k-1\end{pmatrix}$ increasing sequences $(c_1, c_2, \ldots, c_{k-1})$ with length $k-1$ satisfying the inequalities

(24) $$1 \le c_1 < c_2 < \ldots < c_{k-1} \le \ell + 1 .$$

If $k = 1$ the set $\begin{pmatrix}[\ell+1]\\ k-1\end{pmatrix}$ only contains the empty sequence.

(25) *Construction of a bijection*

$$\Phi : f \to ((c_1, c_2, \ldots, c_{k-1}), g)$$

of $A_{n,.,.,k}$ *onto* $\bigcup_{k-1\le\ell\le n-1} \begin{pmatrix}[\ell+1]\\ k-1\end{pmatrix} \times A_{n-1,.,.,\ell}$ *for* $n > 1$.

For each f in A_n with $n > 1$ define the following integral-valued map g by

(26) $$g(x) = \min(2n-2, f(x)) \quad \text{for } 1 \le x \le 2n-2 .$$

Clearly g is an exceedant surjection of $[2n-2]$ onto $\{2, 4, 6, \ldots, 2n-2\}$. Let $t_1, t_2, \ldots, t_\ell$ be the indices of the ℓ maximum occurrences of g, labeled from left to right, so that $1 \le t_1 < t_2 < \ldots < t_\ell = 2n-3$. Also let $t_{\ell+1} = 2n-2$. The equality $f(x) = 2n$ cannot hold, unless x is equal to $2n$, $2n-1$ or one of the indices $t_1, t_2, \ldots, t_{\ell+1}$. Let f have k maximum occurrences, whose indices are denoted by $s_1, s_2, \ldots, s_k$, with $1 \le s_1 < s_2 < \ldots < s_k = 2n-1$. As $s_1, s_2, \ldots, s_k$ occur in the sequence $(t_1, t_2, \ldots, t_{\ell+1})$, the map g belongs to $A_{n-1,.,.,\ell}$ with $k - 1 \le \ell$. Also $\ell \le n - 1$, because every exceedant surjection in A_{n-1} cannot have more than $n - 1$ maximum occurrences. If $k > 1$ define the sequence $(c_1, c_2, \ldots, c_{k-1})$ as being the increasing sequence of the subscripts c such that

(27) $$f(t_c) = 2n .$$

By construction

(28) $$1 \le c_1 < c_2 < \ldots < c_{k-1} \le \ell + 1 .$$

If $k = 1$ let $(c_1, c_2, \ldots, c_{k-1})$ be the empty sequence. In this case $(2n-1 , f(2n-1))$ is the only maximum occurrence of f and g is simply the restriction of f to the interval $[2n-2]$.

In the following example we determine the pair $((c_1, c_2, \ldots, c_{k-1}) , g)$ associated with f defined by:

$x =$	1	2	3	4	5	6	7	8
$f(x) =$	6	2	4	8	6	8	8	8
$g(x) =$	6	2	4	6	6	6		

Here $K(f) = k = 3$, $K(g) = \ell = 3$, $(t_1, \ldots, t_{\ell+1}) = (1, 4, 5, 6)$ and $(c_1, \ldots, c_{k-1}) = (2, 4)$.

Clearly Φ is injective. To prove that Φ is also bijective and give the *construction of its inverse* Φ^{-1} we proceed as follows. Let g belong to $A_{n-1,.,.,\ell}$ with $k-1 \le \ell \le n-1$ and let $(c_1, c_2, \ldots, c_{k-1})$ satisfy (28). The map g has ℓ maximum occurrences that will be denoted by $t_1, t_2, \ldots, t_\ell$. With $t_{\ell+1} = 2n-2$ define $f = \Phi^{-1}((c_1, c_2, \ldots, c_{k-1}), g)$ by

$$(29) \qquad \begin{aligned} f(x) &= 2n \quad \text{if} \quad x = t_{c_1}, t_{c_2}, \ldots, t_{c_{k-1}}, 2n-1, 2n \\ &= g(x) \quad \text{if} \quad 1 \le x \le 2n-2 \quad \text{and} \quad x \ne t_{c_1}, t_{c_2}, \ldots, t_{c_{k-1}} . \end{aligned}$$

Clearly (26) holds. The inequality $k-1 \le \ell$ together with (20) imply that the set $[\ell+1]\backslash\{c_1, c_2, \ldots, c_{k-1}\}$ is non-empty. Let j be any element of this set. Then (29) implies that $f(t_j) = g(t_j) = 2n-2$. This proves that f is also an exceedant surjection. Finally $(c_1, c_2, \ldots, c_{k-1})$ is the increasing sequence of the c_i's with $f(t_{c_i}) = 2n$.

The next proposition shows how the numbers $I(f)$ and $J(f)$ are related with the sequence $(c_1, c_2, \ldots, c_{k-1})$.

3. <u>Proposition</u>. *Let* f *belong to* $A_{n,.,.,k}$ *and* $\Phi(f) = ((c_1, c_2, \ldots, c_{k-1}), g)$. *Then* $I(g) = I(f)$ *or* $I(f) - 1$ *(resp.* $J(g) = J(f)$ *or* $J(f) - 1$*)* *according as* 1 *(resp.* $\ell + 1$*)* *does or does not occur in* $(c_1, c_2, \ldots, c_{k-1})$.

<u>Proof</u>. From (29) it follows that $f(x) = g(x)$ if $x < t_{c_1}$. Hence f and g have the same upper records within the interval $[1, t_{c_1} - 1]$. If

$c_1 = 1$, then $(t_1, g(t_1))$ (resp. $(t_1, f(t_1))$ is the rightmost upper record of g (resp. f). Hence $I(g) = I(f)$. If $1 < c_1$ and $1 < k$, the pair $(t_1, 2n-2)$ is an upper record for both f and g. It is also the rightmost upper record for g. The pair $(t_{c_1}, 2n)$ is also an upper record, but for f only. As there are no upper records of f between $(t_1, 2n-2)$ and $(t_{c_1}, 2n)$, we conclude that $I(g) = I(f) - 1$. Finally, if $k = 1$, the map g is the restriction of f to $[2n-2]$ and the relation $I(g) = I(f) - 1$ is evident.

Clearly, the maps f and g have the same fixed points within the interval $[1, 2n-3]$. Moreover $(2n, 2n)$ is always a fixed point of f. In the interval $[2n-2, 2n]$ there are two possibilities to consider (i) $g(2n-2) = 2n-2 < f(2n-2) = 2n$; (ii) $g(n-2) = f(2n-2) = 2n-2$. In case (i) $(2n-2, 2n-2)$ is a fixed point of g, but not of f. Hence $J(g) = J(f)$ since $(2n, 2n)$ is a fixed point of f. In case (ii) $(2n-2, 2n-2)$ is a fixed point of both f and g. Hence $J(g) = J(f) - 1$. Now (i) corresponds to the case where $c_{k-1} = \ell + 1$ and (ii) to the case where $\ell + 1$ does not occur in $(c_1, c_2, \ldots, c_{k-1})$.

Q.E.D.

4. Corollary. *The sequence* $(a_{n,i,j,k})$ *satisfies recurrence* (9), (10).

Proof. First $a_{1,1,1,1} = 1$, since A_1 only contains the element $f = f(1)\, f(2) = 2\ 2$. For $n > 1$ it follows from proposition 3 that the set $A_{n,i,j,k}$ is mapped under Φ onto the union of the following sets

$$\cup\{((c_1,\dots,c_{k-1})\ ,\ g)\ :\ 1,\ \ell+1\notin\{c_1,\dots,c_{k-1}\}\ ,\ g\in A_{n-1,i-1,j-1,\ell}\}$$

$$\cup\{((c_1,\dots,c_{k-1})\ ,\ g)\ :\ 1 < c_1 \le c_{k-1} = \ell+1\ ,\ g\in A_{n-1,i-1,j,\ell}\}$$

$$\cup\{((c_1,\dots,c_{k-1})\ ,\ g)\ :\ 1 = c_1 \le c_{k-1} < \ell+1\ ,\ g\in A_{n-1,i,j-1,\ell}\}$$

$$\cup\{((c_1,\dots,c_{k-1})\ ,\ g)\ :\ 1 = c_1 \le c_{k-1} = \ell+1\ ,\ g\in A_{n-1,i,j,\ell}\}\ ,$$

where the range of ℓ is the interval $[k-1\ ,\ n-1]$. Clearly, the cardinalities of these four unions are the four summations occurring in the right hand side member of relation (10) .

Q.E.D.

4. TIED UPPER RECORDS.

A new combinatorial construction is used to prove identity (15) and will be described in this section. The keyrole will be played by the notion of *tied upper record*. Let f belong to A_n $(n>1)$ and have i upper records, whose indices are $s_1, s_2, \dots, s_i$ *labeled from right to left*, so that

$$1 = s_i < s_{i-1} < \dots < s_1 < 2n\ .$$

Clearly, s_1 is the smallest integer x such that $f(x) = 2n$. By convention $s_0 = 2n + 1$. The upper record $(s_k\ ,\ f(s_k))$ of f is called *tied*, if there exists an integer x with the property that

$$s_k < x < s_{k-1} \quad \text{and} \quad f(s_k) = f(x)\ . \tag{30}$$

Otherwise, the upper record is said to be untied. As $f(2n-1) = f(2n) = 2n$, the map f always has a tied upper record. Let $(s_p , f(s_p))$ denote the tied upper record of f *with least value* $f(s_p)$. Put $P(f) = p$ so that

$$I(f) = i \geq p = P(f) \geq 1 .$$

For any $x = 3, 4, \ldots, 2n$ define

$$(31) \qquad \begin{aligned} h(x) &= f(x) \quad \text{if} \quad x \neq s_{i-1}, \ldots, s_{p+1} , s_p \\ &= f(s_{k+1}) \quad \text{if} \quad x = s_k \quad \text{and} \quad i > k \geq p . \end{aligned}$$

In particular, when $P(f) = p = i$, the upper record $(s_i, f(s_i))$ is already tied, and h is simply the restriction of f to the interval $[3,2n]$.

5. Lemma. *If* f *is an exceedant surjection of* $[2n]$ *onto* $\{2, 4, 6, \ldots, 2n\}$ *no fixed point of* f *can be an upper record of* f .

Proof. If $f(x) = x$, then x is even, say $2k$. Then $f(2k-1) \geq 2k$, since f is exceedant. But then $(2k , f(2k))$ cannot be an upper record.
Q.E.D.

6. Proposition. *Let* f *belong to* A_n *and* h *defined by* (31) . *Then* h *is an exceedant surjection of* $[3 , 2n]$ *onto* $\{4, 6, 8, \ldots, 2n\}$. *Moreover* h *has the same fixed points as the restriction of* f *to* $[3 , 2n]$.

<u>Proof</u>. Let $3 \leq x \leq 2n$. When $x \neq s_{i-1}, \ldots, s_{p+1}, s_p$, then $h(x) = f(x) \geq x$. When $x = s_k$ with $i < k \leq p$, two cases are to be considered: (i) $s_{k+1} = s_k - 1$; (ii) $s_{k+1} < s_k - 1$. In case (i) $h(s_k) = f(s_{k+1}) = f(s_k-1) > s_k - 1$. Then $h(s_k) \geq s_k$. If $h(s_k) = s_k$, then $s_k = 2j$ and $s_{k+1} = 2j-1$ with $j > 1$. The map f has the upper record $(s_{k+1}, f(s_{k+1})) = (2j-1, 2j)$. Each $f(x)$ with $x < 2j-1$ is then less than $2j$. Hence, the restriction of f to the non-empty interval $[2j-2]$ is itself an exceedant surjection. Accordingly, it has a tied upper record and this contradict the definition of $P(f)$. Hence $h(s_k) > s_k$. In case (ii) $h(s_k) = f(s_{k+1}) > f(s_k-1) \geq s_k - 1$, because $(s_{k+1}, f(s_{k+1}))$ is an untied upper record. Thus $h(s_k) \geq s_k$. If $h(s_k) = s_k$, again $s_k = 2j$ with $j > 1$. On the one hand, $s_k = h(s_k) = f(s_{k+1}) > f(2j-1)$ because $(s_{k+1}, f(s_{k+1}))$ is untied. On the other hand, $f(2j-1) \geq 2j = s_k$, leading to a contradiction. In both cases the strict inequality $h(s_k) > s_k$ holds. It then follows from lemma 5 and (31) that h and the restriction of f to $[3, 2n]$ have the same fixed points. It remains to show that the range of h is $\{4, 6, 8, \ldots, 2n\}$. Let y be an integer of the latter set. As f is surjective, there is an integer x with $f(x) = y$. If $x = 1$, then $x = s_i = 1$ and $f(s_i) = y \geq 4 > 2 = f(2)$, because at least one of the values $f(1)$, $f(2)$ is equal to 2 . Then $s_{i-1} \geq 3$ and $h(s_{i-1}) = f(s_i) = y$. If $x = 2$, then $s_i = 1$, $s_{i-1} = x = 2$, $3 \leq s_{i-2}$ and $2 = f(s_i) < f(s_{i-1}) = y$. Also $h(s_{i-2}) = f(s_{i-1}) = y$. Suppose now $x \geq 3$. If $x \neq s_{i-1}, \ldots, s_{p+1}, s_p$, then $h(x) = f(x) = y$. If $x = s_{k+1}$ with $i < k \leq p$, then $3 \leq s_k$ and $h(s_k) = f(s_{k+1}) = y$. Finally, if $x = s_p$, there is an integer z with $s_p < z$ and $f(s_p) = f(z)$. As $h(z) = f(z)$, again $h(z) = y$.

Q.E.D.

Let us keep the same notations as in (30) and (31) . Let $i > k \geq p$ and assume that $3 \leq s_k$. Then $h(s_k) = f(s_{k+1}) > f(x) \geq h(x)$ for any $x < s_{k+1}$, since $(s_{k+1} , f(s_{k+1}))$ is an upper record of f . The same inequality holds for any x with $s_{k+1} < x < s_k$, because $(s_{k+1} , f(s_{k+1}))$ is untied. As $h(s_k) = f(s_{k+1}) > f(s_{j+1}) = h(s_j)$ for each $j > k$, we conclude that $h(x) < h(s_k)$ for $3 \leq x \leq s_k$. Hence $(s_k , h(s_k))$ is an upper record of h for each $k = i - 1, \ldots, p + 1 , p$ when $3 \leq s_k$. Of course, there may be other upper records $(x , h(x))$ with $x < s_p$. On the other hand, $(s_{p-1} , h(s_{p-1})), \ldots, (s_1 , h(s_1))$ are the only upper records of h with indices at least equal to s_{p-1} . Finally, as $(s_p , f(s_p))$ is a tied upper record of f , there is an integer x with the property that $s_p < x < s_{p-1}$ and $f(s_p)=f(x)$. Let x be the smallest integer with this property. Then $(x , h(x)) = (x , f(x))$ is the upper record of h with the p-th greatest value.

For each $n > 1$ denote by A'_{n-1} the set of all exceedant surjections of $[3 , 2n]$ onto $\{4, 6, 8, \ldots, 2n\}$. Notations such as $A'_{n-1,i,.,.}, \ldots$ have obvious meanings. Put $I(h) = \ell$ and denote by $t_1, t_2, \ldots, t_\ell$ the indices of the upper records of h labeled from right to left, so that

(32) $$3 = t_\ell < \ldots < t_2 < t_1 \leq 2n-1 .$$

Also let $t_{\ell+1} = 2$. From the above remarks it follows that the indices $s_1, s_2, \ldots, s_{i-1}$ occur in the sequence $(t_1, t_2, \ldots, t_{\ell+1})$. It makes sense to define the sequence $(d_1, d_2, \ldots, d_{i-1})$ by the equation

(33) $$(s_1, s_2, \ldots, s_{i-1}) = (t_{d_1}, t_{d_2}, \ldots, t_{d_{i-1}}) .$$

If $i = 1$ the sequence $(d_1, d_2, \ldots, d_{i-1})$ is empty. Furthermore

(34) $$1 \le d_1 < d_2 < \ldots < d_{i-1} \le \ell + 1 ,$$

with

(35) $$(d_1, d_2, \ldots, d_{p-1}) = (1, 2, \ldots, p - 1)$$

and

(36) $$p + 1 \le d_p < \ldots < d_{i-1} \le \ell + 1 .$$

When $f(1) \neq f(2)$, the upper record of h with the p-th greatest value is $(t_p, h(t_p))$. Hence

(37) $$h(t_p) = f(s_p) .$$

When $f(1) = f(2)$, the map h is the restriction of f to $[3,2n]$ and so

(38) $$2 = f(s_p) .$$

Finally

(39) $$h \in A'_{n-1,\ell,.,.} \quad \text{with} \quad i - 1 \le \ell \le n - 1 .$$

Note that (35) and (36) mean that p is the smallest integer not occurring in $(d_1, d_2, \ldots, d_{i-1})$.

(40) *Construction of a bijection*

$$\Psi : f \to ((d_1, d_2, \ldots, d_{i-1}) , h)$$

$$\textit{of } A_{n,i,.,.} \textit{ onto } \bigcup_{i-1 \le \ell \le n-1} \binom{[\ell+1]}{i-1} \times A'_{n-1,\ell,.,.} \quad (n>1) .$$

With f in $A_{n,i,..}$ and $P(f) = p$ the map h id defined by (31) and the sequence $(d_1, d_2, \ldots, d_{i-1})$ by (33) .

7. <u>Example</u>. Consider the following map f

x =	$\underline{1}$	2	$\underline{3}$	4	$\underline{5}$	6	7	$\underline{8}$	9	10	11	$\underline{12}$	$\underline{13}$	14	15	16
f(x) =	$\underline{4}$	2	$\underline{8}$	6	$\underline{10}$	6	8	$\underline{12}$	10	12	12	$\underline{14}$	$\underline{16}$	14	16	16

Here $I(f) = i = 6$. Indices and values of upper records of f are underlined. Thus

$$(s_{i-1}, \ldots, s_2, s_1) = (1, 3, 5, 8, 12, 13) .$$

The pair $(s_p , f(s_p)) = (s_3 , f(s_3)) = (8 , 12)$ is the tied upper record with least value. Hence $P(f) = p = 3$. According to (31)

x =	$\underline{3}$	$\underline{4}$	$\underline{5}$	6	7	$\underline{8}$	9	$\underline{10}$	11	$\underline{12}$	$\underline{13}$	14	15	16
h(x) =	$\underline{4}$	$\underline{6}$	$\underline{8}$	6	8	$\underline{10}$	10	$\underline{12}$	12	$\underline{14}$	$\underline{16}$	14	16	16

and $I(h) = \ell = 7$. Then $(t_{\ell+1}, \ldots, t_2, t_1) = (2, 3, 4, 5, 8, 10, 12, 13)$. As $(s_{i-1}, \ldots, s_2, s_1) = (3, 5, 8, 12, 13)$, the $(d_1, d_2, \ldots, d_{i-1}) = (1, 2, 4, 5, 7)$.

8. <u>Proposition</u>. *The mapping* $\Psi : f \quad ((d_1, d_2, \ldots, d_{i-1}) , h)$ *defined in* (40) *is bijective. Furthermore* $J(h) = J(f)$ *or* $J(f) - 1$ *(resp.* $K(h) = K(f)$ *or* $K(f) - 1$*) according as* $\ell + 1$ *(resp.* 1*) does or*

does not occur in $(d_1, d_2, \ldots, d_{i-1})$. *Finally, the inverse* Ψ^{-1} *of* Ψ *is defined by the following algorithm* (i) - (vi) .

Let $((d_1, d_2, \ldots, d_{i-1}), h)$ satisfy (34) and (39) , and denote by $t_1, t_2, \ldots, t_\ell$ the indices of the ℓ upper records of h , so that (32) holds. The algorithm for Ψ^{-1} is the following:

(i) determine p by (35) and (36);

(ii) put $s_i = 1$, $t_{\ell+1} = 2$ and define $(s_1, s_2, \ldots, s_{i-1})$ by (33);

(iii) $f(1)$ (resp. $f(2)) = 2$ if $\ell + 1$ is equal to d_{i-1} (resp. does not occur in $(d_1, d_2, \ldots, d_{i-1})$;

(iv) $f(s_{k+1}) = h(s_k)$ if $s_k \geq 3$ and $i > k \geq p$;

(v) $f(x) = h(x)$ if $x \geq 3$ and $x \neq s_{k-1}, \ldots, s_{p+1}$;

(vi) $f(s_p) = 2$ or $h(t_p)$ according as $p = i = 1 + 1$ or not.

<u>Proof</u>. The injectivity of Ψ is obvious. Let $\Psi(f) = ((d_1, d_2, \ldots, d_{i-1}), h)$. If $d_{i-1} = \ell + 1$, then $s_{i-1} = t_{d_{i-1}} = t_{\ell+1} = 2$. The pair $(s_{i-1}, f(s_{i-1})) = (2, f(2))$ is an upper record of f . In particular, $f(1) = 2 < f(2)$. When $\ell + 1$ does not occur in $(d_1, d_2, \ldots, d_{i-1})$, either $i = 1$, or $i > 1$ and $d_{i-1} < \ell + 1$. If $i = 1$ the map f has only one upper record and necessarily $f(1) = 2n$, $f(2) = 2$. If $i > 1$ and $d_{i-1} < \ell + 1$, then $s_{i-1} = t_{d_{i-1}} \geq t_\ell = 3$. In this case $2n > f(1) \geq 2 = f(2)$. Thus

(41) $\qquad d_{i-1} = \ell + 1$ if $f(1) = 2 < f(2)$, and $s_{i-1} = 2$,

and

(42) $\ell + 1$ does not occur in $(d_1, d_2, \ldots, d_{i-1})$

if $f(1) \geq 2 = f(2)$ and $s_{i-1} \geq 3$.

Taking into accout proposition 6 the number $J(h)$ of fixed points of h is equal to $J(f)$ or $J(f) - 1$, according as $\ell + 1$ does or does not occur in $(d_1, d_2, \ldots, d_{i-1})$.

In the same manner, if $d_1 = 1$, then (35) and (36) imply that $1 < p = P(f)$. The maps f and h have the same maximum occurrences. If 1 does not occur in $(d_1, d_2, \ldots, d_{i-1})$, then $P(f) = p = 1$. In this case $h(s_1) = f(s_2) < 2n = f(s_1)$. As $f(x) = h(x)$ for every $x > s_1$, we conclude that $K(h) = K(f) - 1$. This proves the first part of the proposition.

Conversely, let $((d_1, d_2, \ldots, d_{i-1}) , h)$ satisfy (34) and (39) , and f defined by (i) - (vi) . It is straightforward to verify that f is an exceedant surjection, and its image under Ψ is the above ordered pair.

Q.E.D.

9. Corollary. *The sequence* $(a_{n,i,j,k})$ *satisfies recurrence* (9) , (15) .

Proof. Let f be in $A_{n,i,j,k}$ and $\Psi(f) = ((d_1, d_2, \ldots, d_{i-1}) , h)$. According to proposition 8 the map h belongs to $A'_{n-1,\ell,j-1,k-1}$,

$A'_{n-1,\ell,j,k-1}$, $A'_{n-1,\ell,j-1,k}$ or $A'_{n-1,\ell,j,k}$, according as neither 1 nor $\ell + 1$ occur in $(d_1, d_2, \ldots, d_{i-1})$, $1 < d_1 \le d_{i-1} = \ell + 1$, $1 = d_1 \le d_{i-1} < \ell + 1$, $1 = d_1 \le d_{i-1} = \ell + 1$.

Q.E.D.

5. INVOLUTIONS.

Let $1 \le c_1 < c_2 < \ldots < c_{k-1} \le \ell + 1$. The mapping

$$(c_1, c_2, \ldots, c_{k-1}) \to (c'_1, c'_2, \ldots, c'_{k-1}) ,$$

where

$$(c'_1, c'_2, \ldots, c'_{k-1}) = (\ell + 2 - c_{k-1} , \ell + 2 - c_{k-2}, \ldots, \ell + 2 - c_1) ,$$

is an involution of the set $\begin{pmatrix}[\ell+1]\\ k-1\end{pmatrix}$. Let $f' = f'' = f$ when f is the single element of A_1 . When $n > 1$, $1 \le k \le n$ (resp. $1 \le i \le n$) and f is in $A_{n,.,.,k}$ (resp. $A_{n,i,.,.}$) form the sequences

$$(43) \quad f \overset{\Phi}{\to} ((c_1, c_2, \ldots, c_{k-1}) , g) \to ((c'_1, c'_2, \ldots, c'_{k-1}) , g') \overset{\Phi^{-1}}{\to} f'$$

$$(44) \quad f \overset{\Phi}{\to} ((d_1, d_2, \ldots, d_{i-1}) , h) \to ((d'_1, d'_2, \ldots, d'_{i-1}) , h'') \overset{\Psi^{-1}}{\to} f'' ,$$

assuming that the involutions $g \to g'$ of A_{n-1} and $h \to h''$ of A'_{n-1} have been defined by induction. Note that (43) and (44) involve the bijections Φ and Ψ defined in (25) and (40) respectively.

10. **Proposition.** *For each* $n > 0$ *the transformations* $f \to f'$ *and* $f \to f''$ *defined by* (43) *and* (44) *respectively are involutions of* A_n *with the following properties*

$$I(f') = J(f)\ ,\ J(f') = I(f)\ ,\ K(f') = K(f)$$

and

$$I(f'') = I(f)\ ,\ J(f'') = K(f)\ ,\ K(f'') = J(f)\ .$$

Proof. By construction 1 (resp. $\ell + 1$) does or does not occur in $(c_1', c_2', \ldots, c_{k-1}')$ if and only if $\ell + 1$ (resp. 1) does or does not occur in $(c_1, c_2, \ldots, c_{k-1})$. Proposition 10 is then a simple consequence of propositions 3 and 8. For instance if $1 = c_1 \le c_{k-1} < \ell + 1$ and f is in $A_{n,i,j,k}$, then g belongs to $A_{n-1,i,j-1,\ell}$. By induction g' is in $A_{n-1,j-1,i,\ell}$. As $1 < c_1' \le c_{k-1}' = \ell + 1$, the map f' is in $A_{n,j,i,k}$.

Q.E.D.

11. **Example.** Let us determine f' and f'' when f is the following element of A_4.

I J K	x =	1	2	3	4	5	6	7	8	$c_1,\ldots,c_{k-1}$	$\ell+1$
3 2 3	f(x) =	4	2	6	8	6	8	8	8		
2 2 3	g(x) =	4	2	6	6	6	6			2,4	4
1 2 2		4	2	4	4					2,3	3
1 1 1		2	2							1	2
										$c'_1,\ldots,c'_{k-1}$	
1 1 1		2	2							2	2
2 1 2		2	4	4	4					1,2	3
2 2 3		2	6	6	4	6	6			1,3	4
2 3 3	f'(x) =	2	8	6	4	8	6	8	8		

I J K	x =	1	2	3	4	5	6	7	8	$d_1,\ldots,d_{i-1}$	$\ell+1$	p
3 2 3	f(x) =	4	2	6	8	6	8	8	8			
3 1 2	h(x) =			4	6	6	8	8	8	2,3	4	1
2 1 2						6	8	8	8	1,3	3	2
1 1 1								8	8	2	2	1
										$d'_1,\ldots,d'_{i-1}$		
1 1 1								8	8	1	2	2
2 2 1						6	6	8	8	1,3	3	2
3 2 1				4	6	6	6	8	8	2,3	4	1
3 3 2	f''(x) =	4	2	6	8	6	6	8	8			

6. TABLES.

Table 2 gives the first values of the coefficients $a_{n,i,j,k}$. Because of the symmetry only the values of $a_{n,i,j,k}$ with $1 \le j \le k$ are shown.

TABLE 2

n	i j k	$a_{n,i,j,k}$	i j k	$a_{n,i,j,k}$	i,j,k	$a_{n,i,j,k}$
1	1 1 1	1				
2	1 2 2	1				
3	1 2 3	1	1 3 3	1	2 2 2	2
	2 2 3	2				
4	1 2 3	1	1 2 4	2	1 3 3	4
	1 3 4	3	1 4 4	1	2 2 2	2
	2 2 3	8	2 2 4	6	2 3 3	12
	2 3 4	3	3 3 3	6		
5	1 2 3	3	1 2 4	8	1 2 5	6
	1 3 3	16	1 3 4	24	1 3 5	11
	1 4 4	22	1 4 5	6	1 5 5	1
	2 2 2	6	2 2 3	32	2 2 4	48
	2 2 5	22	2 3 3	84	2 3 4	74
	2 3 5	18	2 4 4	36	2 4 5	4
	3 3 3	126	3 3 4	60	3 3 5	6
	3 4 4	12				

REFERENCES

1. L. Carlitz, The Staudt-Clausen Theorem, *Math. Magazine* 35 (1961), 131-146.

2. L. Carlitz, A conjecture concerning Genocchi numbers, *K. norske Vidensk. Selsk. Sk.* 9 (1972), 1-4.

3. L. Carlizt, Private communication (1974).

4. D. Dumont, Interprétations combinatoires des nombres de Genocchi, *Duke Math. J.* 41 (1974), 305-318.

5. J.M. Gandhi, A conjectured representation of Genocchi numbers, *Amer. Math. Monthly* 77 (1970), 505-506.

6. G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, London, 1938.

7. J. Riordan and P. Stein, Proof of a conjecture on Genocchi numbers, *Discrete Math.* 5 (1973), 381-388.

Present address of the two authors:

Département de Mathématique
Université de Strasbourg
7, rue René-Descartes
67084 Strasbourg
FRANCE