EXPECTATIONS AND ASYMPTOTIC DISTRIBUTIONS FOR THE j-th esf's OF TWO MATRICES

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1. INTRODUCTION

Let $X(p \times q)$ be distributed $N(M; \Sigma \otimes I_q)$, then for $q \geq p$ $B = XX'$ is distributed $W(\Sigma,q,\Omega)$, $\Omega = \Sigma^{-1}MM'$. Let $A(p \times p)$ be distributed independently as $W(\Sigma,n)$, then the first matrix we are interested in is

1.1 $$V_1 = A^{-1}B \ .$$

Since

$$\operatorname*{plim}_{n\to\infty} nV_1 = \Omega$$

we will write the j-th elementary symmetric function of nV_1 , i.e. $\mathrm{tr}_j nV_1$, under the assumption of linearity (see Madansky and Oklin (1969)) as

1.2 $$\xi_1 = \mathrm{tr}_j nV_1 = \mathrm{tr}_j\Omega + \mathrm{tr}(nV_1 - \Omega)\,\frac{\partial \mathrm{tr}_j\Omega}{\partial\Omega} \ .$$

The second matrix we are interested in is the matrix

1.3 $$V_2 = A_{11.2}^{-\frac{1}{2}}GA_{11.2}^{-\frac{1}{2}}$$

where A is partitioned as

$$A = \begin{pmatrix} A_{11}(q \times q) & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

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and $A_{11.2} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $G = A_{12}A_{22}^{-1}A_{21}$. It is well known that $A_{11.2} \sim W(\Sigma_{11.2},n)$ and conditional on A_{22} independent of $A_{11.2}$ that $G \sim W(\Sigma_{11.2}, p - q, \Delta)$ $\Delta = \Sigma_{11.2}^{-\frac{1}{2}}\beta A_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}$, $\beta = \Sigma_{12}\Sigma_{22}^{-1}$. Σ is partitioned in a similar way than A . If we assume (Sugiura (1969)) that

$$\Sigma_{11}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-\frac{1}{2}} = \frac{1}{n}\theta$$

it follows that

$$\underset{n\to\infty}{\text{plim}}\ nA_{11.2}^{-\frac{1}{2}}GA_{11.2}^{-\frac{1}{2}} = \theta .$$

Hence, under the assumption of linearity we will write

1.4 $$\xi_2 = \text{tr}_j nV_2 = \text{tr}_j\theta + \text{tr}(nV_2 - \theta)\,\frac{\partial \text{tr}_j\theta}{\partial\theta} .$$

It may be pointed out that V_1 is the matrix associated with test of means and V_2 with the test of independence. Asymptotic distributions for the j-th esf's in these two cases were considered in de Waal (1975) but due to an overlooked error in equation 2.7 of de Waal (1974) these results have to be reconsidered.

We shall consider here the asymptotic distributions of ξ_1 and ξ_2 . It is however interesting to know more about $E(\xi_i)$, $i = 1,2$, and therefore we shall first, in section 3, consider these expectations before we derive the asymptotic distributions in section 4. In section 2 a few preliminary results will be given.

2. PRELIMINARY RESULTS

Lemma 2.1. de Waal (1973)

2.1 $$\frac{\partial \mathrm{tr}_j \Sigma}{\partial \Sigma} = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i} \Sigma^{i-1} \mathrm{tr}_{j-i} \Sigma .$$

Lemma 2.2. Let $F = X'A^{-1}X$ for X defined in section 1 and $q < p$, $R(q \times q)$ any positive definite symmetric matrix, then

2.2 $$E \operatorname{etr}(it\, nRF) = |I - 2itR|^{-\frac{1}{2}p} \operatorname{etr}\{itR\Omega(1 - 2itR)^{-1}\} + O(n^{-1})$$

where $\Omega = M'\Sigma^{-1}M$.

Proof: Since $F = X'A^{-1}X = X^{*\prime}A^{*-1}X^*$ where $X^* \sim N(\Sigma^{-\frac{1}{2}}M, I_q \otimes I_p)$ and $A^* \sim W(I_q,n)$ we can assume $\Sigma = I$ and M as $\Sigma^{-\frac{1}{2}}M$ in the densities of X and A. From Crowther (1974) the density of $F = X'A^{-1}X$ conditional on A is given by

2.3 $$\frac{1}{\Gamma_q(\frac{1}{2}p)2^{\frac{1}{2}qp}} |A|^{\frac{1}{2}q} \operatorname{etr}(-\tfrac{1}{2}M'M) |F|^{\frac{1}{2}(p-q-1)} \sum_{k=0}^{\infty} \sum_K \frac{1}{(\frac{1}{2}p)_K k!} E_Y C_K \left(-\tfrac{1}{2}F(Y - \frac{i}{\sqrt{2}} M)'A(Y - \frac{i}{\sqrt{2}} M)\right) , \quad F > 0$$

where the expectation is taken w.r.t. the density

$$\pi^{-\frac{1}{2}pq} \operatorname{etr}(-Y'Y) , \quad Y(p \times q) .$$

Hence

$$2.4 \quad E\,\mathrm{etr}(itnRT) = E_A\, E_F\{\mathrm{etr}(itnRF)|A\} = E_A\, 2^{-\frac{1}{2}pq}\, |A|^{\frac{1}{2}q}(-int)^{-\frac{1}{2}pq}\, |R|^{-\frac{1}{2}p}$$

$$\mathrm{etr}(-\tfrac{1}{2}M'M)E_Y\,\mathrm{etr}\left\{\frac{1}{2int}R^{-1}(Y - \frac{i}{\sqrt{2}}M)'A(Y - \frac{i}{\sqrt{2}}M)\right\}$$

$$= \frac{\Gamma_p\left(\frac{1}{2}(n+q)\right)}{\Gamma_p(\frac{1}{2}n)(-int)^{\frac{1}{2}pq}}\, |R|^{-\frac{1}{2}p}\, \mathrm{etr}(-\tfrac{1}{2}M'M)$$

$$E_Y\left|I - \frac{1}{int}(Y - \frac{i}{\sqrt{2}}M)R^{-1}(Y - \frac{i}{\sqrt{2}}M)'\right|^{-\frac{1}{2}(n+q)}$$

$$= (-2it)^{-\frac{1}{2}pq}\, |R|^{-\frac{1}{2}p}\, \mathrm{etr}(-\tfrac{1}{2}M'M)E_Y\,\mathrm{etr}\left\{\frac{1}{2it}(Y - \frac{i}{\sqrt{2}}M)'\right.$$

$$\left.(Y - \frac{1}{\sqrt{2}}M)R^{-1}\right\} + 0(n^{-1})$$

$$= (-2it)^{-\frac{1}{2}pq}\, |R|^{-\frac{1}{2}p}\, \mathrm{etr}(-\tfrac{1}{2}M'M)\mathrm{etr}(-\frac{1}{4it}M'MR^{-1})$$

$$\left|I - \frac{1}{2it}R^{-1}\right|^{-\frac{1}{2}p}\mathrm{etr}\left\{-\frac{1}{8it}\left(R^{-1}M'MR^{-1}(I - \frac{1}{2it}R^{-1}\right)^{-1}\right\} + 0(n^{-1})$$

$$= |I - 2itR|^{-\frac{1}{2}p}\, \mathrm{etr}\{it\, M'M(I - 2itR)^{-1}R\} + 0(n^{-1}) .$$

Substitute $\Sigma^{-\frac{1}{2}}M$ for M and the lemma is proved.

It is interesting to note from 2.2 that ignoring terms $0(n^{-1})$ nRF has a noncentral Wishart distribution with p degrees of freedom, noncentrality parameter $M'\Sigma^{-1}M$ and covariance matrix R. The distribution of tr nRF ignoring terms $0(n^{-1})$ will be denoted by $\chi^2_{pq}(R,\frac{1}{2}\Omega)$ where $\Omega = M'\Sigma^{-1}M$. If $R = I$ then it follows that $\chi^2_{pq}(I,\frac{1}{2}\Omega) = \chi^2_{pq}(\frac{1}{2}\mathrm{tr}\Omega)$, the noncentral chi-square distribution with pq degrees of freedom and noncentrality parameter $\frac{1}{2}\mathrm{tr}\Omega$. The density of $Y = \mathrm{tr}\, nRF$ ignoring terms $0(n^{-1})$ can be written in the following form:

$$2.5 \quad \frac{\Gamma_q(\frac{1}{2}p)y^{\frac{1}{2}pq-1}}{|2R|^{\frac{1}{2}p}\Gamma(\frac{1}{2}pq)} \frac{2^{\frac{1}{2}q(q-1)}}{(2\pi i)^{\frac{1}{2}q(q+1)}} \int_{R(Z)>0} \mathrm{etr}(Z)|Z|^{-\frac{1}{2}p} \sum_{k=0}^{\infty} \sum_K \frac{(\frac{1}{2}p)_K}{(\frac{1}{2}pq)_k k!} y^k C_K\left(-\tfrac{1}{2}(I - Z^{-1}\Omega)R^{-1}\right)dZ \;, \quad y > 0 \;,$$

where the integration is taken over $Z = V + iW$ symmetric with $V > V_0 > 0$ fixed and W ranges over all symmetric matrices. The density is not hard to derive if the integral (Khatri and Pillai (1968))

$$2.6 \quad \int_D |U|^{\frac{1}{2}(p-q-1)} \alpha_q(U)C_K(U)du_2 \ldots du_q = \frac{\Gamma_q(\frac{1}{2}p)\Gamma_q(\frac{1}{2}q)(\frac{1}{2}p)_K C_K(I_q)}{\pi^{\frac{1}{2}q^2}\Gamma(\frac{1}{2}pq)(\frac{1}{2}pq)_k}$$

where $U = \mathrm{diag}(u_1, \ldots, u_q)$, $u_1 = 1 - u_2 \ldots - u_q$, $D = \{0 < u_q < \ldots < u_1\}$ and $\alpha_q(U) = \prod_{i<j}^{q} (u_i - u_j)$, is used after expressing the ${}_0F_1$ function in the noncentral Wishart density as

$$2.7 \quad {}_0F_1(\tfrac{1}{2}p;\, \tfrac{1}{4}\Omega R^{-1}S) = \frac{2^{\frac{1}{2}q(q-1)}}{(2\pi i)^{\frac{1}{2}q(q+1)}} \int_{R(Z)>0} \mathrm{etr}(Z)\; |Z|^{-\frac{1}{2}p}\; \mathrm{etr}(\tfrac{1}{4}Z^{-1}\Omega R^{-1}S)dZ \;.$$

<u>Corollary 2.1.</u> Let $V = B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}}$ if $q \geq p$, $R(p \times p)$ any p.d. symmetric matrix and $\Omega = \Sigma^{-\frac{1}{2}}MM'\Sigma^{-\frac{1}{2}}$, then

$$2.8 \quad E\ \mathrm{tr}(nitVR) = |I - 2itR|^{-\frac{1}{2}q}\ \mathrm{etr}\left(it\Omega(I - 2itR)^{-1}R\right) + O(n^{-1}) \;.$$

<u>Proof:</u> This follows from 2.2 using the fact that $F(q \times q)$ is distributed as a noncentral multivariate beta distribution with $(p, n + q - p)$ degrees of freedom while $V(p \times p)$ is distributed as a noncentral multivariate beta distribution with (q,n) degrees of freedom.

3. EXPECTATIONS

Theorem 3.1. If $F(q \times q)$ is distributed as a noncentral multivariate beta distribution with $(p, n + q - p)$ degrees of freedom and noncentrality parameter $\Omega = M'\Sigma^{-1}M$, then

$$E(\mathrm{tr}_j F) = \frac{1}{(n+q-p-2)^{(j)}} \sum_{i=0}^{j} \binom{q-i}{j-i} (p - i)^{(j-i)} \mathrm{tr}_i \Omega \qquad (3.1)$$

where

$$(a)^{(j)} = a(a - 1) \dots (a - j + 1) .$$

Proof: We consider $F = X'A^{-1}X$ as was defined in lemma 2.2. Let $X = (X_J X_{\bar{J}})$ such that $X_J(p \times j)$. Then according to the definition of X the columns of X_J are independently normally distributed with covariance Σ and $E(X_J) = M_J$. F now becomes

$$F = \begin{pmatrix} X_J'A^{-1}X_J & X_J'A^{-1}X_{\bar{J}} \\ X_{\bar{J}}'A^{-1}X_J & X_{\bar{J}}'A^{-1}X_{\bar{J}} \end{pmatrix} = \begin{pmatrix} F_{JJ} & F_{J\bar{J}} \\ F_{\bar{J}J} & F_{\bar{J}\bar{J}} \end{pmatrix} .$$

It is obvious that F_{JJ} is distributed as a noncentral multivariate beta distribution with p and $n + j - p$ degrees of freedom and noncentrality parameter $M_J'\Sigma^{-1}M_J$. But (de Waal (1972))

$$E|F_{JJ}| = \frac{1}{(j+n-p-2)^{(j)}} \sum_{i=0}^{j} (p - i)^{(j-i)} \mathrm{tr}_i(M_J'\Sigma^{-1}M_J) . \qquad (3.2)$$

Substitute in

$$E(\mathrm{tr}_j F) = \textstyle\sum_{JJ} E|F_{JJ}| \qquad (3.3)$$

where $\sum_{JJ} |A_{JJ}|$ is the summation over all principal minors of order j of a matrix A and (Saw (1973))

$$3.4 \qquad \sum_{JJ} \mathrm{tr}_i(M_J' \Sigma^{-1} M_J) = \binom{q-i}{j-i} \mathrm{tr}_i \Omega ,$$

the theorem follows.

<u>Corollary 3.1.</u> If $V(p \times p)$ is distributed as a noncentral multivariate beta distribution with (q,n) degrees of freedom, then

$$3.5 \qquad E(\mathrm{tr}_j V) = \frac{1}{(n-2)^{(j)}} \sum_{i=0}^{j} \binom{p-i}{j-i} (q-i)^{(j-i)} \mathrm{tr}_i \Omega .$$

<u>Theorem 3.2.</u> If V_2 is defined as in 1.3 and $P = \Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$, then

$$3.6 \quad E(\mathrm{tr}_j V_2) = \frac{1}{(n-2)^{(j)}} \sum_{i=0}^{j} (n)^{(i)} \binom{q-i}{j-i} (p-q-i)^{(j-i)} \mathrm{tr}_i\left(P(I-P)^{-1}\right) .$$

(Note that if R is defined as $R = A_{11}^{-1}A_{12}A_{22}^{-1}A_{21}$, then $\mathrm{tr}_j V_2 =$
$= \mathrm{tr}_j\left(R(I-R)^{-1}\right)$.)

<u>Proof:</u> Since $A_{11.2}^{-1}G$ given A_{22} is distributed as a noncentral multivariate beta distribution with $(p-q, n)$ degrees of freedom and noncentrality parameter $\Delta = \Sigma_{11.2}^{-\frac{1}{2}}\beta A_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}$, it follows from 3.5 that

$$3.7 \quad E\left(\mathrm{tr}_j(V_2|A_{22})\right) = \frac{1}{(n-2)^{(j)}} \sum_{i=0}^{j} \binom{q-i}{j-i} (p-q-i)^{(j-i)} \mathrm{tr}_i(\beta'\Sigma_{11.2}^{-1}\beta A_{22}) .$$

But $A_{22}(p-q \times p-q) \sim W(\Sigma_{22},n)$ and hence (de Waal (1972))

$$E\ \mathrm{tr}_i(\beta'\Sigma_{11.2}^{-1}\beta A_{22}) = (n)^{(i)}\ \mathrm{tr}_i(\beta'\Sigma_{11.2}^{-1}\beta\Sigma_{22}) .$$

The unconditional expectation 3.6 follows easily.

4. ASYMPTOTIC DISTRIBUTIONS

Theorem 4.1. Ignoring terms $0(n^{-1})$

4.1 $$(\xi_1 - \mathrm{tr}_j\Omega + \mathrm{tr}\Omega\Gamma_j) \sim \chi^2_{pq}(\Gamma_j, \tfrac{1}{2}\Omega)$$

where

$$\Gamma_j = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i}\, \Omega^{i-1}\, \mathrm{tr}_{j-i}\Omega\ .$$

Proof: The characteristic function of ξ_1 is given by

4.2 $$\begin{aligned}\phi_{\xi_1}(t) &= E\exp(it\xi_1)\\ &= \exp(it\,\mathrm{tr}_j\Omega)\mathrm{etr}(-it\Omega\Gamma_j)E\,\mathrm{etr}(itnV_1\Gamma_j) + 0(n^{-1})\ .\end{aligned}$$

Using 2.8 with R replaced by Γ_j , the theorem follows.

Theorem 4.2. Ignoring terms $0(n^{-1})$

4.3 $$(\xi_2 - \mathrm{tr}_j\theta + \mathrm{tr}\theta\Lambda_j) \sim \chi^2_{q(p-q)}(\Lambda_j, \tfrac{1}{2}\theta)$$

where

$$\Lambda_j = (-1)^{j-1} \sum_{i=1}^{j} (-1)^{j-i}\, \theta^{i-1}\, \mathrm{tr}_{j-i}\theta\ .$$

Proof: From 1.4 and 2.8 the conditional c.f. of ξ_2 given A_{22} is given by

4.4 $\phi_{\xi_2|A_{22}}(t) = \exp(it\ tr_j\theta)etr(-it\theta\Lambda_j)\ |I - 2it\Lambda_j|^{-\frac{1}{2}(p-q)}$

$$etr\left(it\Delta(I - 2it\Lambda_j)^{-1}\Lambda_j\right) + 0(n^{-1})$$

where $\Delta = \Sigma_{11.2}^{-\frac{1}{2}}\beta A_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}$. Hence the unconditional c.f. of ξ_2 becomes

4.5 $\phi_{\xi_2}(t) = \exp(it\ tr_j\theta)etr(-it\theta\Lambda_j)\ |I - 2it\Lambda_j|^{-\frac{1}{2}(p-q)}$

$$E_{A_{22}}etr\{itA_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}(I - 2it\Lambda_j)^{-1}\Lambda_j\Sigma_{11.2}^{-\frac{1}{2}}\beta\} + 0(n^{-1}) .$$

But

4.6 $E_{A_{22}}etr\{itA_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}(I - 2it\Lambda_j)^{-1}\Lambda_j\Sigma_{11.2}^{-\frac{1}{2}}\beta\}$

$$= |I - 2it\Sigma_{22}\beta'\Sigma_{11.2}^{-\frac{1}{2}}(I - 2it\Lambda_j)^{-1}\Lambda_j\Sigma_{11.2}^{-\frac{1}{2}}\beta|^{-\frac{1}{2}n}$$

$$= etr\{it\theta(I - 2it\Lambda_j)^{-1}\Lambda_j\} + 0(n^{-1})$$

since $|I - \frac{1}{n}\xi|^{-nx} = etr(x\xi) + 0(n^{-1})$ and under the assumption $\Sigma_{11}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-\frac{1}{2}} = \frac{1}{n}\theta$ it follows that $\Sigma_{11.2}^{-\frac{1}{2}}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{11.2}^{-\frac{1}{2}} = \frac{1}{n}\theta + 0(n^{-1})$. Substitute 4.6 in 4.5 and the theorem follows.

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