AN INVARIANT SEQUENTIAL TEST FOR ZERO DRIFT BASED ON THE FIRST PASSAGE TIMES IN BROWNIAN MOTION*

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1. INTRODUCTION

Let $W(t)$ be a separable Brownian Motion process with $W(0) = 0$, drift a, and diffusion constant δ^2. It is well known (see, for example, Cox and Miller [3, p. 210] that the first passage time T to the state $A > 0$ has the density function

$$p(t) = (\lambda/t^3)^{\frac{1}{2}}\phi(\lambda^{\frac{1}{2}}(1-\mu t)/t^{\frac{1}{2}}) , \quad t > 0 \tag{1.1}$$

where ϕ is the standard normal density function, $\mu = a/A$ and $\lambda = A^2/\delta^2$. We shall construct an invariant sequential test for the hypothesis of zero drift against the alternative of positive drift, based on the observations $t_1, t_2, \ldots$ on the first passage times. Nádas [5] has given a fixed sample size test for this problem, Seshadri and Shuster [6] have evaluated the exact critical region based on Nádas' test statistic.

Section 2 contains the derivation of an invariantly sufficient sequence of statistics for testing the hypothesis of zero drift. A proof for the monotonicity of and an asymptotic formula for the likelihood ratio are given in sections 3 and 4 respectively. It is proved in section 5 that the probability is one that the sequential test comes to a conclusion with the acceptance of either the null or the alternative hypothesis.

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2. DERIVATION OF THE TEST SEQUENCE

Since $\lambda > 0$, the drift is zero if and only if $\tau = \mu\lambda = 0$. We shall first develop an invariant sequential test for testing H_0: $\tau = 0$ against H_1: $\tau = \Delta$, where Δ is a positive number. It shall be shown in the sequel that the test remains valid for testing H_0 against the alternatives H_1': $\tau \geq \Delta$. For a positive number c, the density function of cT is the first passage density (1.1) with μ replaced by μ/c and with λ replaced by λc. Thus the problem of testing H_0 vs. H_1 remains invariant under the group G of positive scale transformations and for a sample of size n, $(t_1/t_n,\ldots,t_n/t_n)$ is a maximal invariant on R^n, the n-dimensional Euclidean space. The induced group $\overline{G}$ on the parameter space also consists of all positive scale transformations on the quadrant $(\mu \geq 0, \lambda > 0)$ so that $\overline{g}(\mu,\lambda) = (\mu/c, \lambda c)$ for $c > 0$. A maximal invariant under $\overline{G}$ is τ. An invariant test is, therefore, based on the sequence $\{x_i = t_i/t_n,\ i=1,\ldots,n\}$; $n=1,2,\ldots$. We shall use the notation $C = (\lambda/2\pi)^{n/2}$, $\Pi(\underline{t}) = \Pi_{i=1}^n t_i^{-3/2}$, $b(\underline{t}) = \sum_{i=1}^n t_i$ and $d(\underline{t}) = \sum_{i=1}^n t_i^{-1}$. Then the joint density of $(t_1,\ldots,t_n)$ is, from (1.1),

$$p(t_1,\ldots,t_n) = C\Pi(\underline{t})\ \exp\{n\tau - \lambda[\mu^2 b(\underline{t}) + d(\underline{t})]/2\} .$$

Making the change of variables from t_i to $x_i = t_i/t_n$, $i=1,\ldots,n$, we obtain the joint density of $(x_1,\ldots,x_n)$ as

$$\begin{aligned} f_\tau = f_\tau(x_1,\ldots,x_n) &= C\Pi(\underline{x})\exp\{n\tau\}\int_0^\infty z^{-\frac{n}{2}-1} e^{-\frac{\lambda}{2}\{\mu^2 b(\underline{x})z + \frac{d(\underline{x})}{z}\}} dz \\ &= [\Pi(\underline{x})b^{n/2}(\underline{x})](4\pi)^{-n/2}\tau^n \exp\{n\tau\}\int_0^\infty s^{-\frac{n}{2}-1} e^{-\left(s+\frac{v_n\tau^2}{4s}\right)} ds \end{aligned} \tag{2.1}$$

where

$$v_n = b(\underline{x})d(\underline{x}) = b(\underline{t})d(\underline{t}) = \sum_{i=1}^n t_i \sum_{i=1}^n t_i^{-1} . \tag{2.2}$$

It follows from the factorization theorem that v_n is sufficient for τ in the joint distribution of $(x_1,\ldots,x_n)$. Thus the sequence $\{v_n\}$ is an invariantly sufficient sequence for the family of distributions (1.1) under the group G of transformations on R^n. An alternative way is to obtain a sufficient sequence, to employ an invariant reduction on this sufficient sequence and then to prove that the resulting sequence is, in fact, invariantly sufficient. An elegant theory has been developed which gives sufficient conditions for the resulting sequence to be an invariantly sufficient sequence. (See Hall, Wijsman and Ghosh [4]). Since we derived v_n directly, we do not resort to this theory in the present context.

To obtain an analogous expression for $f_0(x_1,\ldots,x_n)$, we note that for $\tau = 0(\mu=0,\lambda>0)$, the density function of T is

$$(\lambda/2\pi t^3)^{\frac{1}{2}}\exp\{-\lambda/2t\} \ , \quad t > 0 \ . \tag{2.3}$$

Proceeding as above, the joint density of $(x_1,\ldots,x_n)$ is obtained as

$$f_0 = f_0(x_1,\ldots,x_n) = [\Pi(\underline{x})b^{n/2}(\underline{x})](4\pi)^{-n/2}\int_0^\infty s^{-\frac{n}{2}-1} e^{-\frac{v_n}{4s}} ds. \tag{2.4}$$

The likelihood ratio f_Δ/f_0 at the n-th stage of sampling depends on the observed values $(t_1,\ldots,t_n)$ only through v_n, and is given by

$$L_n(v_n) = \Delta^n\exp\{n\Delta\}\int_0^\infty h_1(s)ds\Big/\int_0^\infty h_0(s)ds \tag{2.5}$$

where $h_1(s) = h(s,\Delta) = s^{-\frac{n}{2}-1}\exp\left\{-s - \frac{v_n\Delta^2}{4s}\right\}$ and $h_0(s) = \exp\{s\}h(s,1)$.

It is easily verified that (2.5) is the ratio of the density functions of v_n at $\tau = \Delta$ and at $\tau = 0$. The continuation region of the sequential test is

$$b < \ell n[L_n(v_n)] < a \tag{2.6}$$

where $b = \ln(\beta/(1-\alpha))$ and $a = \ln((1-\beta)/\alpha)$. It shall be proved later that the test terminates with probability one. Then, on the assumption that the excess of $\ln[L_n(v_n)]$ over the decision boundaries is negligible on the termination of the test, the test has the desired strength (α,β). We shall first show that $L_n(v_n)$ is a decreasing function of v_n.

3. MONOTONICITY OF THE LIKELIHOOD RATIO

Let $\ell_n(v_n)$ denote the log likelihood ratio $(\ell_n(v_n) = \ln[L_n(v_n)])$ at the n-th stage of sampling. The derivative of $\ell_n(v_n)$ with respect to v_n is obtained from (2.5) as

$$4I_0I_1\frac{d\ell_n}{dv_n} = -\Delta^2 I_0\int_0^\infty s^{-1}h_1(s)ds + I_1\int_0^\infty s^{-1}h_0(s)ds$$

where $I_0 = \int_0^\infty h_0(s)ds$ and $I_1 = \int_0^\infty h_1(s)ds$, and thus $I_0I_1 > 0$. Hence

$$4I_0I_1\frac{d\ell_n}{dv_n} = \int_0^\infty\int_0^\infty h_1(s)h_0(t)(t^{-1}-\Delta^2 s^{-1})ds\,dt$$

$$= \iint_{\{s>\Delta^2 t\}} h_1(s)h_0(t)(t^{-1}-\Delta^2 s^{-1})ds\,dt$$

$$+ \iint_{\{s<\Delta^2 t\}} h_1(s)h_0(t)(t^{-1}-\Delta^2 s^{-1})ds\,dt\ .$$

If we substitute $\Delta^2 t = u$, $s = v$ in the first integral and $s = u$, $\Delta^2 t = v$ in the second integral, we obtain

$$4I_0I_1\frac{d\ell_n}{dv_n} = \iint_{\{v>u\}} (u^{-1}-v^{-1})g(u,v)du\,dv\ ,$$

where $g(u,v) = (\Delta^2/uv)^{\frac{n}{2}+1} e^{-\frac{v_n\Delta^2}{4}(u^{-1}+v^{-1})}(e^{-v}-e^{-u})$. $g(u,v)$ is negative over the whole range of integration. It follows that $\ell_n(v_n)$ and hence $L_n(v_n)$ is a strictly decreasing function of v_n. Thus the decision rule of the test can be written as

$$(H_1): \underline{v}_n < v_n < \overline{v}_n : (H_0) \quad \text{for } n=2,3,\ldots$$

where $\underline{v}_n$ and $\overline{v}_n$ are the solutions of

$$\ell_n(\underline{v}_n) = a \quad \text{and} \quad \ell_n(\overline{v}_n) = b \ .$$

The above monotonicity property of $L_n(v_n)$ also implies that the test is valid for testing H_0 against all alternatives $H_1': \tau \geq \Delta$.

4. ASYMPTOTIC FORMULA FOR THE LIKELIHOOD RATIO

We consider first the integral $I_1 = \int_0^\infty h_1(s)ds$. Substituting $s = 2^{-1}v_n^{\frac{1}{2}}\Delta \exp\{S\}$, we obtain

$$I_1 = (2/v_n^{\frac{1}{2}}\Delta)^{n/2}\int_{-\infty}^{\infty} \exp\{-v_n^{\frac{1}{2}}\Delta \cosh S - \frac{nS}{2}\}dS$$

where $\cosh(\circ)$ is the hyperbolic cosine function. Thus I_1 may be written as

$$I_1 = 2(2/v_n^{\frac{1}{2}}\Delta)^{n/2}K_{n/2}(v_n^{\frac{1}{2}}\Delta) \tag{4.1}$$

where $K_\nu(z) = (\frac{1}{2})\int_{-\infty}^{\infty} \exp\{-z \cosh S - \nu S\}dS$, is the modified Bessel function of second kind (see Abramowitz and Stegun [1]). Also, we have

$$I_0 = \int_0^\infty s^{-\frac{n}{2}-1} e^{-\frac{v_n}{4s}}ds = \int_0^\infty S^{\frac{n}{2}-1} e^{-\frac{Sv_n}{4}} dS$$

$$= 2^n\Gamma(n/2)/v_n^{n/2} \ . \tag{4.2}$$

Substituting (4.1) and (4.2) in (2.5), we obtain

$$L_n(v_n) = 2^{-\frac{n}{2}+1}\Delta^{n/2}e^{n\Delta}v_n^{n/4}K_{n/2}(v_n^{\frac{1}{2}}\Delta)[\Gamma(n/2)]^{-1} . \quad (4.3)$$

We shall write $\omega_n = 4v_n/n^2$, $\underline{\omega}_n = 4\underline{v}_n/n^2$ and $\overline{\omega}_n = 4\overline{v}_n/n^2$, and shall denote the likelihood ratio by $M_n(\omega_n)$ and its logarithm by $m_n(\omega_n)$. It is easily seen that

$$M_n(\omega_n) = 2^{-n+1}(n\Delta)^{n/2}e^{n\Delta}\omega_n^{n/4}K_{n/2}(n\Delta\omega_n^{\frac{1}{2}}/2)[\Gamma(n/2)]^{-1}. \quad (4.4)$$

Now Stirling's asymptotic formula for the Gamma function is

$$\Gamma(\nu) = (2\pi/\nu)^{\frac{1}{2}}\nu^{\nu}e^{-\nu}[1 + O(\nu^{-1})] \quad (4.5)$$

as $\nu \to \infty$, and an asymptotic formula for $K_\nu(\nu z)$ (see Abramowitz and Stegun [1]) is

$$K_\nu(\nu z) = (\pi/2\nu)^{\frac{1}{2}}(1+z^2)^{-\frac{1}{4}}e^{-\nu\eta}\{1 + O(\nu^{-1})\} \quad (4.6)$$

as $\nu \to \infty$, where $\eta = (1+z^2)^{\frac{1}{2}} + \log\{z/(1+\sqrt{1+z^2})\}$. Substituting these asymptotic forms of $\Gamma(\nu)$ and $K_\nu(\nu z)$ in (4.4) and taking logarithms, we obtain, after a routine calculation,

$$m_n(\omega_n) = \frac{n}{2}[G(\omega_n) + O(n^{-1})] \quad (4.7)$$

where

$$G(\omega) = \log[1+(1+\omega\Delta^2)^{\frac{1}{2}}] - \log 2 - (1+\omega\Delta^2)^{\frac{1}{2}} + 1+2\Delta . \quad (4.8)$$

5. THE TERMINATION PROPERTY

To prove that the above sequential test terminates with probability one, we shall first show that there exists a unique number ℓ lying in a bounded open interval (a',b') such that

(i) $m_n(\omega) \to \infty$ for $\omega < \ell$

and (ii) $m_n(\omega) \to -\infty$ for $\omega > \ell$. (5.1)

Writing $a' = 4(\Delta^{-1}+1)$ and $b' = 2(a'+4)$, it follows from (4.8) that

$$G(a') = \log(1+\Delta) \quad \text{and} \quad G(b') = \log(1+2\Delta)-2\Delta \; . \tag{5.2}$$

Using the inequality $\log(1+x)-x < 0$ for $x > 0$, it follows from (4.7) and (5.2) that

$$\lim_{n\to\infty} m_n(a') = \infty \quad \text{and} \quad \lim_{n\to\infty} m_n(b') = -\infty \; . \tag{5.3}$$

Also, from (4.8) ,

$$\frac{d}{d\omega}[G(\omega)] = -\Delta^2/2\{1+(1+\omega\Delta^2)^{\frac{1}{2}}\} < 0 \quad \text{for} \quad \omega \geq 0 \; . \tag{5.4}$$

This implies that

$$\lim_{n\to\infty} \frac{d}{d\omega}[m_n(\omega)] < 0 \quad \text{for all} \quad \omega > 0 \; . \tag{5.5}$$

The existence and uniqueness of ℓ follows from (5.3) and (5.5). We do not need an explicit form of ℓ for the present purpose. However, it is easily seen that ℓ is a solution of $G(\ell) = 0$.

We shall follow the argument of Cox [2] to show that the probability of a decision is one. The probability that a terminal decision is not reached by stage n is less than or equal to the probability that v_n lies in the open interval $(\underline{v}_n, \overline{v}_n)$ which, in turn is equal to the probability that ω_n lies in the interval $(\underline{\omega}_n, \overline{\omega}_n)$. We call this probability P_n.
It is easily seen that the maximum likelihood estimator E_n of τ^{-1} is $(\omega_n/4) - 1$. The asymptotic variance of $n^{\frac{1}{2}}E_n$ is $V = \tau^{-2}(\tau^{-1}+2)$. Thus the asymptotic normality of the maximum likelihood estimator implies that $n^{\frac{1}{2}}E_n$ has asymptotically a normal distribution with mean $n^{\frac{1}{2}}\tau^{-1}$ and variance V. Thus, for large values of n,

$$P_n = \Phi(n^{\frac{1}{2}}(\overline{E}_n-\tau^{-1})/V^{\frac{1}{2}}) - \Phi(n^{\frac{1}{2}}(\underline{E}_n-\tau^{-1})/V^{\frac{1}{2}}) \; , \tag{5.6}$$

where $\underline{E}_n = (\underline{\omega}_n/4)-1$ and $\overline{E}_n = (\overline{\omega}_n/4)-1$. As n tends to infinity, $\overline{\omega}_n$ and

$\underline{\omega}_n$ both tend to ℓ. Thus, unless $\ell = 4(1+\tau^{-1})$, P_n tends to zero as $n \to \infty$ since both terms in (5.6) approach either one or zero.

The above argument does not apply when $\ell = 4(1+\tau^{-1})$. It follows from (4.7) and (5.4) that the derivative of $m_n(\omega)$ with respect to ω is

$$\frac{d}{d\omega}[m_n(\omega)] = (n/2)[g(\omega) + O(n^{-1})] \ , \tag{5.7}$$

where $g(\omega) = -\Delta^2/2\{1+(1+\omega\Delta^2)^{\frac{1}{2}}\}$. Since $m_n(\bar{\omega}_n) = b$ and $m_n(\underline{\omega}_n) = a$, we have, by the mean value theorem,

$$(a-b)/n^{\frac{1}{2}}(\underline{\omega}_n - \bar{\omega}_n) = (n^{\frac{1}{2}}/2)g(\omega) + O(n^{-\frac{1}{2}}) \tag{5.8}$$

for some $\omega \in (\underline{\omega}_n, \bar{\omega}_n)$. The right hand side of (5.8) has the limit $-\infty$ as $n \to \infty$, since $g(\omega)$ is negative for all ω and all $\Delta > 0$. Thus $n^{\frac{1}{2}}(\bar{\omega}_n - \underline{\omega}_n)$ has the limit zero as $n \to \infty$. Observing that

$$|\Phi(c) - \Phi(d)| < |c-d|, \quad \text{for } -\infty < d < c < \infty \ ; \tag{5.9}$$

we obtain, from (5.6),

$$|P_n| < \tau n^{\frac{1}{2}} |\bar{\omega}_n - \underline{\omega}_n| / 4(\tau^{-1}+2)^{\frac{1}{2}} \to 0 \quad \text{as} \quad n \to \infty \ . \tag{5.10}$$

Thus the test terminates with probability one also for the case $\ell = 4(1+\tau^{-1})$.

REFERENCES

[1] Abramowitz, M. and Stegun, I.A. (Ed.), *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series No. 55, Washington, D.C.: U.S. Government Printing Office, 1964.

[2] Cox, D.R., "Sequential Tests for Composite Hypotheses," *Proceedings of the Cambridge Philosophical Society, 48*, (1952), 290-99.

[3] Cox, D.R. and Miller, H.D., *The Theory of Stochastic Processes*, London: Methuen and Co. Ltd., (1965).

[4] Hall, W.J., Wijsman, R.A. and Ghosh, J.K., "The Relationship between Sufficiency and Invariance with Applications in Sequential Analysis," *Annals of Mathematical Statistics, 36*, (1965), 575-614.

[5] Nádas, A., "Best Tests for Zero Drift Based on First Passage Times in Brownian Motion," *Technometrics, 15,* (1973), 125-32.

[6] Seshadri, V. and Shuster, J.J., "Exact Tests for Zero Drift Based on First Passage Times in Brownian Motion," *Technometrics, 16,* (1974), 133-34.