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SUMMATION METHODS, I

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A. Introduction. The objective of this work is to present new and little known series operations that are particularly applicable in obtaining multiple generating functions for combinatorial and probability problems. Infinite series appearing herein are to be regarded as formal power series or else appropriate assumptions regarding convergence must be made; no further reference to convergence will appear.

We are concerned with methods of changing the bounds, the order of summation and/or the summand of such series as $\sum_A f(i_1,\ldots,i_n)$ (A is a set of conditions for the indices) so that the result is in some 'desirable' form. What constitutes a desirable form depends on the problem; frequently, attempts are made at (a) interchanging the order of summation while keeping the summand unchanged, (b) making the lower bounds constant (usually zero or one) or the upper bounds ∞, and compensating by making appropriate changes in the argument of the summand. Usually (a) is used in achieving (b) and vice versa. Looking at it another way we see that the emphasis is on delineating useful arrangements of the terms of multiple series.

Part of the results are independent of the nature of the summand; in other cases the results depend on general properties of the summand; e.g., it is symmetric, the coefficients are cyclic. For the most part we consider cases where the only restriction on an index is that it

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ranges consecutively from one integer to another (or to ∞). However, many cases occur where the conditions on the indices are more complicated. As alluded to earlier, there often occurs a kind of reciprocity between a sum having a complicated summand with simple conditions on the indices and that of a sum having a less complicated summand but a more involved system of conditions for the indices. For example, we often have identities such as $\sum_A 1 = \sum_{i,j=0}^{n} f(i,j)$ where A is a set of conditions on i and j (more complicated than the conditions on the right, $0 \leq i,\ j \leq n$) and where $f(i,j)$ is a function (more complicated than the summand on the left).

In order to maintain reasonable brevity, we shall not consider familiar properties of the usual moment, enumerative, etc. generating functions; our work is more general. Nor shall we exhibit the many summation methods found in books on finite differences (Poisson's formula, method of inverse differences, etc.) and the many interesting formulas found in a variety of works including Gould [13], Knuth [32], Rainville [35], Riordan [38], Schwatt [41], Jolley [31], and Mangulis [33]. The stress here is of a different, though complementary, nature.

A few words about the arrangement of the material is in order. Bear in mind that this work represents a collection of formulas of certain types. The formulas are contained in Part B; there are 67 of them (not counting ramifications). Some of the latter formulas include some introduction or discussion and are therefore lengthy. We continue to regard these more lengthy statements as formulas (rather than sections, etc.) in order to maintain continuity of the format; the earlier formulas are shorter and would not be considered sections. Some formulas contain several equations. These equations are referred to by small letters (numbers identify formulas). If an equation of an earlier formula is referred to, the formula number followed by the equation letter is given, e.g., (32.C).

No claim to completeness is made. Clearly, no claim can be made in such a work as this. In fact, several aspects suggest important avenues for further research.

A note regarding any errors found here will be appreciated by the writer.

This writer hopes that the reader will find at least a few new results that are interesting and/or applicable.

Before listing the formulas, we introduce the following notation:

Symbol	Meaning of Symbol
*(n)	If found next to a formula, it means that n may be replaced by ∞ in that formula.
[x]	The largest integer $\le$ x.
{x}	The smallest integer $\geq$ x.
$\delta_{i,j}$	Kronecker data = $\begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i\neq j \end{cases}$
α_m	A primitive m^{th} root of unity ($\alpha_m \neq 1$).

Other notation is given in context.

Some conditions are understood. For example, the sum $\sum_{k=a}^{n}$ is nonsensical if n < a. Care should be taken in applications of more complicated sums to see that all conditions are satisfied.

B. Formulas.

1. $\sum_{j=a}^{n} f(j) = \sum_{j=a}^{n} f(n+a-j)$.

2. $\sum_{j=a}^{n} f(j) = \sum_{j=a+b}^{n+b} f(j-b) \quad , \qquad *(n)\ .$

3. $\sum_{j=a}^{n} f(j) = \sum_{j=a}^{r} f(j) + \sum_{j=r+1}^{n} f(j) \quad , \quad a \le r < n, \quad *(n).$

4. $\sum_{j=0}^{n} f(j) = \sum_{j=0}^{[n/2]} f(j) + \sum_{j=0}^{[(n-1)/2]} f(n-j)\ .$

5. $\sum_{j=a}^{n} f(j) = \sum_{j=[(a+1)/2]}^{[n/2]} f(2j) + \sum_{j=[a/2]}^{[(n-1)/2]} f(2j+1)\ , \quad *(n)\ .$

6. $\sum_{j=0}^{n} (-1)^j f(j) = \sum_{j=0}^{[n/2]} f(2j) - \sum_{j=0}^{[(n-1)/2]} f(2j+1), \ *(n)\ .$

7. $\sum_{j=1}^{n} (-1)^j f(j) = \sum_{k=0}^{b-1} (-1)^k \sum_{j=1}^{[(n+k)/b]} (-1)^{bj} f(bj-k)\ .$

(See [42]; there are a number of interesting similar formulas in this work.

8. $\sum_{j=1}^{nb} f(j) = \sum_{k=0}^{b-1} \sum_{j=1}^{n} f(kn+j)\ .$

9. $\sum_{\substack{3j-1 \equiv r (\mathrm{mod}\ 2) \\ j \le m}} f(j) = \sum_{\substack{j\ \mathrm{odd} \\ j \le m}} f(j) = \sum_{j=1}^{[(m+1)/2]} (2j-1).$

10. $\sum_{\substack{3j \equiv n (\mathrm{mod}\ 2) \\ i \le m}} f(j) = \sum_{\substack{j\ \mathrm{odd} \\ j \le m}} f(j) = \sum_{j=1}^{[m/2]} f(2j)\ .$

(9–10: r even)

$$\left.\begin{aligned}
&11.\quad \sum_{\substack{3j-1\equiv r \pmod 2\\ j\leq m}} f(j) = \sum_{\substack{j \text{ even}\\ j\leq m}} f(j) = \sum_{j=1}^{[n/2]} f(2j)\\[2ex]
&12.\quad \sum_{\substack{3j\equiv r \pmod 2\\ j\leq m}} f(j) = \sum_{\substack{j \text{ odd}\\ j\leq m}} f(j) = \sum_{j=1}^{[(m+1)/2]} f(2j-1)
\end{aligned}\right\}\ r \text{ odd}$$

13. $$\sum_{j=0}^{\infty} a_{k+jm}\, x^{k+jm} = m^{-1} \sum_{j=1}^{m} \alpha_m^{m-kj} \sum_{t=0}^{\infty} a_t (\alpha_m^j\, x)^t \quad .$$

(This is the multisection of series; α_m is an i^{th} root of unity and $\alpha_m \neq 1$.)

14. $$\Big(\sum_{j=1}^{n} f(j)\, g(j)\Big)^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} f(i)^2\, g(k)^2 - \sum_{1\leq k<j\leq n} \big(f(k)\, g(j) - f(j)\, g(k)\big)^2 ,$$

15. $$2\sum_{j=0}^{n} \sum_{k=0}^{j} f(j)\, f(k) = \Big(\sum_{j=0}^{n} f(j)\Big)^2 + \sum_{j=0}^{n} f(j)^2 \quad .$$

16. $$\sum_{j+ak=n} f(j,k) = \sum_{k=0}^{[n/a]} f(n-ak,k) \quad .$$

17. $$\sum_{j=a}^{n} \sum_{k=b}^{j} f(j,k) = \sum_{k=b}^{n} \sum_{j=\max(a,k)}^{n} f(j,k) \quad , *(n).$$

18. $$\sum_{j=a}^{n} \sum_{k=a(a-1)}^{j(j-1)} f(j,k) = \sum_{k=a(a-1)}^{n(n-1)} \sum_{j=1+[(1+\sqrt{4k-3})/2]}^{n} f(j,k) \quad , \qquad *(n) \quad .$$

19. $$\sum_{j=a}^{n}\sum_{k=a}^{rj} f(j,k) = \sum_{k=a}^{ra-1}\sum_{j=a}^{n} f(j,k) + \sum_{k=ra}^{rn}\sum_{j=1+[(k-1)/r]}^{n} f(j,k) ,$$

$*(n)$. $r \geq 1$, $a \geq 1$.

20. $$\sum_{j=0}^{n}\sum_{k=0}^{rj} f(j,k) = \sum_{k=0}^{nr}\sum_{j=\{k/r\}}^{n} f(j,k) , \quad *(n).$$

21. $$\sum_{j=0}^{n}\sum_{k=j}^{rj} f(j,k) = \sum_{k=0}^{rn}\sum_{j=[(k+r-1)/r]}^{\min(k,n)} f(j,k) , \quad *(n)$$

22. $$\sum_{j=a}^{n}\sum_{k=a}^{[j/r]} f(j,k) = \sum_{k=a}^{[n/r]}\sum_{j=rk}^{n} f(j,k) , \quad *(n).$$

23. $$\sum_{j=a}^{n}\sum_{k=b}^{[j/r]} f(j,k) = \sum_{k=b}^{[a/r]}\sum_{j=a}^{tr-1} f(j,k)$$

$$+ \sum_{k=b}^{t+s-1}\sum_{j=tr+r\delta_{k,t+1}+2r\delta_{k,t+2}+\ldots+(s-1)r\delta_{k,s-1}}^{r(t+s)-1} f(j,k)$$

$$+ \sum_{k=b}^{t+s}\sum_{j=r(t+2)}^{n} f(j,k) ,$$

where t and s are determined by $t = [a/r] + 1$ and

$n-tr = sr + p$, $(0 \leq p < sk)$. $*(n)$.

24. $$\sum_{j=a}^{n}\sum_{\substack{k=b \\ k|j}}^{j} f(j,k) = \sum_{k=b}^{n}\sum_{j=\{a/k\}}^{[n/k]} f(jk,k), \quad *(n) .$$

25. $\sum_{j=a}^{n} \sum_{\substack{k=b \\ k|j}}^{j} f\left(\frac{j}{k}, k\right) = \sum_{k=b}^{n} \sum_{j=\{a/k\}}^{[n/k]} f(j,k), \quad *(n).$

26. $\sum_{j=a}^{\infty} \sum_{k=j+a}^{\infty} f(j,k) = \sum_{k=2a}^{\infty} \sum_{j=a}^{k-a} f(j,k) .$

27. $\sum_{j=0}^{n} \sum_{k=r-j}^{r} f(j,k) = \sum_{k=r-n}^{r} \sum_{j=r-k}^{n} f(j,k), \quad *(n).$

28. $\sum_{j=a}^{n} \sum_{k=j}^{n} f(j,k) = \sum_{k=a}^{n} \sum_{j=k}^{n} f(k+n-j,\ n-j+a) .$

29. $\sum_{j=a}^{\infty} \sum_{k=j+a}^{\infty} f(j,k) = \sum_{k=2a}^{\infty} \sum_{j=a}^{k-a} f(j,k) .$

30. $\sum_{j=1}^{n} \sum_{k=1}^{[\sqrt[a]{j}]} f(j,k) = \sum_{k=1}^{[\sqrt[a]{n}]} \sum_{k=j^a}^{n} f(jk) .$

(See [12] .)

31. $\sum_{j=1}^{n} \sum_{k=1}^{2^i-1} f(j,k) = \sum_{j=1}^{2^n-1} \sum_{k=1+[\log_2 j]}^{n} f(j,k) .$

(See [12].)

32. $\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f(j,k) = \sum_{j=0}^{\infty} \sum_{k=0}^{n} f(j-k,\ k).$

33. $$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} f(j,k) = \sum_{j=0}^{\infty} \sum_{k=0}^{[j/2]} f(j-2k,\ k)\ .$$

34. $$\sum_{j=0}^{\infty} \sum_{k=0}^{j} f(j,k) = \sum_{j=0}^{\infty} \sum_{k=0}^{[j/2]} f(j-k,\ k)\ .$$

35. $$\sum_{j_1=a_1}^{n} \sum_{j_2=a_2}^{j_1} \sum_{j_3=a_3}^{j_2} \cdots \sum_{j_r=a_r}^{j_{r-1}} f(j_1,\ j_2, \ldots,\ j_r)$$

$$= \sum_{j_r=a_r}^{n} \sum_{j_{r-1}=\max(a_{r-1},j_r)}^{n} \sum_{j_{r-2}=\max(a_{r-2},j_r,j_{r-1})}^{n}$$

$$\cdots \sum_{j_1=\max(a_1,j_r,j_{r-1},\ldots,j_2)}^{n} f(j_1,\ j_2,\ \ldots,\ j_r), \quad *(n).$$

36. $$\sum_{j_1=0}^{\infty} \sum_{j_2=0}^{j_1} \sum_{j_3=0}^{j_2} \cdots \sum_{j_r=0}^{j_{r-1}} f(j_1,\ j_2,\ \ldots,\ j_r)$$

$$= \sum_{j_r=0}^{\infty} \sum_{j_{r-1}=j_r}^{\infty} \cdots \sum_{j_1=j_2}^{\infty} f(j_1,\ j_2,\ \ldots,\ j_r)$$

$$= \sum_{j_1,\ldots,j_r=0}^{\infty} f(j_1+j_2+\ldots+j_r,\ j_2+j_3+\ldots+j_r,\ \ldots,\ j_{r-1}+j_r,\ j_r).$$

37. $$\sum_{j_1+j_2+\cdots+j_n=r} f(j_1, j_2, \ldots, j_n)$$

$$= \sum_{j_n=0}^{r} \sum_{j_{n-1}=0}^{r-j_n} \cdots \sum_{j_2=0}^{r-j_n-\cdots-j_3} f(r-j_n-\cdots-j_2, j_2, \ldots, j_n) .$$

38. $$\sum_{0\le j_1\le j_2\le\cdots\le j_n\le r} f(j_1, j_2, \ldots, j_n)$$

$$= \sum_{j_1=0}^{r} \sum_{j_2=j_1}^{r} \cdots \sum_{j_n=j_{n-1}}^{r} f(j_1, j_2,\ldots, j_n)$$

$$= \sum_{j_n=0}^{r} \sum_{j_{n-1}=0}^{j_n} \cdots \sum_{j_1=0}^{j_2} f(j_1, j_2, \ldots, j_n), \quad *(r) .$$

39. $$\sum_{j_1,j_2,\ldots,j_n=0}^{\infty} \sum_{k=0}^{j_1+j_2+\cdots+j_n} f(j_1, j_2, \ldots, j_n, k)$$

$$= \sum_{j_1,\ldots,j_n,k=0}^{\infty} \{f(j_1+k, j_2, j_3, \ldots, j_n, k)$$

$$+ f(j_1, j_2 + k + 1, j_3, \ldots, j_n, k + j_1 + 1)$$

$$+ f(j_1, j_2, j_3 + k + 1, j_4, \ldots, j_n, k + j_1 + j_2 + 1)$$

$$+ \ldots + f(j_1, j_2, \ldots, j_{n-1}, j_n + k + 1, k + j_1 + \ldots + j_{n-1} + 1)\} .$$

40. $$\sum_{j_1,j_2,\ldots,j_n=0}^{\infty} \sum_{k=0}^{\min(j_1,j_2,\ldots,j_n)} f(k, j_1, j_2,\ldots, j_n)$$

$$= \sum_{k=0}^{\infty} \sum_{j_1,j_2,\ldots,j_n=k}^{\infty} f(k, j_1, j_2, \ldots, j_n) .$$

41. $$\sum_{j=0}^{\infty} \sum_{k_1=0}^{j} \sum_{k_2=0}^{j} \cdots \sum_{k_n=0}^{j} f(j, k_1, \ldots, k_n)$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} \sum_{j=\max(k_1,\ldots,k_n)}^{\infty} f(j, k_1, k_2, \ldots, k_n)$$

$$= \sum_{j,k_1,k_2,\ldots,k_n=0}^{\infty} f(j - \max(k_1,\ldots,k_n), k_1, k_2, \ldots, k_n) .$$

42. $$\sum_{i,j=0}^{\infty} \sum_{k=0}^{\max(i,j)} f(i,j,k)$$

$$= \sum_{i,j,k=0}^{\infty} \{f(i+k, i+j+k, k) + f(i, i+j+k+1, i+k+1)$$

$$+ f(i+j+k, j, k) + f(i+j+k+1, j+k+1, i+k+1)\}$$

$$- \sum_{i,k=0}^{\infty} f(i+k, i+k, k) .$$

(There are other similar expressions for
$\sum_{i,j=0}^{\infty} \sum_{k=0}^{\max(i,j)} f(i,j,k)$.)

43. Let $s(j_1,j_2,\ldots,j_k)$ be symmetric in $j_1,j_2,\ldots,j_k$. Then

$$\sum_{j_1,\ldots,j_k=0}^{\infty} s(j_1,\ldots,j_k) = \sum_{i=1}^{k} \binom{k}{i} \sum_{j_1,\ldots,j_i=1}^{\infty} s(j_1,\ldots j_i,0,\ldots,0) + S(0,\ldots,0).$$

44. Let $s(j_1,j_2,\ldots,j_k)$ be symmetric in $j_1,j_2,\ldots,j_k$ and let $\sum^* s(j_1,\ldots,j_k)$ be the symmetric sum over $j_1,\ldots,j_k,\ldots,j_n$ (the sum of $S(j_1,\ldots,j_k)$ over all k-combinations of $j_1,\ldots,j_n$). Then

$$\sum_{j_1,\ldots,j_n=0}^{r} \sum^* s(j_1,\ldots,j_k) = \binom{n}{k}(r+1)^{n-k} \sum_{j_1,\ldots,j_k=0}^{r} s(j_1,\ldots,j_k).$$

45. Let $\sum^*$ be the symmetric sum over s out of k characters.

If f is symmetric, then

$$\sum^* \sum_{n_1,\ldots,n_k=0}^{\infty} f(n_1,\ldots,n_s)\, x_1^{n_1} \ldots x_k^{n_k}$$

$$= \sum_{n_1,\ldots,n_k=0}^{\infty} \{\sum^* f(n_1,\ldots,n_s)\}\, x_1^{n_1} \ldots x_k^{n_k}$$

where the symmetric sum on the left is over s out of k x's and the symmetric sum on the right is over s out of k n's and <u>does not extend beyond the braces</u>.

46. Suppose that $n \geq 1$ and $a_1, \ldots, a_n$ are positive integers that are relatively prime in pairs. Let $A = a_1 a_2 \ldots a_n$. Let $f_1(x), \ldots, f_n(x)$ be functions of x of period 1 such that there exists the relation $\sum_{r=0}^{k-1} f_i(X + r/k) = c_i^{(k)} f_i(kx)$ where $c_i^{(k)}$ is independent of x. Then

$$\sum_{r=0}^{kA-1} f_1\left(\frac{r}{a_1 k}\right) f_2\left(\frac{r}{a_2 k}\right) \ldots f_n\left(\frac{r}{a_n k}\right) = c_1^{(a_1)} c_2^{(a_2)} \ldots c_n^{(a_n)} \sum_{r=0}^{k-1} f_1\left(\frac{r}{k}\right) f_2\left(\frac{r}{k}\right) \ldots f_n\left(\frac{r}{k}\right)$$

and, more generally,

$$\sum_{r=0}^{kA-1} f_1\left(x_1 + \frac{r}{a_1 k}\right) \ldots f_n\left(x_n + \frac{r}{a_n k}\right) = c_1^{(a_1)} \ldots c_n^{(a_n)} \sum_{r=0}^{k-1} f_1\left(a_1 x_1 + \frac{r}{k}\right) \ldots f_n\left(a_n x_n + \frac{r}{k}\right) .$$

This is due to Carlitz [4] and [5] where there appear some applications.

47. If g is a continuous increasing function and if g(i) is integral for $i \in \{a-1, a, \ldots, n\}$ and if g^{-} denotes the inverse function of g then,

$$\text{(a)} \quad \sum_{i=a}^{n} \sum_{j=g(a)}^{g(i)} f(i,j) = \sum_{j=g(a)}^{g(n)} \sum_{i=1+[g^{-}(j-1)]} f(i,j) \quad , \quad *(n) \quad .$$

This formula is useful in some problems requiring the interchanging of the order of summation. H. W. Gould (see [14]) and probably others are familiar with it. A proof, one which we adopt here because of its revealing applicable structure, is given by Towe [43].

Proof of (a). The left side may be expressed as the sum of the entries of the following table where x denotes f(i,j) evaluated for the indicated values of i and j:

i \ j	g(a)	g(a)+1	g(a)+2	...	$g(a)+n_0$	g(a+1)	g(a+1)+1	...	$g(a+1)+n_1$	g(a+2)	...	g(n)
a	x											
a+1	x	x	x	...	x	x						
a+2	x	x	x	...	x	x	x	...	x	x		
.												
.												
.												
n	x	x	x	...	x	x	x	...	x	x	...	x

Note that

(b) $$g(a+k)+n_k+1=g(a+k+1).$$

Reversing the order of summation, the sum may be expressed as

(c) $$\sum_{j=g(a)}^{g(n)} \sum_{i=b(j)}^{n} f(i,j)$$

where

(d) $$b(j)=a \text{ if } j=g(a)$$

and

(e) $b(j)=a+k+1$ if $g(a+k)+1 \leq j \leq g(a+k)+n_k+1$, $k=0,1,2,\ldots,n-a-1$.

Then, in order to obtain the right side of (c) we need to show that $b(j)=1+[g^{-}(j-1)]$. We do so by considering (d) and (e) separately.

From (e) we must show that if $g(a+k) \le j-1 \le g(a+k)+n_k$, then $a+k=[g^{-}(j-1)]$. If $a+k>[g^{-}(j-1)]$, then $a+k>g^{-}(j-1)$ because a+k is an integer. Since g is increasing, $g(a+k)>j-1$. But this is a contradiction. Next, if $a+k<[g^{-}(j-1)]$ then $a+k+1 \le g^{-}(j-1)$ so that $g(a+k+1) \le j-1$. Then by (b), $g(a+k)+n_k+1 \le j-1$; we have another contradiction. Therefore, $a+k=[g^{-}(j-1)]$ as required.

On the other hand, from (d) we consider the case $j=g(a)$. Since g is increasing and since $g(a-1)$ and $g(a)$ are integers

$$g(a-1) \le g(a)-1<g(a).$$

Then $a-1 \le g^{-}(g(a)-1)<a$ because g^{-} is increasing on $[g(a-1), g(n)]$. From this we see at once that

$$a=1+[g^{-}(g(a)-1)].$$

Thus (a) is proved.

The formula, with only minor changes in the above proof, is also valid if n is replaced by ∞; this case frequently occurs.

Note that if $b < g(a)$, then

$$\text{(f)} \quad \sum_{j=a}^{n} \sum_{k=b}^{g(j)} f(j,k) = \sum_{j=a}^{n} \sum_{k=g(a)}^{g(j)} f(j,k) + \sum_{j=a}^{n} \sum_{k=b}^{g(a)-1} f(j,k).$$

Equation (a) now may be employed to evaluate the first sum on the right side.

A sum of the form

$$\text{(g)} \qquad \sum_{j=a}^{n} \sum_{k=h(j)}^{g(j)} f(j,k)$$

is usually difficult to work with. One approach is to put

$r(j) = g(j) - h(j)$ and $e(j,k) = f(j,k+h(j))$ so that it becomes

(h) $$\sum_{j=a}^{n} \sum_{k=0}^{r(j)} e(j,k) \quad .$$

Now, perhaps (f) can be applied; however, $e(j,k)$ may be unwieldy. An alternate approach may be possible; (g) is

$$\sum_{j=a}^{n} \sum_{k=h(a)}^{g(j)} f(j,k) - \sum_{j=a+1}^{n} \sum_{k=h(a)}^{h(j)-1} f(j,k)$$

$$= \sum_{j=a}^{n} \sum_{k=h(a)}^{g(j)} f(j,k) - \left(\sum_{j=a+1}^{n} \sum_{k=h(a)}^{h(j)} f(j,k) - \sum_{j=a+1}^{n} f(j, h(j)) \right)$$

$$= \sum_{j=a}^{n} \sum_{k=h(a)}^{g(j)} f(j,k) - \sum_{j=a}^{n} \sum_{k=h(a)}^{h(j)} f(j,k) + f(a,h(a)) + \sum_{j=a+1}^{n} f(j,h(j)).$$

Perhaps (a) and (f) can be applied to some of these sums.

For a simple example, suppose we are interested in the sum $s=\sum_{j=0}^{n} \sum_{k=j}^{j^2} f(j,k)$. Then, as in going from (g) to (h), $s=\sum_{j=0}^{n} \sum_{k=0}^{j(j-1)} f(j,k+j)$ and the order of summation may be inverted according to formula 18.

Under the same conditions of function (g) above, we note the following [12]: If $g^*(n) = \text{card}\ \{m \mid f(m) < n\}$ then

$$\sum_{i=1}^{n} \sum_{j=1}^{g(i)} f(i,j) = \sum_{j=1}^{g(n)} \sum_{i=g^*(j)+1}^{n} f(i,j) \ .$$

Formula (a) is sometimes useful in interchanging the order of summation of multiple sums, provided the conditions on g are met. Suppose g(i,j) is continuous and increasing in both variables. Then by two applications of (a) we obtain

$$\sum_{i,j=a}^{n} \sum_{h=g(a,a)}^{g(i,j)} f(i,j,k) = \sum_{h=g(a,a)}^{g(n,n)} \sum_{i=1+[g_h^-(h-1,n)]}^{n} \sum_{j=1+[g_h^-(i,h-1)]}^{n} f(i,j,k) ,$$

where g_h^- denotes the inverse of g^- with respect to h. Other similar formulas may be obtained for multiple sums.

48. Let γ denote an increasing continuous function on the real numbers such that $\gamma(n)$ is a positive integer if n is a positive integer and $\gamma(0) = 0$. Also let γ^- be the inverse function of γ. Then

$$\sum_{j_1,\ldots,j_n=0}^{\infty} \sum_{k=0}^{(\min(j_1,\ldots,j_n))} f(k,j_1,\ldots,j_n) = \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n \geq \{\gamma^-(k)\}} f(k,j_1,\ldots,j_n).$$

(See [18]and [25].)

49. Under the conditions of formula 48,

$$\sum_{j_1,\ldots,j_n=0}^{\infty} \sum_{k=\gamma(\max(j_1,\ldots,j_n))}^{\infty} f(k,j_1,\ldots,j_n) = \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n \leq [\gamma^-(k)]} f(k,j_1,\ldots,j_n).$$

50. Under the conditions of formula 48,

$$\sum_{j_i,\ldots,j_n=0}^{\infty} \sum_{k=0}^{[\gamma^-(\min(j_1,\ldots,j_n)]} f(k,j_1,\ldots,j_n) =$$

$$\sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n \geq \gamma(k)} f(k,j_1,\ldots,j_n) \quad .$$

51. Under the conditions of formula 48,

$$\sum_{j_1,\ldots,j_n=0}^{\infty} \sum_{k=\{\gamma^-(\max(j_1,\ldots,j_n))\}}^{\infty} f(k,j_1,\ldots,j_n)$$

$$= \sum_{k=0}^{\infty} \sum_{j_1,\ldots,j_n \le \gamma(k)} f(k,j_1,\ldots,j_n) \quad .$$

52. The q-Eulerian function $H_k(x|q_1,\ldots,q_k)$ may be defined symbolically by $x^{-1} \Pi_{j=1}^{k}(1 - q_j H)$ if $k \geq 1$; in addition, $H_0 = 1$. Roselle [39; (3.2) and (3.9)] proved the following:

(a) $xH_k(x|q_1,\ldots,q_k) = \sum_{n_1,\ldots,n_k=0}^{\infty} q_1^{n_1} \ldots q_k^{n_k} x^{-\max(n_1,\ldots,n_k)}$,

(b) $H_k(x^{-1}|q_1^{-1}, \ldots , q_k^{-1}) = (-1)^k x\, q_1 \ldots q_k\, H_k(x|q_1, \ldots, q_k)$.

For further properties of H, see [3], [8] and [39].

We let γ denote an increasing continuous function on the real numbers such that $\gamma(0) = 0$ and $\gamma(n)$ is a positive integer if n is a positive integer. Also, we let γ^- be the inverse function of γ and recall that $\{x\}$ is the least integer $\geq x$. Then

(c) $$\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \sum_{j=0}^{[\gamma^-(\min(n_1,\ldots,n_k))]} c_j\, q_1^{n_1} \ldots q_k^{n_k}\, z^t$$

$$= (1-z)^{-1}(-1)^k q_1^{-1}\ldots q_k^{-1}\, H_k(z|q_1^{-1},\ldots,q_k^{-1}) \sum_{j=0}^{\infty} c_j(q_1\ldots q_k z)^{\gamma(j)}$$

Proof is as follows:

$$\sum_{t=0}^{\infty} \sum_{n_1,\dots,n_k=0}^{t} \sum_{j=0}^{[\gamma^-(\min(n_1,\dots,n_k))]} c_j\, q_1^{n_1} \dots q_k^{n_k} z^t$$

$$= \sum_{n_1,\dots,n_k+0}^{\infty} \sum_{t\geq\max(n_1,\dots,n_k)} \sum_{j=0}^{[\gamma^-(\min(n_1,\dots,n_k))]} c_j q_1^{n_1}\dots q_k^{n_k} z^t \quad \text{(formula 41)}$$

$$= (1-z)^{-1} \sum_{n_1,\dots,n_k=0}^{\infty} \sum_{j=0}^{[\gamma^-(\min(n_1,\dots,n_k))]} c_j q_1^{n_1}\dots q_k^{n_k} z^{\max(n_1,\dots,n_k)} \quad \text{(formula 41)}$$

$$= (1-z)^{-1} \sum_{n_1,\dots,n_k=0}^{\infty} \sum_{j\leq[\gamma^-(n_1)],\dots,[\gamma^-(n_k)]} c_j q_1^{n_1}\dots q_k^{n_k} z^{\max(n_1,\dots,n_k)}$$

(properties of γ, min, [ ])

$$= (1-z)^{-1} \sum_{j=0}^{\infty} c_j \sum_{n_1,\dots,n_k\geq\gamma(j)}^{\infty} q_1^{n_1}\dots q_k^{n_k} z^{\max(n_1,\dots,n_k)} \quad \text{(property of } \gamma)$$

$$= (1-z)^{-1} z^{-1} H_k(z^{-1}|q_1,\dots,q_k) \sum_{j=0}^{\infty} c_j (q_1 \dots q_k\, z)^{\gamma(j)} \quad \text{(equation (a))}$$

$$= (1-z)^{-1}(-1)^k q_1^{-1} \dots q_k^{-1} H_k(z|q_1^{-1},\dots,q_k^{-1}) \sum_{j=0}^{\infty} c_j (q_1 \dots q_k\, z)^{\gamma(j)}$$

(equation (b)).

Application 1. Let $f(n)$ be a nondecreasing sequence of positive integers and define the distribution function, $D(f)$, by

$$D(f(n)) = \text{card}\,\{k|f(k) \leq n;\ k = 1, 2, \dots\}.$$

Note that

(d) $$\sum_{n=1}^{\infty} D(f(n))\, x^n = \sum_{n=1}^{\infty} \sum_{f(k)\leq n} x^n = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} x^{n+f(k)} = (1-x)^{-1} \sum_{k=1}^{\infty} x^{f(k)}\ .$$

Therefore, if we put $c_0 = 0$ and $c_j = 1$ in (c) and then use (d), we have

$$\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} [\gamma^{-}(\min(n_1,\ldots,n_k))]\, q_1^{n_1} \ldots q_k^{n_k}\, z^t$$

$$= (1-z)^{-1}(-1)^k\, q^{-1} \ldots q_k^{-1}\, H_k(z|q^{-1},\ldots,q_k^{-1}) \sum_{j=1}^{\infty} (q_1 \ldots q_k\, z)^{\gamma(j)}$$

$$= (1-q_1q_2\ldots q_k z)(1-z)^{-1}(-1)^k q_1^{-1}\ldots q_k^{-1} H_k(z|q_1^{-1},\ldots,q_k^{-1}) \sum_{j=1}^{\infty} D(\gamma(j)(q_1\ldots q_k z)^j$$

This gives an interesting q-generating formula for distribution functions:

$$\sum_{j=1}^{\infty} D(\gamma(j))(q_1\ldots q_k z)^j$$

$$=(1-q_1q_2\ldots q_k z)^{-1}(1-z)(-1)^k q_1\ldots q_k (H_k(z|q_1^{-1},\ldots,q_k^{-1}))^{-1}$$

$$\cdot \sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} [\gamma^{-}(\min(n_1,\ldots,n_k))] q_1^{n_1}\ldots q_k^{n_k}\, z^t.$$

Application 2. In (c) put $\gamma(j) = j^m$, $c_0 = 0$, $c_j = 1 (j \geq 1)$. Then

$$\text{(e)} \quad \sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} [\min(n_1,\ldots,n_k)^{1/m}]\, q_1^{n_1} \ldots q_k^{n_k}\, z^t$$

$$= (1-z)^{-1}(-1)^k\, q_1^{-1} \ldots q_k^{-1}\, H_k(z|q_1^{-1}, \ldots , q_k^{-1}) \sum_{j=1}^{\infty} (q_1 \ldots q_k\, z)^{j^m} .$$

In the case that k=1, (e) may be compared with some results in [12], [44] and other similar formulas of number theoretic interest.

53. Under the same conditions of the summation formula (52.c), we have

$$\text{(a)} \quad \sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} \sum_{j=0}^{\gamma(\min(n_1,\ldots,n_k))} c_j\, q_1^{n_1} \cdots q_k^{n_k}\, z^t$$

$$= (1-z)^{-1}\,(-1)^k\, q_1^{-1}\cdots q_k^{-1}\, H_k(z|q_1^{-1},\ldots,q_k^{-1}) \sum_{j=0}^{\infty} c_j (q_1\cdots q_k z)^{\{\gamma^-(j)\}}.$$

This formula may be compared with the partition formulas of [18]. Proof of (a) is as follows. The left side becomes

$$\sum_{n_1,\ldots,n_k=0} \sum_{t\geq \max(n_1,\ldots,n_k)} \sum_{j=0}^{\gamma(\min(n_1,\ldots,n_k))} c_j q_1^{n_1} \cdots q_k^{n_k}\, z^t$$

$$= (1-z)^{-1} \sum_{n_1,\ldots,n_k=0}^{\infty} \sum_{j\leq\gamma(n_1),\ldots,\,(n_k)} c_j q_1^{n_1} \cdots q_k^{n_1}\, z^{\max(n_1,\ldots,n_k)}$$

$$= (1-z)^{-1} \sum_{z=0}^{\infty} c_j \sum_{n_1,\ldots,n_k\geq\{\gamma^-(j)\}} q_1^{n_1} \cdots q_k^{n_k}\, z^{\max(n_1,\ldots,n_k)}$$

$$= (1-z)^{-1}\, z^{-1}\, H_k(z^{-1}|q_1,\ldots,q_k) \sum_{j=0}^{\infty} c_j (q_1\cdots q_k\, z)^{\{\gamma^-(j)\}}.$$

The last step is obtained by applying (52.a). Using (52.b) this becomes the right side of (a); this completes the proof.

Application. If, in (a), we put $c_0 = 0$, $c_j = 1$ $(j \geq 1)$ and $\gamma(x) = x^m$, then

(b) $\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} (\min(n_1,\ldots,n_k))^m q_1^{n_1} \ldots q_k^{n_k} z^t$

$$= (1-z)^{-1} (-1)^k q_1^{-1} \ldots q_k^{-1} H_k(z|q_1^{-1} \ldots q_k^{-1}) \sum_{j=1}^{\infty} (q_1 \ldots q_k z)^{\{j^{1/m}\}}.$$

The Eulerian polynomial, $a_k(x)$, may be expressed as (see [18, p. 303]):

$$a_k(x) = (1 - x)^{-1} \sum_{j=1} x^{j^{1/k}} .$$

Also, $x(x - 1)^k H_k(x) = a_k(x)$ where $H_k(x)$ is the ordinary Eulerian function $H_k(x|1,1, \ldots , 1)$. Hence, from (b) we have the interesting relation

(c) $\sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} (\min(n_1,\ldots,n_k))^m q_1^{n_1} \ldots q_k^{n_k} z^t$

$$= (1-z)^{-1} (-1)^{k+m} z H_k(z|q_1^{-1}, q_2^{-1}, \ldots , q_k^{-1}) H_m(q_1 q_2 \ldots q_k z).$$

If we put $q_1 = q_2 = \ldots = q_k = 1$, then (c) becomes a new product formula for Eulerian functions (and hence polynomials):

(d) $H_k(z) H_m(z) = (-1)^{k+m} \frac{1-z}{z} \sum_{t=0}^{\infty} \sum_{n_1,\ldots,n_k=0}^{t} (\min(n_1,\ldots,n_k))^m z^t.$

Notice the curious symmetry in m and k.

Dividing (d) by $(-1)^{k+1} (1 - z)/z$, expressing the left side as a power series in z and comparing coefficients we get

(e) $$\sum_{n_1,\ldots,n_k=0}^{t-1} (\min(n_1,\ldots,n_k))^m = \sum_{i+j+h=t} \binom{k+m+h}{h} A_{k,i} A_{n,j}$$

where the $A_{n,j}$ are the Eulerian numbers defined by $(x - 1)^n H_n(x) = \sum_{j=1}^{n} A_{n,j} x^{j-1}$ $(n \geq 1)$. Evidently (e) is a new interesting relation for the Eulerian numbers. Special cases of (e) include Worpitzky's well known result $x^k = \sum_{s=1}^{k} A_{k,s} \binom{x+s-1}{k}$.

Expressing the right side of (e) as $\sum_{h=0}^{t} \binom{k+m+h}{h} \sum_{i+j=t-h} A_{k,i} A_{m,j}$ and then inverting [38; p. 106] we find a convolution formula for Eulerian numbers:

(f) $$\sum_{i+j=t} A_{k,i} A_{m,j} = \sum_{h=0}^{t} (-1)^h \binom{k+m+1}{h} \sum_{n_1,\ldots,n_k=0}^{t-h-1} (\min(n_1,\ldots,n_k))^m.$$

Combinatorial significance of Eulerian numbers is well known (see [37; p. 214] and [8]): $A_{n,k}$ is the number of permutations of $\{1, 2, \ldots, n\}$ with k rises where it is agreed that a rise appears on the left. Thus there is a direct interpretation of (f); it gives the number of permutations of $\{1, 2, \ldots, k\}$ and $\{1, 2, \ldots, m\}$ for which there are a total of exactly t rises.

54. The results of (52) and (53) can easily be generalized in some directions. Consider the following. Let γ be a function from a p dimensional (real) space to the real numbers. Define the distribution function of γ as follows:

$$D(\gamma:n_1,\ldots,n_k) = \text{card } \{i_1,\ldots,i_A | \gamma(i_1,\ldots,i_p) \leq n_1, \ldots, n_k\} .$$

Then,

$$(1-z)^{-1}(-1)^k H_k(z|q_1^{-1}\dots q_k^{-1})(q_1\dots q_k)^{-1} \sum_{i_1,\dots,i_p=0}^{\infty} c_{i_1,\dots,i_p}(q_1\dots q_k\, z)^{\gamma(i_1,\dots,i_p)}$$

$$= (1-z)^{-1} \sum_{i_1,\dots,i_p=0}^{\infty} \sum_{n_1,\dots,n_k \geq \gamma(i_1,\dots,i_p)} c_{i_1,\dots,i_p}\, q_1^{n_1}\dots q_k^{n_k} z^{\max(n_1,\dots,n_k)}$$

$$= (1-z)^{-1} \sum_{n_1,\dots,n_k=0}^{\infty} \sum_{\{i_1,\dots,i_p|\gamma(i_1,\dots,i_p)\leq \min(n_1,\dots,n_k)\}} c_{i_1,\dots,i_p} q_1^{n_1}\dots q_k^{n_k} z^t .$$

A q-generating function for the distribution function is found by conveniently defining c. Thus, we may put $c_{i_1,\dots,i_p} = 1$ or

$$c_{i_1,\dots,i_p} = \begin{cases} 0 \text{ if at least one of the } i\text{'s is zero} \\ 1 \text{ otherwise.} \end{cases}$$

Special note. In much of the remainder of this work we discuss some general methods for determining power series which possess one or both the following properties: (a) the sequence of coefficients are periodic, (b) the terms of the sequence of coefficients, when arranged in their natural order according to the expansion of the series, change only at specified positions. For instance, we may be interested in the series $a(0)x^0 + a(0)x^1 + a(0)x^2 + a(1)x^3 + a(1)x^4 + a(2)x^5 + a(2)x^6 + a(2)x^7 + a(3)^8 + a(3)x^9 + a(4)x^{10} + \dots$ where, further $a(k)$ is periodic in k of period r. Or we may be interested in this series with x^k replaced by x^k/k or by some other indicator. We point out that property (b) can usually be effected by an appropriate composite of a function with the greatest integer function; this idea was examined by Schwatt [41] and [42].

In some areas, our work is an extension of Schwatt's. Also see the bibliography.

55. $\sum_{j=0}^{n} f(j,[\frac{j}{r}])$

$$= \sum_{k=0}^{[n/r]} \sum_{j=0}^{r-1} f(j+rk,k) - \sum_{j=0}^{r[n/r]+r-n-2} f(j+n+1,[\frac{n}{r}]), \quad r \geq 1.$$

56. $\sum_{j=0}^{\infty} f(j, [\frac{j}{r}]) = \sum_{k=0}^{\infty} \sum_{j=0}^{r-1} f(j+rk,k), \quad r \geq 1.$

57. $\sum_{j=0}^{n} f ([\frac{j}{r}]) = r \sum_{k=0}^{[n/r]} f(k) - (r[n/r] + r-n-1) f([\frac{n}{r}]), \quad r \geq 1 .$

58. $\sum_{j=0}^{\infty} f ([\frac{j}{r}]) = r \sum_{k=0}^{\infty} f(k) , \quad r \geq 1 .$

59. $\sum_{k=0}^{\infty} f(k,[\frac{k}{a}],[\frac{k}{b}])$

$$= \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{k=\max(0,[(r-s)/b])}^{\infty} f(bk+s, (bk+s-r)/a,k)$$

if $f(x,y,z) = 0$ whenever $y \neq$ integer)

$$= \sum_{r=0}^{a-1} \sum_{s=0}^{b-1} \sum_{k=\max(0,[(s-r)/a])} f(ak+r,k,(ak+r-s)/b)$$

if $f(x,y,q) = 0$ whenever $z \neq$ integer) .

(See [17].)

60. Suppose $f_k > 0$, $k = 0, 1, 2, \ldots$. A series in which the first p terms are positive and the next q terms are negative, and in which this pattern of alternating signs by groups is continued throughout, may be represented by

$$\sum_{k=0}^{\infty} (-1)^{[\frac{q+k}{p+q}]+[\frac{k}{p+q}]} f_k$$

$$= \sum_{j=0}^{\infty} \sum_{k=jm}^{jm+p-1} f_k - \sum_{j=0}^{\infty} \sum_{k=jm+p}^{jm+m-1} f_k \qquad (m = p+q).$$

(See [21], [24] and [42].)

61. A formula for the series that results from $\sum_{k=1}^{n} f(k)$ if the first a terms are retained, the following b terms are removed, the next a terms are retained and the following b terms are removed, etc., is given by $\sum_{k\geq 1} f(k+b[(k-1)/a])$. (See [42; p. 189].)

62. $$\sum_{j=0}^{\infty} \sum_{i=0}^{g([j/r])} f(j,i) = \sum_{k=0}^{\infty} \sum_{i=0}^{g(k)} \sum_{j=rk}^{(k+1)r-1} f(j,i) \quad .$$

(Furthermore, the methods of formula 47 may be applied in order to interchange the order of summation of the sums of k and i on the right side.)

63. Consider the sum

$$s = \sum_{k} f(qx + py)$$

where k extended over the lattice points

$\{(x,y)|0<x<p,\ 0<y<q,\ qx+py<pq\}$, where p and q are relatively prime positive integers and where f is a polynomial. It is shown by L. J. Mordell [34] that s is determined by the two formulas

$$\sum_{y=1}^{q-1}\sum_{x=1}^{p-1} f(qx+py) = \sum_k f(gx+py) + \sum_k f(2pq-qx-py),$$

$$\sum_{l=1}^{pq-1} f(l) - \sum_{l=1}^{p-1} f(lq) - \sum_{l=1}^{q-1} f(lp) = \sum_k f(qx+pq) + \sum_k f(pq-qx-py).$$

64. We present this formula because it falls into an interesting class of sums, sums of the general form $\sum_k f([k/m],[k/n])$ where m and n are positive integers. Recall that α_m is a primitive root of unity ($\alpha_m \neq 1$).

If a(k,i,j:r) is a function in i, j, and k and if it is periodic in j with period r then

$$\sum_{k=0}^{\infty} a(k,[\tfrac{k}{m}],[\tfrac{k}{n}]:\ r)\ x^k$$

$$= \sum_{j=0}^{r-1}\sum_{h=0}^{\infty}\sum_{k=(j+hr)n}^{(j+hr+1)n-1} a(k,[\tfrac{k}{m}],\ j:r)\ x^k \text{ (periodicity: Formula 66a)}$$

$$= \sum_{j=0}^{r-1}\sum_{h=0}^{\infty}\sum_{k=0}^{n-1} a(k+(j+hr)n,[\tfrac{k+(j+hr)n}{m}]\ ,\ j:r)\ x^{k+(j+hr)n}$$

$$= \sum_{j=0}^{r-1}\sum_{k=0}^{n-1} \{\sum_{k=0}^{\infty} a(k+jn+hrn,\ [\tfrac{k+jn+hrn}{m}]\ ,\ j:r)\ x^{k+jn+hrn}\}$$

$$= \sum_{j=0}^{r-1}\sum_{k=0}^{n-1} (rn)^{-1} \sum_{k=1}^{rn} \alpha_{rn}^{rn-(k+jn)h} \sum_{t=0}^{\infty} a(t,[\tfrac{t}{m}],\ j:r)(\alpha_{rn}^{h}\ x)^t \text{ (multisection)}$$

$$= \sum_{j=0}^{r-1} \sum_{k=0}^{n-1} (rn)^{-1} \sum_{h=1}^{rn} \alpha_{rn}^{rn-(k+jn)h} \sum_{t=0}^{\infty} \sum_{l=0}^{m-1} a(l+mt,t,j:r)(\alpha_{rn}^{h} x)^{l+mt} \quad \text{(Formula 56)}$$

$$= (rn)^{-1} \sum_{j=0}^{r-1} \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \sum_{h=1}^{rn} \alpha_{rn}^{rn-(k+jn)h} \{\sum_{t=0}^{\infty} a(l+mt,t,j:r)(\alpha_{rn}^{h} x)^{l+mt}\}$$

$$= (rn)^{-1} \sum_{j=0}^{r-1} \sum_{k=0}^{n-1} \sum_{s=0}^{\infty} \sum_{h=1}^{rn} \sum_{\substack{l+mt=s \\ l\leq m-1}} a(s,t,j:r)\alpha_{rn}^{rn-kh-jnh+hs} x^{s} .$$

(Compare with formula 59.)

65. If $a(i,j:r)$ is a function in i and j and periodic in i with period r then

$$\sum_{k=0}^{\infty} a(k,[\tfrac{k}{n}]:r)x^{k} = \frac{1}{r} \sum_{l,j=0}^{r-1} \sum_{k=0}^{n-1} \sum_{t=0}^{\infty} \alpha_{r}^{r-j(l+1)+l(k+nt)} a(j,t:r)x^{nt} .$$

66. If $c(x:r)$ is a function of x of period r then

$$\sum_{k=0}^{\infty} c([\tfrac{q+k}{p+q}] +[\tfrac{k}{p+q}]:r) f_{k}$$

$$= p \sum_{k=0}^{\infty} c(2k:r) \sum_{j=Mk+1}^{Mk+p} f_{j} + q \sum_{k=0}^{\infty} c(2k+1:r) \sum_{j=M(k+1)-q+1}^{M(k+1)} f_{j} \quad \text{(where } M=p+q)$$

$$= p \sum_{t=0}^{\infty} \sum_{k=tr}^{(t+1)r-1} c(2k:r) \sum_{j=Mk+1}^{Mk+p} f_{j} + q \sum_{t=0}^{\infty} \sum_{k=tr}^{(t+1)r-1} c(2k+1:r) \sum_{j=M(k+1)-q+1}^{M(k+1)} f_{j}$$

$$= p \sum_{t=0}^{\infty} \sum_{k=0}^{r-1} c(2(k+tr):r) \sum_{j=M(k+tr)+1}^{M(k+tr)+p} f_j + q \sum_{t=0}^{\infty} \sum_{k=0}^{r-1} c(2(k+tr+1):r) \sum_{j=M(k+tr+1)-q+1}^{M(k+tr+1)} f_j$$

$$= p \sum_{k=0}^{r-1} c(2k:r) \sum_{t=0}^{\infty} \sum_{j=M(k+tr)+1}^{M(k+tr)+p} f_r +q \sum_{k=0}^{r-1} c(2k+1:r) \sum_{t=0}^{\infty} \sum_{j=M(k+tr+1)-q+1}^{M(k+tr+1)} f_j$$

(Compare with formula 60. M=p+q.)

67. Let a(i, j:r) be a function in i and j, periodic in j with period r. If n and r are positive integers then

$$\sum_{k=0}^{\infty} a(k, \frac{k}{n} :r) = \sum_{j=0}^{\infty} \sum_{k=0}^{n-1} a(k + nj, j:r)$$

$$= \sum_{k=0}^{n-1} \sum_{h=0}^{\infty} \sum_{j=hr}^{(h+1)r-1} a(k + nj, j:r)$$

$$= \sum_{k=0}^{n-1} \sum_{h=0}^{\infty} \sum_{j=0}^{r-1} a(k + n(j + hr), j + hr:r)$$

$$= \sum_{j=0}^{r-1} \sum_{h=0}^{\infty} \sum_{k=(j+hr)n}^{(j+hr+1)n-1} a(k,j:r).$$

(There is a direct extension to multiple sums of $a(i_1,\ldots, i_m, j_1,\ldots, j_m:r_1,\ldots, r_m)$.)

Application. Let i be $(-1)^{1/2}$ and consider

$$\sum_{k=0}^{\infty} \exp \left(\frac{2\pi i}{r} [\frac{k}{n}]\right) f_k(x).$$

If $f_k(x) = x^k$, then this series becomes $(1-x^n)(1-x)^{-1}(1-x^n \exp(2\pi i/r))^{-1}$.
The more interesting case is where $f_k(x) = x^k/k!$; so we consider

(b) $$E(r, n; x) = \sum_{k=0}^{\infty} \exp\left(\frac{2\pi i}{r}\left[\frac{k}{n}\right]\right)x^k/k! .$$

The following definitions are useful:

$$S(n, r, j; x) = \sum_{h=0}^{\infty} \sum_{k=(j+hr)n}^{(j+hr+1)n-1} x^k/k! \quad ,$$

$$\hat{S}(n, r, j; s) = \int_0^{\infty} e^{-sx}\, S(n, r, j; x)\, dx.$$

We find that $\hat{S}(n, r, j; s) = s^{rn-jn-n}\,(s^n - 1)(s^{nr} - 1)^{-1}\,(s - 1)^{-1}$.
In order to invert the Laplace transformation we apply formula 21 of [1, p. 232]. This gives

$$S(n, r, j; x) = \sum_{q=1}^{2} \frac{\Phi_{1,q}(W_1)}{(2-q)!(q-1)!}\, x^{2-q}\, e^{W_1 x} + \sum_{u=2}^{rn} \Phi_{u,1}(W_u)\, e^{W_u x}$$

where the function $\Phi_{a,b}(s)$ are rational; with a little manipulation they may be determined using the formulas of [1]. Then

(c) $$S(n, r, j; x) = e^x/r + \frac{1}{rn}\sum_{k=2}^{rn} \frac{w_k^{1-n-jn}(1-w_k^n)}{1 - w_k} \exp(w_k\, x),\ (0 \le j < r)$$

where $w_2, w_3, \ldots, w_{rn}$ are primitive $(rn)^{th}$ roots of unity.

Applying (a) and (c) to (b) we get

$$E(r, n; x) = \sum_{j=0}^{r-1} \exp\left(\frac{2\pi i j}{r}\right) S(n, r, j; x)$$

$$= \sum_{j=0}^{r-1} \exp\left(\frac{2\pi i j}{r}\right) \frac{e^x}{r} + \frac{1}{rn} \sum_{k=2}^{rn} \frac{w_k^{1-n-jn}(1 - w_k^n)}{1 - w_k} \exp(w_k x) .$$

$$= e^x \delta_{1,r} + \frac{1}{rn} \sum_{j=0}^{r-1} \exp\left(\frac{2\pi i j}{r}\right) \sum_{k=2}^{rn} \frac{w_k^{1-n-jn}(1-w_k^n)}{1-w_k} \exp(w_k x)$$

where $\delta_{1,r}$ is the Kronecker delta. This equation has several interesting consequences; for instance, it generalizes some results of [21; section 2] concerning alternating exponential series. Note the following special cases.

$$E(1, 1; x) = e^x$$

$$E(2, 1; x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k!} = e^{-x}$$

$$E(2, n; x) = \sum_{k=0}^{\infty} (-1)^{[k/n]} \frac{x^k}{k!}$$

$$E(4, n; x) = \sum_{k=0}^{\infty} i^{[k/n]} \frac{x^k}{x!}$$

Also, note that $E(r, 1; x)$ is the exponential series for roots of unity. While the specific results are interesting, we emphasize the general method used in obtaining the results. This method may be applied to other power series with cyclic coefficients.

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