# LIKELIHOOD RATIO TESTS FOR ORDER RESTRICTIONS IN EXPONENTIAL FAMILIES

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## ABSTRACT

This report is on a continuation of the work discussed in Robertson and Wegman (1975). The results of a Monte-Carlo study of the power of the likelihood ratio statistic considered in the previous paper are discussed. The asymptotic distributions for the likelihood ratio statistics for testing homogeneity against trend and trend against "otherwise" when the sampled distributions belong to an exponential family are given.

1. Introduction: This report is on a continuation of the work discussed in Robertson and Wegman (1975). In Section 2 we discuss the results of a Monte-Carlo study of the power of the likelihood ratio statistic considered in the previous paper. In order to be able to make comparisons we included in this study a statistic proposed by Van Eeden (1958) for testing the same hypothesis.

In Section 3, we also consider tests for trend in parameters when the parameters involved arise from a distribution of the exponential type. The distribution for the likelihood ratio statistic for testing a trend hypothesis about normal means is shown to be the asymptotic distribution for the likelihood ratio statistic for an analogous test whenever the sampled distributions are members of the exponential family.

2. Monte Carlo Study: In this section, we report the results of a Monte-Carlo study of the power of the likelihood ratio statistic considered by Robertson and Wegman (1975). Following the notation of Barlow, Bartholomew, Bremner and Brunk (1972), suppose we have independent random samples from each of $k$ normal populations indexed by $x_i$: $i = 1, 2, \ldots, k$. Suppose $\mu(x_i)$ is the mean of the population indexed by $x_i$ and $<<$ is a partial order on $S = \{x_1, x_2, \ldots, x_k\}$. A function $r(\cdot)$ on $S$ is isotone provided $r(x_i) \le r(x_j)$ whenever $x_i << x_j$. Robertson and Wegman consider the likelihood ratio for testing the null hypothesis

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$$H_1: \mu(\cdot) \text{ is isotone}$$

versus $H_2 - H_1$ where $H_2$ places no restriction on $\mu(\cdot)$. In this Monte-Carlo study, we restrict our attention to a null hypothesis which specifies a linear order, i.e. $H_1^*: \mu(x_1) \geq \mu(x_2) \geq \ldots \geq \mu(x_k)$. We also take the variances $\sigma^2(x_i)$ to be one and draw the same number of items from each population. Computation of the likelihood ratio test statistic is discussed and a table of critical values for this null hypothesis is given in Robertson and Wegman.

Van Eeden (1958) proposed another statistic for testing $H_1^*$ against $H_2 - H_1^*$, namely $T_{12}^* = \max_{1 \leq i \leq k-1} \left(\overline{X}(x_{i+1}) - \overline{X}(x_i)\right)$ where $\overline{X}(x_i)$ is the sample mean of the $i^{th}$ population. Let $\alpha$ be a "target" significance level and let $\alpha^* = \alpha/(k-1)$. The critical point for Van Eeden's test when $\sigma^2(x_i) = 1$ is

$$t_{\alpha^*} = \sqrt{2/n} \cdot \xi_{\alpha^*}$$

where $n$ is the number of observations made on each population and where $\xi_{\alpha^*}$ is defined by

$$(1/\sqrt{2\pi}) \int_{\xi_{\alpha^*}}^{\infty} e^{-\frac{1}{2}x^2} \, dx = \alpha^* .$$

The true significance level $\alpha_0 = \sup_{\mu(\cdot) \in H_1^*} P[T_{12}^* \geq t_{\alpha^*} | \mu(\cdot)]$ is bounded above by $\alpha$ and below by $\alpha - \frac{1}{2}\alpha^2$. For this study we chose $\alpha = .05$ and $.01$ so that $.04875 \leq \alpha_0 \leq .05$ for $\alpha = .05$ and $.00995 \leq \alpha_0 \leq .01$ for $\alpha = .01$.

Normal pseudo-random variates were generated according to the well-known Box-Mueller transform and sample means, $\overline{X}(x_i)$, based on $n = 100$ were calculated. For three different types of alternate hypotheses estimates of the power and the standard error of the estimate of the power were calculated based on 1000 replications of the Monte-Carlo experiment. In the first study the means were taken to be linear according to the rule $\mu(x_i) = \beta \cdot i$ ; $i = 1, 2, \ldots, k$ , $k = 3, 6, 9, 12$ ; $\beta = 1, 1/2, 1/3, \ldots, 1/10, 1/20, \ldots, 1/80$ and finally for $\alpha = .01$ and $.05$ . Results of this study are given in Tables 1 and 2 and Figures 1 and 2.

As we might reasonably expect, the likelihood ratio statistic beats $T_{12}^*$ , often impressively so, as illustrated by Figures 1 and 2. For example for $k = 12$, $\beta = 1/10$ and $\alpha = .05$ , $T_{12}^*$'s power is approximately .27 while the power of the likelihood ratio statistic is still 1 . For alternatives of this type, the powers of both tests increase as $k$ increases and, of course as $\beta$ increases. The case $\beta = 0$ corresponds to the null hypothesis $H_1^{**}$: $\mu(x_i) = 0$ ; $i = 1, 2, \ldots, k$ and, hence, here the power is an estimate of the significance level. These estimates of the significance levels for $T_{12}^*$ generally underestimate the "target" significance level as Van Eeden's theory predicts but in most cases this estimate is within two standard deviations of the target level.

Since $T_{12}^*$ is based on differences between adjacent sample means one might reasonably expect it to be more sensitive to alternatives where one or more of the differences between adjacent population means is large. Table 3 gives estimated power for slippage alternatives of the type $\mu(x_2) = \mu(x_3) = \ldots = \mu(x_k) = 0$ and $\mu(x_1) = -1/90, -2/80, -3/70, -4/60, -5/50, -6/40, -7/80, -8/20$ and $-9/10$ . Table 4 gives further data for slippage alternatives:

$\mu(x_i) = -.35$ while $\mu(x_j) = 0$ for $j \neq i$ ; $i = 1, 2, \ldots, 12$ . Also given in Table 4 is a step type alternative for which $\mu(x_i) = -.35$ for $j \leq i$ and $\mu(x_i) = 0$ for $j > i$ ; $i = 1, 2, \ldots, 12$ .

In Table 3 the likelihood ratio statistic is more powerful than $T_{12}^*$ except for $\mu(x_i) = -(1/90)$ or $-(2/80)$ . In these two instances the slippage is so small that the power is essentially equal to the size of the test. As expected, the differences in power for $T_{12}^*$ and the likelihood ratio statistic in Table 3 are not nearly so dramatic as those in Tables 1 and 2. Heuristically one might predict this since the likelihood ratio statistic is based on all the means simultaneously and hence should be more sensitive to the sorts of alternatives in Tables 1 and 2 compared to those in Table 3.

We may, in Table 4, compare powers as the location, $i$ , of the slipped mean ranges from 1 through 12 . The power of the likelihood ratio monotonically decreases with this location shift whereas the power of $T_{12}^*$ stays relatively constant. For $i = 10, 11$ $T_{12}^*$ beats the likelihood ratio and significantly for $i = 11$ . Notice that as the location of the slippage increases the alternative comes closer to satisfying the null and in fact for $i = 12$ , $H_1^*$ is satisfied so that the powers approximate the size of the test.

Finally for the step alternatives, the likelihood ratio statistic has maximum power near $k/2$ and its power decreases in both directions while $T_{12}^*$ has essentially constant power. Again notice that the case $i = 12$ satisfies $H_1^*$ so we have another estimate of the size of the test.

| β \ k | Power for Likelihood Ratio Test Statistic, $T_{12}$: 3 | 6 | 9 | 12 | Power for $T^*_{12}$: 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) |
| $\frac{1}{2}$ | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) |
| $\frac{1}{3}$ | .997 (.002) | 1 (0) | 1 (0) | 1 (0) | .910 (.009) | .999 (.001) | 1 (0) | 1 (0) |
| $\frac{1}{4}$ | .951 (.007) | 1 (0) | 1 (0) | 1 (0) | .670 (.015) | .863 (.011) | .921 (.008) | .940 (.008) |
| $\frac{1}{5}$ | .810 (.012) | 1 (0) | 1 (0) | 1 (0) | .466 (.016) | .648 (.015) | .723 (.014) | .774 (.013) |
| $\frac{1}{6}$ | .666 (.015) | 1 (0) | 1 (0) | 1 (0) | .337 (.015) | .485 (.016) | .568 (.016) | .593 (.016) |
| $\frac{1}{7}$ | .545 (.016) | 1 (0) | 1 (0) | 1 (0) | .258 (.014) | .363 (.015) | .436 (.016) | .453 (.016) |
| $\frac{1}{8}$ | .461 (.016) | .997 (.002) | 1 (0) | 1 (0) | .226 (.013) | .315 (.015) | .346 (.015) | .368 (.015) |
| $\frac{1}{9}$ | .397 (.016) | .989 (.003) | 1 (0) | 1 (0) | .187 (.012) | .264 (.014) | .282 (.014) | .298 (.014) |
| $\frac{1}{10}$ | .342 (.015) | .968 (.006) | 1 (0) | 1 (0) | .170 (.012) | .229 (.013) | .244 (.014) | .269 (.014) |
| $\frac{1}{20}$ | .151 (.011) | .461 (.016) | .887 (.010) | .997 (.002) | .076 (.008) | .110 (.010) | .114 (.010) | .121 (.010) |
| $\frac{1}{30}$ | .096 (.009) | .233 (.013) | .552 (.016) | .856 (.011) | .054 (.007) | .075 (.008) | .089 (.009) | .086 (.009) |
| $\frac{1}{40}$ | .087 (.009) | .181 (.012) | .362 (.015) | .614 (.015) | .050 (.007) | .067 (.008) | .079 (.008) | .079 (.008) |
| $\frac{1}{50}$ | .081 (.009) | .151 (.011) | .240 (.014) | .416 (.016) | .055 (.007) | .059 (.008) | .074 (.008) | .073 (.008) |
| $\frac{1}{60}$ | .080 (.009) | .127 (.010) | .183 (.012) | .305 (.015) | .049 (.007) | .065 (.008) | .070 (.008) | .067 (.008) |
| $\frac{1}{70}$ | .052 (.007) | .121 (.010) | .164 (.012) | .241 (.014) | .034 (.006) | .049 (.007) | .063 (.008) | .065 (.008) |
| $\frac{1}{80}$ | .068 (.008) | .091 (.009) | .143 (.011) | .209 (.013) | .038 (.006) | .046 (.007) | .048 (.007) | .060 (.008) |
| 0 | .060 (.008) | .051 (.007) | .054 (.007) | .056 (.007) | .036 (.006) | .043 (.006) | .038 (.006) | .041 (.006) |

TABLE 1

Monte Carlo estimates of power and standard errors (in parentheses) for the likelihood ratio test and for the test statistic $T^*_{12}$. Significance level, $\alpha = .05$ and alternatives $\mu(x_i) = \beta \cdot i$.

| $\beta$ \ k | Power for Likelihood Ratio Test Statistic, $T_{12}$: 3 | 6 | 9 | 12 | Power for $T^*_{12}$: 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) |
| $\frac{1}{2}$ | 1 (0) | 1 (0) | 1 (0) | 1 (0) | .992 (.003) | 1 (0) | 1 (0) | 1 (0) |
| $\frac{1}{3}$ | .983 (.004) | 1 (0) | 1 (0) | 1 (0) | .642 (.015) | .901 (.009) | .955 (.007) | .973 (.005) |
| $\frac{1}{4}$ | .829 (.012) | 1 (0) | 1 (0) | 1 (0) | .351 (.015) | .512 (.016) | .590 (.016) | .653 (.015) |
| $\frac{1}{5}$ | .601 (.016) | 1 (0) | 1 (0) | 1 (0) | .197 (.013) | .320 (.015) | .374 (.015) | .401 (.016) |
| $\frac{1}{6}$ | .412 (.016) | 1 (0) | 1 (0) | 1 (0) | .124 (.010) | .200 (.013) | .239 (.014) | .250 (.014) |
| $\frac{1}{7}$ | .300 (.014) | .998 (.001) | 1 (0) | 1 (0) | .088 (.009) | .124 (.010) | .148 (.011) | .161 (.012) |
| $\frac{1}{8}$ | .228 (.013) | .979 (.004) | 1 (0) | 1 (0) | .065 (.008) | .095 (.009) | .107 (.010) | .119 (.010) |
| $\frac{1}{9}$ | .178 (.012) | .949 (.007) | 1 (0) | 1 (0) | .052 (.007) | .091 (.009) | .095 (.009) | .103 (.010) |
| $\frac{1}{10}$ | .145 (.011) | .882 (.010) | 1 (0) | 1 (0) | .050 (.007) | .073 (.008) | .072 (.008) | .091 (.009) |
| $\frac{1}{20}$ | .050 (.007) | .261 (.014) | .709 (.014) | .991 (.003) | .018 (.004) | .030 (.005) | .026 (.005) | .023 (.005) |
| $\frac{1}{30}$ | .027 (.005) | .075 (.008) | .301 (.014) | .694 (.015) | .009 (.003) | .016 (.004) | .018 (.004) | .020 (.004) |
| $\frac{1}{40}$ | .021 (.004) | .055 (.007) | .139 (.007) | .366 (.015) | .014 (.004) | .015 (.003) | .012 (.003) | .012 (.003) |
| $\frac{1}{50}$ | .024 (.005) | .036 (.006) | .075 (.008) | .189 (.012) | .015 (.004) | .017 (.004) | .017 (.004) | .020 (.004) |
| $\frac{1}{60}$ | .023 (.005) | .042 (.006) | .063 (.008) | .127 (.010) | .007 (.003) | .018 (.004) | .018 (.004) | .015 (.004) |
| $\frac{1}{70}$ | .013 (.004) | .026 (.005) | .055 (.007) | .083 (.009) | .009 (.003) | .017 (.004) | .017 (.004) | .015 (.004) |
| $\frac{1}{80}$ | .010 (.003) | .023 (.005) | .037 (.006) | .075 (.008) | .004 (.002) | .007 (.003) | .008 (.003) | .012 (.003) |
| 0 | .008 (.003) | .013 (.004) | .009 (.003) | .009 (.003) | .005 (.002) | .004 (.002) | .005 (.002) | .005 (.002) |

TABLE 2

Monte Carlo estimates of power and standard errors (in parentheses) for the likelihood ratio test and the test statistic, $T^*_{12}$ . Significance level, $\alpha = .01$ and alternatives $\mu(x_i) = \beta \cdot i$ .

| | $\mu(x_1)$ \ k | Power for the Likelihood Ratio Test Statistic, $T_{12}$ | | | | Power for $T_{12}^*$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 3 | 6 | 9 | 12 | 3 | 6 | 9 | 12 |
| α=.05 | $-\frac{1}{90}$ | .055 (.007) | .054 (.007) | .050 (.007) | .056 (.007) | .037 (.006) | .055 (.007) | .052 (.007) | .048 (.007) |
| | $-\frac{2}{80}$ | .065 (.008) | .055 (.007) | .049 (.007) | .041 (.006) | .036 (.006) | .039 (.006) | .042 (.006) | .043 (.006) |
| | $-\frac{3}{70}$ | .086 (.009) | .075 (.008) | .075 (.008) | .059 (.008) | .060 (.008) | .053 (.007) | .056 (.007) | .056 (.007) |
| | $-\frac{4}{60}$ | .115 (.010) | .085 (.009) | .081 (.009) | .072 (.008) | .064 (.008) | .053 (.007) | .055 (.007) | .048 (.007) |
| | $-\frac{5}{50}$ | .188 (.012) | .148 (.011) | .105 (.010) | .091 (.009) | .104 (.010) | .074 (.008) | .064 (.008) | .063 (.008) |
| | $-\frac{6}{40}$ | .277 (.014) | .229 (.013) | .189 (.012) | .168 (.012) | .171 (.012) | .140 (.011) | .116 (.010) | .099 (.009) |
| | $-\frac{7}{30}$ | .479 (.016) | .447 (.016) | .383 (.015) | .336 (.015) | .320 (.015) | .258 (.014) | .226 (.013) | .200 (.013) |
| | $-\frac{8}{20}$ | .916 (.009) | .909 (.009) | .873 (.010) | .839 (.012) | .761 (.014) | .688 (.015) | .638 (.015) | .597 (.016) |
| | $-\frac{9}{10}$ | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) |
| α=.01 | $-\frac{1}{90}$ | .014 (.004) | .011 (.003) | .011 (.003) | .003 (.002) | .008 (.003) | .008 (.003) | .005 (.002) | .004 (.002) |
| | $-\frac{2}{80}$ | .010 (.003) | .005 (.002) | .003 (.002) | .009 (.003) | .007 (.003) | .007 (.003) | .007 (.003) | .009 (.003) |
| | $-\frac{3}{70}$ | .023 (.005) | .019 (.004) | .012 (.003) | .018 (.004) | .011 (.003) | .009 (.003) | .011 (.003) | .014 (.004) |
| | $-\frac{4}{60}$ | .026 (.005) | .022 (.005) | .019 (.004) | .017 (.004) | .014 (.004) | .016 (.004) | .020 (.004) | .018 (.004) |
| | $-\frac{5}{50}$ | .058 (.007) | .040 (.007) | .027 (.005) | .021 (.004) | .027 (.005) | .023 (.005) | .014 (.004) | 0.13 (.004) |
| | $-\frac{6}{40}$ | .101 (.010) | .087 (.009) | .069 (.008) | .051 (.007) | .057 (.007) | .040 (.006) | .031 (.006) | .027 (.005) |
| | $-\frac{7}{30}$ | .260 (.014) | .215 (.013) | .153 (.011) | .133 (.011) | .147 (.011) | .102 (.010) | .084 (.009) | .071 (.008) |
| | $-\frac{8}{20}$ | .757 (.014) | .742 (.014) | .694 (.015) | .633 (.015) | .550 (.016) | .483 (.016) | .440 (.016) | .409 (.016) |
| | $-\frac{9}{10}$ | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) | 1 (0) |

TABLE 3

Monte Carlo estimates of power and standard errors (in parentheses) for the likelihood ratio test and the test statistic, $T_{12}^*$ , with slippage alternatives $\mu(x_i) = 0$ , $i = 2, 3, \ldots, k$ and $\mu(x_1)$ as indicated.

| | Slippage Alternative | | | | Step Alternative | | | |
|---|---|---|---|---|---|---|---|---|
| | Likelihood Ratio Test | | $T^*_{12}$ | | Likelihood Ratio Test | | $T^*_{12}$ | |
| $i \backslash \alpha$ | .05 | .01 | .05 | .01 | .05 | .01 | .05 | .01 |
| 1 | .664 (.015) | .434 (.016) | .470 (.016) | .272 (.014) | .664 (.015) | .434 (.016) | .470 (.016) | .272 (.014) |
| 2 | .640 (.015) | .411 (.016) | .449 (.016) | .250 (.014) | .937 (.008) | .813 (.012) | .455 (.016) | .250 (.014) |
| 3 | .629 (.015) | .368 (.015) | .476 (.016) | .271 (.014) | .987 (.004) | .944 (.007) | .478 (.016) | .271 (.014) |
| 4 | .625 (.015) | .369 (.015) | .486 (.016) | .285 (.014) | .995 (.002) | .981 (.004) | .489 (.016) | .285 (.014) |
| 5 | .600 (.016) | .353 (.015) | .460 (.016) | .270 (.014) | .998 (.001) | .985 (.004) | .461 (.016) | .270 (.014) |
| 6 | .558 (.016) | .322 (.015) | .433 (.016) | .267 (.014) | 1 (0) | .988 (.003) | .435 (.016) | .269 (.014) |
| 7 | .520 (.016) | .297 (.014) | .459 (.016) | .262 (.014) | .995 (.002) | .977 (.005) | .461 (.016) | .262 (.014) |
| 8 | .529 (.016) | .275 (.014) | .449 (.016) | .249 (.014) | .996 (.002) | .988 (.003) | .450 (.016) | .249 (.014) |
| 9 | .483 (.016) | .255 (.014) | .468 (.016) | .243 (.014) | .990 (.003) | .950 (.007) | .470 (.016) | .243 (.014) |
| 10 | .411 (.016) | .205 (.013) | .447 (.016) | .253 (.014) | .927 (.008) | .813 (.012) | .449 (.016) | .253 (.014) |
| 11 | .293 (.014) | .124 (.010) | .438 (.016) | .268 (.014) | .667 (.015) | .430 (.016) | .442 (.016) | .268 (.014) |
| 12 | .030 (.005) | .008 (.003) | .051 (.007) | .012 (.003) | .056 (.007) | .009 (.003) | .058 (.007) | .014 (.004) |

TABLE 4

Monte Carlo estimates of power and standard error (in parentheses) for the likelihood ratio test and the test statistic, $T^*_{12}$ with slippage or step located at $i$ . Size or slippage of jump is $-.35$ and $k$ is 12 .

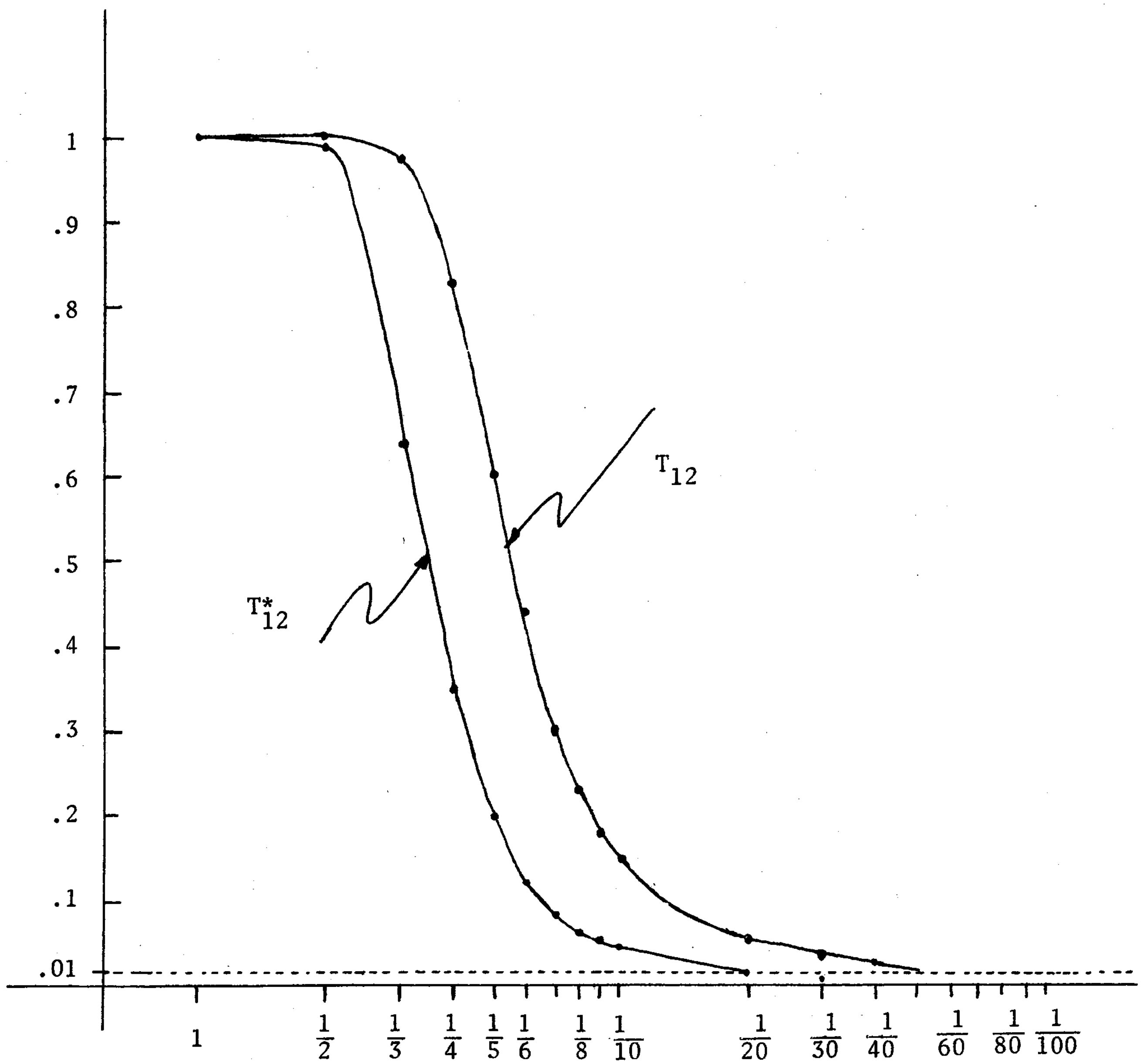


FIGURE 1

Power as a function of $\beta$ $\left(\mu(x_i) = \beta \cdot i\right)$ for the Likelihood Ratio Test Statistic, $T_{12}$ and for $T^*_{12}$ . $\alpha = .01$ and $k = 3$ .

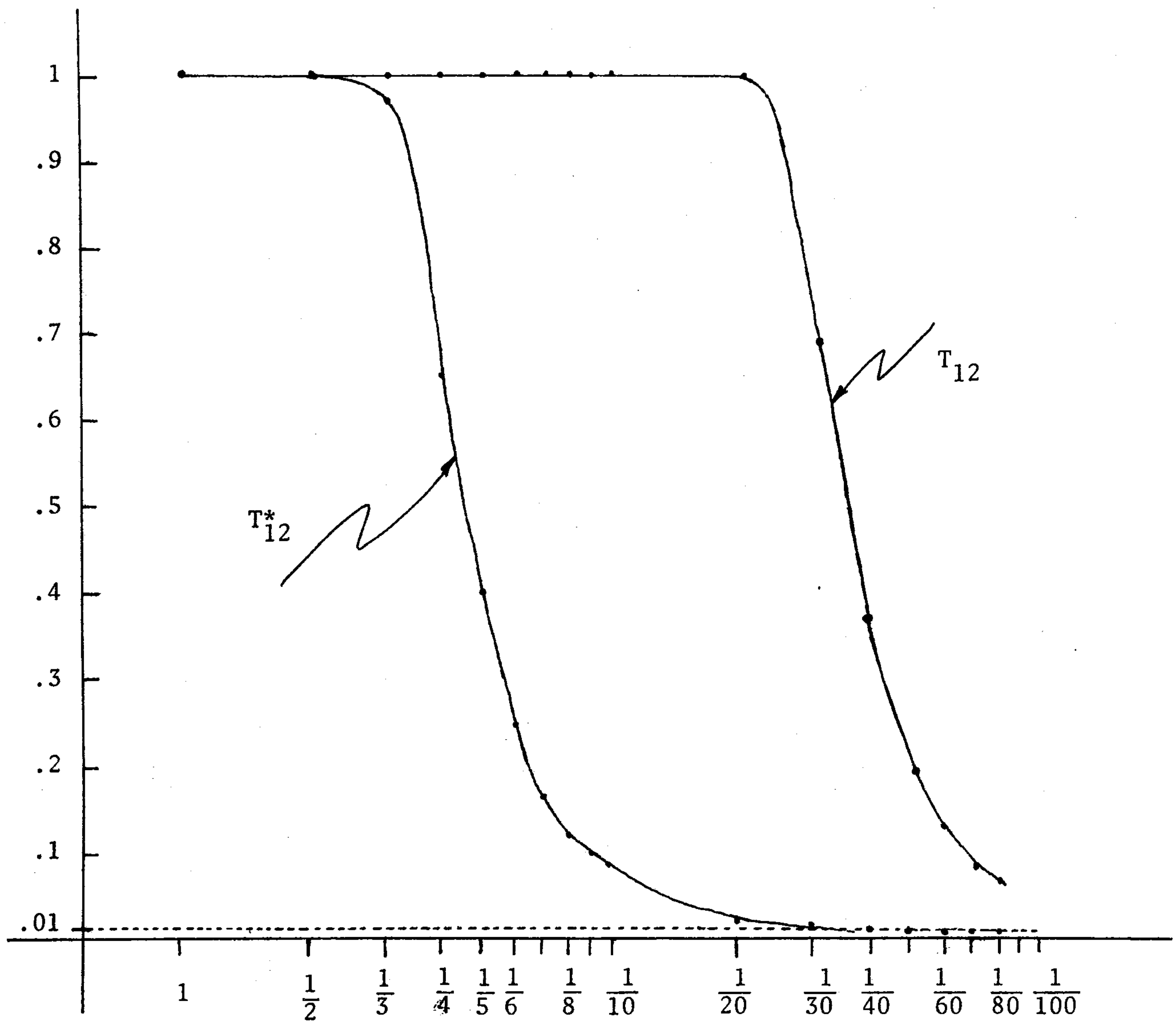


FIGURE 2

Power as a function of $\beta$ $\left(\mu(x_i) = \beta \cdot i\right)$ for the Likelihood Ratio Test Statistic, $T_{12}$ and for $T^*_{12}$ . $\alpha = .01$ and $k = 12$ .

3. Tests of Trend for an Exponential Class of Distributions: Now let us turn our attention to extensions of the likelihood ratio test to distributions of the exponential type. Suppose $\gamma(\circ)$ is a $\sigma$-finite measure on the Borel subsets of the real line and consider a regular exponential family of distributions defined by the probability densities of the form

$$(3.1)\quad f(x;\ \theta,\tau) = \exp[p_1(\theta)p_2(\tau)K(x) + S(x,\tau) + q(\theta;\tau)]\ ; \qquad \theta \in (\theta_1,\theta_2) \text{ and } \tau \in T$$

with respect to $\gamma$ and with $-\infty \le \theta_1 < \theta_2 \le \infty$ . We make the following assumptions:

$$(3.2)\quad p_1(\circ) \text{ and } q(\circ;\tau) \text{ both have continuous second derivatives on } (\theta_1,\theta_2) \text{ for all } \tau \in T ,$$

$$(3.3)\quad p_1'(\theta) > 0 \text{ for all } \theta \in (\theta_1,\theta_2) ,\quad p_2(\tau) > 0 \text{ for all } \tau \in T$$

and

$$(3.4)\quad q'(\theta;\tau) = -\theta p_1'(\theta)p_2(\tau) \text{ for all } \theta \in (\theta_1,\theta_2) \text{ and } \tau \in T .$$

We are thinking of $\tau$ as fixed so that all derivatives are with respect to $\theta$ . If $X$ is any random variable having density function $f(x;\ \theta,\tau)$ then using Theorem 9 on page 52 of Lehmann (1959), the integral, $\int f(x;\ \theta,\tau)d\gamma(x) = 1$ , can be twice differentiated, with respect to $\theta$ , under the integral sign, obtaining $E[K(X)] = \theta$ and $V[K(X)] = [p_1'(\theta)p_2(\tau)]^{-1}$ .

Suppose we have independent random samples from each of $k$ populations belonging to the above exponential family where the $i^{\underline{th}}$ population has parameters $\theta(x_i)$ and $\tau_i$ ( $\tau_i$ is known). Let the items of the random sample from the $i^{\underline{th}}$ population be denoted by $X_{ij}$: $j = 1, 2, \ldots, n_i$ and suppose $<<$ is a partial order on $S = \{x_1, x_2, \ldots, x_k\}$ . Consider the following hypothesès:

$$H_0 \colon \theta(x_1) = \theta(x_2) = \ldots = \theta(x_k) \ ,$$

$$H_1 \colon \theta(\circ) \text{ is isotone with respect to } <<$$

and $H_2$ places the restriction on $\theta(\circ)$ . We consider a likelihood ratio statistic for testing $H_1$ against $H_2 - H_1$ . The maximum likelihood estimate of $\theta(\circ)$ under $H_2$ is given by $\hat{\theta}(\circ)$ where $\hat{\theta}(x_i) = n_i^{-1} \sum_{j=1}^{n_i} K(X_{ij})$ . Furthermore, the maximum likelihood estimate of the common value of $\theta(\circ)$ under $H_0$ is given by

$$\hat{\theta}_0 = [\textstyle\sum_{i=1}^{k} n_i p_2(\tau_i)]^{-1} \circ \sum_{i=1}^{k} n_i p_2(\tau_i)\hat{\theta}_i$$

so that from Robertson and Wright (1975) it follows that the maximum likelihood estimate of $\theta(\circ)$ under $H_1$ is $\overline{\theta}(\circ) = E[\hat{\theta}(\circ)|L]$ where $L$ is the $\sigma$-lattice of subsets of $S$ induced by $<<$ (cf. Barlow et. al. (1972)). The expectation is taken with respect to the space $(S, 2^S, \delta)$ where $\delta$ is the probability measure on the collection, $2^S$ , of all subsets of $S$ which assigns mass $n_i \circ p_2(\tau_i) \div [\sum_{j=1}^{k} n_j p_2(\tau_j)]$ to the singleton $\{x_i\}$ . Maximum likelihood estimation of parameters of distributions belonging to an exponential family were first discussed by Brunk (1955). For a discussion of this and related work see Barlow et. al. (1972).

If $\lambda_{12}$ is the likelihood ratio for testing $H_1$ against $H_2 - H_1$ and $T_{12} = -2\ln\lambda_{12}$ then

$$T_{12} = 2\sum_{i=1}^{k} \{n_i\hat{\theta}(x_i)p_2(\tau_i)[p_1(\hat{\theta}(x_i)) - p_1(\overline{\theta}(x_i))] + n_i[q(\hat{\theta}(x_i);\tau_i) - q(\overline{\theta}(x_i);\tau_i)]\}$$

Expanding $p_1(\circ)$ and $q(\circ;\tau_i)$ about $\hat{\theta}(x_i)$ by using Taylor's Theorem with second degree remainder term and substituting for $p_1(\overline{\theta}(x_i))$ and $q(\overline{\theta}(x_i);\tau_i)$ we obtain

$$T_{12} = 2\sum_{i=1}^{k} \{[n_i\hat{\theta}(x_i)p_2(\tau_i)p_1'(\hat{\theta}(x_i)) + n_i q'(\hat{\theta}(x_i);\tau_i)] \circ (\overline{\theta}(x_i) - \hat{\theta}(x_i)) - [n_i\hat{\theta}(x_i)p_2(\tau_i)p_1''(\alpha_i) \circ 2^{-1} + n_i q''(\beta_i;\tau_i) \circ 2^{-1}] \circ [\overline{\theta}(x_i) - \hat{\theta}(x_i)]^2$$

where $\alpha_i$ and $\beta_i$ converge almost surely to $\theta(x_i)$. This convergence follows from well-known properties of $\overline{\theta}(x_i)$ and $\hat{\theta}(x_i)$. Now from (3.4), $q(\hat{\theta}(x_i);\tau_i) = -\hat{\theta}(x_i)p_1'(\hat{\theta}(x_i))p_2(\tau_i)$ so that

$$(3.5)\quad T_{12} = -\sum_{i=1}^{k} n_i[\hat{\theta}(x_i)p_2(\tau_i)p_1''(\alpha_i) + q''(\beta_i;\tau_i)] \circ [\overline{\theta}(x_i) - \hat{\theta}(x_i)]^2 .$$

Theorem 3.1. If $f(x;\theta,\tau)$ is of the form (3.1) where $p_1(\circ)$ and $q(\circ;\tau)$ satisfy (3.2) - (3.4) and if $\theta(x_1) = \theta(x_2) = \ldots = \theta(x_k)$ and $n_1 = n_2 = \ldots = n_k = n$ then as $n \to \infty$,

$$T_{12} \xrightarrow{L} \sum_{i=1}^{k} p_2(\tau_i)[E(X(\circ)|L)(x_i) - X(x_i)]^2$$

where $X(x_1), X(x_2), \ldots, X(x_k)$ are independent normal random variables having zero means and $V(X(x_i)) = p_2(\tau_i)^{-1}$. The expectation $E(X(\cdot)|L)$ is taken regarding $X(\cdot)$ as a function on the space $(S, 2^S, \delta)$. (Note that $\delta(\{x_i\}) = p_2(\tau_i) \div \sum_{j=1}^k p_2(\tau_j)$.)

<u>Proof</u>: Let the common value of $\theta(\cdot)$ be $\theta_0$. Then from (3.5), using well-known properties of the conditional expectation operator

$$T_{12} = -\sum_{i=1}^k [\hat{\theta}(x_i)p_1''(\alpha_i)p_2(\tau_i) + q''(\beta_i; \tau_i)] \cdot \Big[ E\Big(\sqrt{n}\,(\hat{\theta}(\cdot) - \theta_0)\Big|L\Big)(x_i) - \sqrt{n}\,\big(\hat{\theta}(x_i) - \theta_0\big) \Big]^2 .$$

Now $\hat{\theta}(x_i)$ is the sample mean of i.i.d. random variables having means $\theta_0$ and variances $[p_1'(\theta_0) \cdot p_2(\tau_i)]^{-1} < \infty$. Let $Z_n$ be the $2k$ dimensional random vector defined by

$$Z_{ni} = \hat{\theta}(x_i)p_1''(\alpha_i)p_2(\tau_i) + q''(\beta_i; \tau_i) \; ; \quad i = 1, 2, \ldots, k$$

$$= \sqrt{n}\,[\hat{\theta}(x_{i-k}) - \theta_0] \; ; \quad i = k+1, k+2, \ldots, 2k .$$

Using the Law of Large Numbers, the Central Limit Theorem and Theorem 4.4 of Billingsley (1968), $\vec{Z}_n$ converges weakly to $\vec{Z}$ where

$$Z_i = \theta_0 p_1''(\theta_0)p_2(\tau_i) + q''(\theta_0; \tau_i) \; ; \quad i = 1, 2, \ldots, k$$

$$= Y(x_{i-k}) \; ; \quad i = k+1, k+2, \ldots, 2k$$

where $Y(x_1), Y(x_2), \ldots, Y(x_k)$ are independent normal random variables having zero means and $V(Y(x_i)) = [p_1'(\theta_0)p_2(\tau_i)]^{-1}$. The conditional expectation operator is continuous so that $T_{12}$ is a continuous function of $\vec{Z}_n$. It follows from Corollary 1 of Theorem 5.1 of Billingsley (1968) that

$$T_{12} \xrightarrow{L} -\sum_{i=1}^{k} [\theta_0 p_1''(\theta_0)p_2(\tau_i) + q''(\theta_0;\tau_i)][E(Y(\cdot)|L)(x_i) - Y(x_i)]^2 .$$

The desired result now follows since $q''(\theta_0;\tau_i) = -\theta_0 p_1''(\theta_0)p_2(\tau_i) - p_1'(\theta_0)p_2(\tau_i)$ from (3.4).

Theorem 5 of Robertson and Wegman (1975) now yields

Corollary 3.2. If the hypotheses of Theorem 3.1 are satisfied then for each real number $t$

$$\lim_{n\to\infty} P[T_{12} \geq t] = \sum_{\ell=1}^{k} P[\chi_{k-\ell}^2 \geq t] \circ P(\ell,k)$$

where $\chi_{k-\ell}^2$ is a $\chi^2$ random variable having $k - \ell$ degrees freedom and as in Barlow et. al. (1972). $P(\ell,k)$ is the probability that $E(X(\cdot)|L)$ takes on $\ell$ levels. The probabilities $P(\ell,k)$ depend on the partial order $<<$ and on the weights $p_2(\tau_i)$.

We now show that Corollary 3.2 provides the large sample approximation to the critical level for testing $H_1$ against $H_2 - H_1$. As with the proof of Theorem 2 of Robertson and Wegman (1975) this property is a consequence of the fact that our isotonic estimators can be viewed as projections on closed convex cones in the Hilbert space of all functions on $S = \{x_1, x_2, \ldots, x_k\}$ with inner product defined by $(\gamma(\cdot), \eta(\cdot)) = \sum_{i=1}^{k} \gamma(x_i) \circ \eta(x_i)w_i$ and $w_i = n_i p_2(\tau_i) \div \sum_{j=1}^{k} n_j p_2(\tau_j)$. Suppose $\theta(\cdot)$ satisfies $H_1$ but not $H_0$, let $v_1, v_2, \ldots, v_h$ be the distinct values among $\theta(x_1), \theta(x_2), \ldots, \theta(x_k)$

and let $S_i = \{x_j;\ \theta(x_j) = v_i\}$ ; $i = 1, 2, \ldots ,H$ . Define the partial order $\ll\!\!\le$ on $S$ by $x_\alpha \ll\!\!\le x_\beta$ if and only if $x_\alpha \ll x_\beta$ and $x_\alpha, x_\beta \in S_i$ for some $i$ . Let $L(\theta)$ be the $\sigma$-lattice of subsets of $S$ induced by $\ll\!\!\le$ and let $I(\theta)(I)$ be the collection of all functions on $S$ which are isotone with respect to $\ll\!\!\le(\ll)$ . The collection $I(\theta)(I)$ is a closed convex cone in the Hilbert space of all functions on $S$ and $E(\eta(\circ)|L(\theta))$ $\left[E(\eta(\circ)|L)\right]$ is the projection on $I(\theta)(I)$ in this space (cf. Brunk (1965)). Furthermore

$$I \subset I(\theta) \tag{3.6}$$

and using Corollary 2.3 of Brunk (1965), if $E(\eta(\circ)|L(\theta)) \in I$ then $E(\eta(\circ)|L(\theta)) = E(\eta(\cdot)|L)$ .

<u>Lemma 3.3</u>. If $\min_{x_i \in S_1} \eta(x_i) \ge \max_{x_i \in S_2} \eta(x_i) \ge \min_{x_i \in S_2} \eta(x_i) \ge \max_{x_i \in S_3} \eta(x_i) \ge \ldots \ge \max_{x_i \in S_h} \eta(x_i)$ then $E(\eta(\circ)|L(\theta)) = E(\eta(\circ)|L)$ .

<u>Proof</u>: It suffices to show that $E(\eta(\circ)|L(\theta)) \in I$ . Suppose $\phi(\circ) = E(\eta(\circ)|L(\theta))$ and $x_\alpha \ll x_\beta$ . If $x_\alpha, x_\beta \in S_i$ for some $i$ then $x_\alpha \ll\!\!\le x_\beta$ and $\phi(x_\alpha) \le \phi(x_\beta)$ . Suppose $x_\alpha \in S_i$ , $x_\beta \in S_j$ and $i \ne j$ . Since $\theta(\circ)$ is isotone with respect to $\ll$ the sets $S_1 + S_2 + \ldots + S_i$ and $S_1 + S_2 + \ldots + S_j$ are in $L$ so $x_\beta \in S_1 + S_2 + \ldots + S_i$ and therefore $i > j$ . Now $\phi(x_\alpha)\left(\phi(x_\beta)\right)$ is an average of the values of $\eta(\circ)$ at points in $S_i(S_j)$ so

$$\phi(x_\alpha) \le \max_{x_\ell \in S_i} \eta(x_\ell) \le \min_{x_\ell \in S_j} \eta(x_\ell) \le \phi(x_\beta)$$

and $\phi(\circ) \in I$ .

For any $\theta(\circ)$ let $P_\theta(E)$ be the probability of the event $E$ computed under the assumption that $\theta(\circ)$ is the true vector of parameter values and let $P_0(E)$ be the probability of $E$ computed under $H_0$.

<u>Theorem 3.4</u>. If $\theta(\circ)$ satisfies $H_1$ and $n_1 = n_2 = \ldots = n_k = n$ then

$$\lim_{n\to\infty} P_\theta[T_{12} \geq t] \leq \lim_{n\to\infty} P_0[T_{12} \geq t] .$$

<u>Proof</u>: Define $\lesssim$, $L(\theta)$ and the sets $S_1, S_2, \ldots, S_h$ as before. Now $\hat{\theta}(x_i) \xrightarrow{a.s.} \theta(x_i)$ so for sufficiently large $n$ with probability one $\min_{x_\ell \in S_1} \hat{\theta}(x_\ell) \geq \max_{x_\ell \in S_2} \hat{\theta}(x_\ell) \geq \ldots \geq \max_{x_\ell \in S_h} \hat{\theta}(x_\ell)$ and from (3.5) and Lemma 3.3

$$T_{12} = -\sum_{i=1}^k n[\hat{\theta}(x_i)p''(\alpha_i)p_2(\tau_i) + q''(\beta_i; \tau_i)] \circ [E(\hat{\theta}(\circ)|L(\theta))(x_i) - \hat{\theta}(x_i)]^2 ,$$

for sufficiently large $n$ with probability one. Using an argument similar to the one used for Theorem 3.1 we obtain

$$T_{12} \xrightarrow{L} \sum_{i=1}^k p_2(\tau_i)[E(X(\circ)|L(\theta))(x_i) - X(x_i)]^2$$

where $X(\circ)$ is defined as in Theorem 3.1. Here it is necessary to use the fact that $p_1'(\theta(\circ))$ is positive and constant on the sets $S_1, S_2, \ldots, S_h$. The desired result follows from (3.6) since

$$\begin{aligned}\sum_{i=1}^k p_2(\tau_i)[E(X(\circ)|L(\theta))(x_i) - X(x_i)]^2 &= ||E(X(\circ)|L(\theta)) - X(\circ)||^2 \\ &\leq ||E(X(\circ)|L) - X(\circ)||^2 \\ &= \sum_{i=1}^k p_2(\tau_i)[E(X(\circ)|L)(x_i) - X(x_i)]^2 .\end{aligned}$$

It seems clear that the hypothesis $n_1 = n_2 = \ldots = n_k$ could be relaxed. It was required to take $n$ inside the conditional expectation. However, the measure on $2^S$, on which the expectations depend, also depends on $n_i$ so that such a relaxation would still require some assumption about the way the $n_i$'s go to infinity.

Likelihood ratio tests for testing $H_0$ against $H_1$ are discussed in Barlow et. al. (1972). However we have been unable to find an analogue to Theorem 3.1 for this test. An argument similar to the argument given for Theorem 3.1 yields

<u>Theorem 3.5.</u> Let $T_{01} = -2\ln\lambda_{01}$ where $\lambda_{01}$ is the likelihood ratio for testing $H_0$ against $H_1$ and assume the hypotheses of Theorem 3.1 are satisfied. Then

$$T_{01} \xrightarrow{L} \sum_{i=1}^{k} p_2(\tau_i)\left[E\left(X(\cdot)\,|\,L\right)(x_i) - \overline{X}\right]^2$$

where $\overline{X} = \sum_{i=1}^{k} p_2(\tau_i)X(x_i) \div \sum_{i=1}^{k} p_2(\tau_i)$ and $X(\cdot)$ is as in Theorem 3.1.

<u>Corollary 3.6</u>. If the hypotheses of Theorem 3.1 are satisfied then

$$\lim_{n\to\infty} P[T_{01} \geq t] = \sum_{\ell=1}^{k} P[\chi^2_{\ell-1} \geq t]P(\ell,k)$$

for all $t$ (cf. Theorem 3.1 of Barlow et. al. (1972)).

In closing it is worthwhile to point out that among families with densities of the form given by (3.1) are the Normal, Binomial, Poisson and Exponential families. In particular, in the normal case if the $\mu(x_i)$ are known and

$\theta(x_i)$ is taken to be $\sigma^2(x_i)$ , then one may form likelihood ratio tests for trend in the variance. Such a test is useful in the analysis of residuals

ocedure to determine, for example, if there

hence if weighted least squares is appropriate.

isotonized variance estimate can be used to

hted least squares procedure.

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