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APPROXIMATE SOLUTION OF AN ACTUARIAL EQUATION IN ITS STEADY STATE

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1. INTRODUCTION

Let A_0 be an initial amount to be lent to several customers. Suppose that the amortization of that debt is made according to a given financial basis, the loans being repaid at a certain interest rate. The loans are due at the end of the n subsequent months. A typical financial institution operates in such a way that, as soon as the loans are paid back, those amounts are again invested. A problem often faced by such institutions is that of knowing how the amount, A_x, available at the end of the x-th month,varies. The complete knowledge of A_x is crucial because new loans will eventually have their financial basis changed according to the behavior of A_x. Kellison ([3]) considers a somewhat similar problem, although he is solely concerned with the determination of a periodic payment, which will repay a loan involving step-rate amounts.

The present work tries to closely look at the appropriate actuarial equation expressing the variation of A_x, for which an approximate value is also suggested.

2. THE EQUATION AND AN APPROXIMATE SOLUTION

Following the standard actuarial notation, (see, for instance, Jordan [2]) let i be interest rate compounded monthly;

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$v = \frac{1}{1+i}$, the present value of an amount worth 1 a month later;

$a_{\overline{n}|} = \frac{1-v^n}{i}$, the present value of an annuity of 1 per period for n periods at the interest rate i (the amortization factor);

$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$, the final value of an annuity of n payments of 1 at the compound rate i per period;

A_x, the amount available at the end of the x-th month.

Under the framework above, one can readily write

$$A_x = a_{\overline{n}|}^{-1} \sum_{k=0}^{x-1} A_k \quad \text{for} \quad 0 < x \leq n \tag{2.1}$$

and

$$A_{n+x} = a_{\overline{n}|}^{-1} \sum_{k=x}^{n+x-1} A_k \quad \text{for} \quad 0 \leq x. \tag{2.2}$$

Equation (2.1) has an immediate solution, namely,

$$A_{x+1} = (1 + a_{\overline{n}|}^{-1})A_x \tag{2.3.1}$$

so that

$$A_x = A_0 a_{\overline{n}|}^{-1}(1 + a_{\overline{n}|}^{-1})^{x-1} \quad \text{for} \quad 0 < x \leq n, \tag{2.3.2}$$

which says that the amount available for new loans is an exponential function during the first n months. On the other hand, equation (2.2), generally known as the steady state equation, is a linear difference equation of order (n + 1) with constant coefficients and homogeneous. It can be rewritten as

$$A_{n+x+1} - (1 + a_{\overline{n}|}^{-1})A_{n+x} + a_{\overline{n}|}^{-1}A_x = 0. \tag{2.4}$$

If $\phi(t)$ is the generating function of the sequence $A_0, A_1, A_2, \ldots, A_x, \ldots$, then the preceding equation can be rewritten as (see, for instance, Goldberg [1], p. 192, Table 2A):

$$\frac{\phi(t) - A_0 - \sum_{k=1}^{n} A_k t^k}{t^{n+1}} - (1 + a_{\overline{n}|}^{-1}) \frac{\phi(t) - A_0 - \sum_{k=1}^{n} A_k t^k}{t^n} + a_{\overline{n}|}^{-1}\phi(t) = 0,$$

which gives, when solving for $\phi(t)$,

$$\phi(t) = \frac{A_0 + \sum_{k=1}^{n} A_k t^k - (1 + a_{\overline{n}|}^{-1})tA_0 - (1 + a_{\overline{n}|}^{-1})t\sum_{k=1}^{n-1} A_k t^k}{1 - (1 + a_{\overline{n}|}^{-1})t + a_{\overline{n}|}^{-1}t^{n+1}} . \tag{2.5}$$

Further manipulating the preceding equation, one can see that the numerator of the right hand side will become

$$A_0 + \sum_{k=1}^{n} A_k t^k - (1 + a_{\overline{n}|}^{-1}) \sum_{k=1}^{n} A_{k-1} t^k;$$

so that

$$\phi(t) = \frac{A_0 + \sum_{k=1}^{n} A_k t^k - (1 + a_{\overline{n}|}^{-1}) \sum_{k=1}^{n} A_{k-1} t^k}{1 - (1 + a_{\overline{n}|}^{-1})t + a_{\overline{n}|}^{-1}t^{n+1}}$$

or yet

$$\phi(t) = \frac{A_0\left\{1 + \sum_{k=1}^{n} [a_{\overline{n}|}^{-1}(1 + a_{\overline{n}|}^{-1})^{k-1} - a_{\overline{n}|}^{-1}(1 + a_{\overline{n}|}^{-1})^{k-1}]t^k\right\}}{1 - (1 + a_{\overline{n}|}^{-1})t + a_{\overline{n}|}^{-1}t^{n+1}} . \tag{2.6}$$

By virtue of equations (2.3.1) and (2.3.2), one has

$$\phi(t) = A_0 \frac{1 - t}{1 - (1 + a_{\overline{n}|}^{-1})t + a_{\overline{n}|}^{-1}t^{n+1}} . \tag{2.7}$$

Now letting t_k $(k = 0,1,2,\ldots,n)$ be the roots of the denominator of the rational function on the right hand side of equation (2.7), one obtains

$$\phi(t) = A_0 \sum_{k=0}^{n} \frac{-(t_k - 1)(t - t_k)^{-1}}{(n + 1)a_{\overline{n}|}^{-1}t_k^n - (1 + a_{\overline{n}|}^{-1})} ,$$

which can be rewritten as

$$\phi(t) = A_0 \sum_{k=0}^{n} \left\{ \frac{1 - t_k}{1 - na_{\overline{n}|}^{-1} t_k^{n+1}} \sum_{x=0}^{\infty} \left(\frac{t}{t_k}\right)^x \right\} . \quad (2.8)$$

Therefore, the solution to equation (2.4) is the coefficient of t^x on the right hand side of equation (2.8). Thus

$$A_x = A_0 \sum_{k=0}^{n} \frac{1 - t_k}{1 - na_{\overline{n}|}^{-1} t_k^{n+1}} t_k^{-x} \quad (2.9)$$

or, since t_k is a root in the denominator of equation (2.7),

$$A_x = A_0 \sum_{k=0}^{n} \frac{1 - t_k}{n + 1 - n(1 + a_{\overline{n}|}^{-1})t_k} .$$

The sum on the right hand side (2.9) contains $(n + 1)$ terms which correspond to the roots $t_0, t_1, \ldots, t_n$ of the equation

$$R(t) = 1 - (1 + a_{\overline{n}|}^{-1})t + a_{\overline{n}|}^{-1} t^{n+1} = 0 . \quad (2.10)$$

Now when the value n gets large, which is the case in most of the practical applications (typical values are $n = 36,48$, and so on), the manipulation of equation (2.9) is very difficult. However, an approximate value for A_x can be obtained with the help of equation (2.10). In fact, note that $t_0 = 1$ is a root for it, but its contribution to equation (2.9) is none. Note also that there is a unique root, namely, $t_1 = v = (1 + i)^{-1}$ on the interval $(0,1)$.

$R(t)$ attains a minimum at

$$t^* = \left(\frac{1 + a_{\overline{n}|}^{-1}}{na_{\overline{n}|}^{-1} + a_{\overline{n}|}^{-1}} \right)^{1/n}$$

and that is easy to see by solving $\frac{dR(t)}{dt} = 0$.

Complex roots for equation (2.10) are easily obtained. Let ρ be the modulus and θ be the argument of a complex root of equation (2.10).

So

$$\gamma = \rho \exp\{i\theta\} \quad \text{for} \quad \rho > 0,\ \theta \neq k\pi. \tag{2.11}$$

Thus one can rewrite equation (2.10) as follows

$$R(\gamma) = 1 - (1 + a_{\overline{n}|}^{-1})\gamma + a_{\overline{n}|}^{-1}\gamma^{n+1} = 0, \tag{2.12}$$

which can also be expressed as

$$\begin{cases} 1 - (1 + a_{\overline{n}|}^{-1})\rho \cos\theta + a_{\overline{n}|}^{-1}\rho^{n+1}\cos(n+1)\theta = 0 & (2.13.1) \\ -(+1 + a_{\overline{n}|}^{-1})\rho \sin\theta + a_{\overline{n}|}^{-1}\rho^{n+1}\sin(n+1)\theta = 0\ . & (2.13.2) \end{cases}$$

From equation (2.13.2) one immediately has by virtue of equation (2.11)

$$\rho^n = (1 + a_{\overline{n}|})\frac{\sin\theta}{\sin(n+1)\theta} \tag{2.14}$$

and from equation (2.13.1)

$$1 - \left\{(1 + a_{\overline{n}|}^{-1})\cos\theta - (1 + a_{\overline{n}|}^{-1})\frac{\sin\theta}{\sin(n+1)\theta}\cos(n+1)\theta\right\}\rho = 0$$

or yet

$$\rho = \frac{1}{1 + a_{\overline{n}|}^{-1}}\,\frac{\sin(n+1)\theta}{\sin n\theta}\ . \tag{2.15}$$

For the sake of brevity, let

$$S(\theta) = \frac{\sin(n+1)\theta}{\sin\theta} \tag{2.16}$$

and

$$\phi(\theta) = \frac{\sin(n+1)\theta}{\sin n\theta}\ . \tag{2.17}$$

From equations (2.14) and (2.15) one can write

$$\rho^n = \frac{1 + a_{\overline{n}|}}{S(\theta)} \tag{2.18}$$

and

$$\rho = \frac{\phi(\theta)}{1 + a_{\overline{n}|}^{-1}} \quad . \tag{2.19}$$

Eliminating ρ in the two preceding equations one obtains

$$\psi(\theta) = S(\theta)[\phi(\theta)]^n = (1 + a_{\overline{n}|})(1 + a_{\overline{n}|}^{-1})^n , \tag{2.20}$$

which is the equation of the arguments (of the complex roots). A lower bound for the modulus, ρ, of the complex root $\gamma = \rho \exp\{i\theta\}$ is obtained at once. In fact, from equation (2.12) one has

$$|a_{\overline{n}|}^{-1}\gamma^{n+1}| = |(1 + a_{\overline{n}|}^{-1})\gamma - 1| > |(1 + a_{\overline{n}|}^{-1})\gamma| - 1$$

or

$$1 - |(1 + a_{\overline{n}|}^{-1})\gamma| + |a_{\overline{n}|}^{-1}\gamma^{n+1}| > 0$$

or yet

$$R(\rho) > 0, \tag{2.21}$$

by virtue of equation (2.12).

On the other hand, since $|S(\theta)| < n + 1$, equation (2.14) gives

$$\rho^n > \frac{1 + a_{\overline{n}|}}{1 + n} = (t^*)^n$$

or simply

$$\rho > t^* .$$

Therefore, looking at $R(t)$, expression (2.21) and $\rho > t^*$ immediately give

$$\rho > 1,$$

a lower bound.

As mentioned earlier, an approximate solution for equation (2.9) in its steady state is now proposed. This equation can be rewritten as

$$A_x = A_0\left(\frac{1 - t_1}{1 - na_{\overline{n|}}^{-1} t_1^{n+1}} t_1^{-x} + \epsilon_x\right) \tag{2.22}$$

where

$$t_1 = v = (1 + i)^{-1}$$

and the error

$$\epsilon_x = \sum_{k=2}^{n} \frac{1 - t_k}{1 - na_{\overline{n|}}^{-1} t_k^{n+1}} t_k^{-x} = \sum_{k=2}^{n} \frac{t_k - 1}{n(1 + a_{\overline{n|}}^{-1})t_k - (n + 1)} t_k^{-x}. \tag{2.23}$$

Note that $\epsilon_x \to 0$ as $x \to 0$, since, for $k \geq 2$, $|t_k| > 1$ and

$$|t_k^{-x}| \to 0 \quad \text{as} \quad x \to 0.$$

Also,

$$\frac{1 - t_1}{1 - na_{\overline{n|}}^{-1} t_1^{n+1}} t_1^{-x} = \frac{1 - v}{1 - \dfrac{niv^{n+1}}{1 - v}} r^x = \frac{i}{r - ns_{\overline{n|}}^{-1}} r^x, \tag{2.24}$$

where $r = 1 + i$.

So that

$$A_x = A_0\left(\frac{i}{r - ns_{\overline{n|}}^{-1}} r^x + \epsilon_x\right) \tag{2.25}$$

and thus

$$A_x^* = A_0 \frac{i}{r - ns_{\overline{n|}}^{-1}} r^x. \tag{2.26}$$

is an asymptotic solution to the equation in the steady state.

A lower bound for the modulus, this time corresponding to a root, say θ_k, of the equation (2.20), can also be obtained along the lines previously suggested. For equation (2.20) admits at least one root, θ_k, in the interval $I_k = (\frac{2k\pi}{n}, \frac{(2k+1)\pi}{n+1})$. The corresponding complex root will be

$$\gamma_k = \rho_k \exp\{i\theta_k\}.$$

On the other hand, equation (2.18) can be rewritten as

$$\rho_k^n \geq \frac{1 + a_{\overline{n|}}}{\delta_k} \tag{2.27}$$

where $\delta_k = \max\limits_{\theta \in I_k} S(\theta)$.

Setting $\frac{dS(\theta)}{d\theta} = 0$, one obtains

$$\frac{\sin(n+1)\theta}{\sin\theta} = \frac{(n+1)\cos(n+1)\theta}{\cos\theta},$$

so that

$$[S(\theta)]^2 = \frac{(n+1)^2\cos^2(n+1)\theta}{\cos^2\theta} = \frac{(n+1)^2}{\cos^2\theta + (n+1)^2\sin^2\theta}$$

or

$$|S(\theta)| = \frac{n+1}{\sqrt{1 + (n^2 + 2n)\sin^2\theta}} \tag{2.28}$$

The interval $I_k = (\frac{2k\pi}{n}, \frac{(2k+1)\pi}{n+1})$ contains the maximum θ_k^* whenever $k < \frac{n}{4}$. So $k < \frac{n}{4}$ implies

$$\delta_k = S(\theta_k^*) = \frac{n+1}{\sqrt{1 + (n^2 + 2n)\sin^2\theta_k^*}} < \frac{n+1}{\sqrt{1 + (n^2 + 2n)\sin^2(\frac{2k\pi}{n})}} \tag{2.29}$$

and $k \geq \frac{n}{4}$ implies

$$\delta_k = S(\frac{2k\pi}{n}) = 1. \tag{2.30}$$

From (2.27), (2.29) and (2.30) one can see that $k < \frac{n}{4}$ implies

$$\rho_k^n > (1 + a_{\overline{n|}}) \frac{\sqrt{1 + (n^2 + 2n)\sin^2(\frac{2k\pi}{n})}}{n + 1} \tag{2.31}$$

and $k \geq \frac{n}{4}$ implies

$$\rho_k^n > 1 + a_{\overline{n|}} . \tag{2.32}$$

Now let, for typographical brevity,

$$\rho_{0k} = (1 + a_{\overline{n|}})^{1/n} \frac{\{1 + (n^2 + 2n)\sin^2(\frac{2k\pi}{n})\}^{1/2n}}{(n + 1)^{1/n}} \tag{2.33}$$

or yet

$$\rho_{0k} = (1 + a_{\overline{n|}})^{1/n} \left\{\sin^2(\frac{2k\pi}{n}) + \frac{1}{(n + 1)^2}\cos^2(\frac{2k\pi}{n})\right\}^{1/2n}, \tag{2.34}$$

so that

$$\text{if} \quad k \leq \frac{n}{4}, \text{ then} \quad \rho_k > \rho_{0k} \tag{2.35}$$

and

$$\text{if} \quad k \geq \frac{n}{4}, \text{ then} \quad \rho_k > (1 + a_{\overline{n|}})^{1/n}. \tag{2.36}$$

Going back to equation (2.10), one can see at once that it admits a negative root for which a lower bound can also be obtained. A straightforward manipulation of that equation shows that it has a negative root, say t_2, such that $t_2 < -1$. Now since $R(-\infty) = \infty$ and

$$R((2a_{\overline{n|}})^{1/n+1}) = 1 + (1 + a_{\overline{n|}}^{-1})(2a_{\overline{n|}})^{1/n+1} - 2 > 0$$

one obtains the lower bound mentioned above:

$$t_2 < - (2a_{\overline{n|}})^{1/n+1} \tag{2.37}$$

or

$$|t_2| > (2a_{\overline{n|}})^{1/n+1}. \tag{2.38}$$

In the sequel is shown the behavior of the error ϵ_x, as expressed by equation (2.23), so that for a given financial basis, one will be able to know how good is the approximation suggested by equation (2.26) as compared to the exact solution given by equation (2.22). From equation (2.23) it is possible to isolate the term corresponding to the negative real root t_2. In fact, note that

$$\left|\frac{t_2 - 1}{n(1 + a_{\overline{n}|}^{-1})t_2 - (n + 1)}\right| = \frac{|t_2| + 1}{n(1 + a_{\overline{n}|}^{-1})|t_2| + n + 1} < \frac{|t_2| + 1}{(n + 1)(|t_2| + 1)} = \frac{1}{n + 1}$$

and from equation (2.38)

$$\left|\frac{t_2 - 1}{n(1 + a_{\overline{n}|}^{-1})t_2 - (n + 1)}\, t_2^{-x}\right| < \frac{1}{n + 1}(2a_{\overline{n}|})^{-x/n+1} . \tag{2.39}$$

The terms in ϵ_x corresponding to the complex roots comprise the sum

$$\delta_x = \sum_{k=1}^{n-2}\left(\frac{\gamma_k - 1}{n(1 + a_{\overline{n}|}^{-1})\gamma_k - (n + 1)}\right)\gamma_k^{-x} ,$$

where $\gamma_k = t_{k+2}$. For the sake of brevity, let

$$\frac{\gamma_k - 1}{n(1 + a_{\overline{n}|}^{-1})\gamma_k - (n + 1)} \equiv \alpha_k \exp\{i\tau_k\} \tag{2.40}$$

whose corresponding conjugate term is

$$\delta_x = \sum_{k=1}^{n/2-1} 2\alpha_k \rho_k^{-x} \cos(\tau_k - \theta_k x) \tag{2.41}$$

where

$$\alpha_k = \frac{|\gamma_k - 1|}{|n(1 + a_{\overline{n}|}^{-1})\gamma_k - (n + 1)|} < \frac{1}{n + 1} .$$

So that equation (2.41) is maximized as follows

$$|\delta_x| \le \frac{2}{n + 1} \sum_{k=1}^{n/2-1} \rho_k^{-x} . \tag{2.42}$$

Now note that from expression (2.35) ρ_{0k} is a lower bound for ρ_k, and from (2.34) it is easy to see that $\rho_{0k}^{n} \to 0$ as $n \to \infty$.

Therefore for large n (say, $n = 48$ months), one will have

$$\rho_{0k}^{n} < 1 \text{ and, thus, } \rho_{0k} < 1 .$$

Example. In the sequel an illustrative example showing the variation of A_x and its approximation, A_x^*, is given. Let $A_0 = 10{,}000$, $n = 48$ months, $x \leq 120$ months, and $i = 12\%$, 10% and 8% compounded yearly.

The amount available during $n = 48$ months will be expressed by (equation (2.3.2))

$$A_x = A_0 a_{\overline{n|}}^{-1}(1 + a_{\overline{n|}}^{-1})^{x-1} \quad \text{for} \quad 0 < x \leq n;$$

the values corresponding to the last months $(48 < x \leq 120)$ can be evaluated by the main equation in the steady state (equation (2.4)):

$$A_{n+x+1} = (1 + a_{\overline{n|}}^{-1})A_{n+x} - a_{\overline{n|}}^{-1} \cdot A_x .$$

The asymptotic expression to be used is (equation (2.26))

$$A_x^* = A_0 \frac{i}{r - ns_{\overline{n|}}^{-1}} r^x .$$

And, finally, the error is evaluated by

$$\epsilon_x = A_x - A_x^* .$$

Note that $\epsilon_x > 0$ for $x = 48$.

The calculations are shown in the table below.

EXACT, APPROXIMATE VALUES OF A_x AND ERROR

x	i = 0.0100 A_x	i = 0.0100 A^*_x	i = 0.0100 ϵ_x	i = 0.0083 A_x	i = 0.0083 A^*_x	i = 0.0083 ϵ_x	i = 0.066 A_x	i = 0.066 A^*_x	i = 0.066 ϵ_x
0	1000000.00			1000000.00			1000000.00		
1	26333.83			25343.38			24375.39		
2	27027.31			25985.66			24969.55		
3	27739.04			26644.23			25578.19		
4	28469.51			27319.48			26201.67		
5	29219.22			28011.85			26840.34		
6	29988.68			28721.77			27494.59		
7	30778.40			29449.67			28164.78		
8	31588.91			30196.03			28851.31		
9	32420.77			30961.30			29554.57		
10	33274.53			31745.96			30274.97		
11	34150.77			32550.51			31012.94		
12	35050.10			33375.45			31768.89		
13	35973.10			34221.30			32543.27		
14	36920.41			35088.58			33336.52		
15	37892.66			35977.84			34149.11		
16	38890.52			36889.64			34981.51		
17	39914.66			37824.55			35834.20		
18	40965.77			38783.15			36707.67		
19	42044.55			39766.05			37602.43		
20	43151.75			40773.85			38519.01		
21	44288.10			41807.20			39457.92		
22	45454.37			42866.74			40419.73		
23	46651.36			43953.13			41404.97		
24	47879.87			45067.05			42414.24		
25	49140.73			46209.20			43448.10		
26	50434.79			47380.29			44507.16		
27	51762.94			48581.07			45592.04		
28	53126.05			49812.28			46703.37		
29	54525.06			51074.69			47841.78		
30	55960.92			52369.10			49007.94		
31	57434.58			53696.31			50202.53		
32	58947.06			55057.15			51426.23		
33	60499.36			56452.49			52679.77		
34	62092.54			57883.18			53963.86		
35	63727.67			59350.14			55279.25		
36	65405.87			60854.27			56626.70		
37	67128.25			62396.52			58007.00		
38	68896.00			63977.86			59420.94		
39	70710.30			65599.28			60869.35		
40	72572.37			67261.78			62353.06		
41	74483.48			68966.43			63872.94		
42	76444.91			70714.27			65429.87		
43	78458.00			72506.41			67024.75		
44	80524.10			74343.96			68658.51		
45	82644.61			76228.09			70332.08		
46	84820.96			78159.97			72046.45		
47	87054.62			80140.81			73802.61		
48	89347.10	71345.04	18002.06	82171.84	64883.70	17288.14	75601.58	59007.02	16594.56
49	65366.12	72058.49	-6692.37	58910.98	65422.23	-6511.26	53069.01	59396.47	-6327.46

x	i = 0.0100			i = 0.0083			i = 0.066		
	A_x	A_x^*	ϵ_x	A_x	A_x^*	ϵ_x	A_x	A_x^*	ϵ_x
50	66393.99	72779.07	-6385.08	59761.70	65965.24	-6203.54	53768.43	59788.49	-6020.05
51	67430.66	73506.86	-6076.20	60617.69	66512.75	-5895.06	54470.41	60183.09	-5712.67
52	68475.90	74241.93	-5766.03	61478.70	67064.81	-5586.11	55174.67	60580.30	-5405.62
53	69529.42	74984.35	-5454.93	62344.41	67621.44	-5277.04	55880.90	60980.13	-5099.23
54	70590.94	75734.19	-5143.25	63214.51	68182.70	-4968.19	56588.78	61382.60	-4793.82
55	71660.15	76491.54	-4831.38	64088.67	68748.62	-4659.95	57297.96	61787.72	-4489.76
56	72736.73	77256.45	-4519.72	64966.54	69319.23	-4352.69	58008.09	62195.52	-4187.43
57	73820.31	78029.01	-4208.71	65847.74	69894.58	-4046.84	58718.80	62606.01	-3887.21
58	74910.52	78809.30	-3898.79	66731.88	70474.71	-3742.82	59429.69	63019.21	-3589.52
59	76006.95	79597.40	-3590.45	67618.54	71059.65	-3441.10	60140.35	63435.14	-3294.79
60	77109.18	80393.37	-3284.19	68507.29	71649.44	-3142.15	60850.34	63853.81	-3003.47
61	78216.76	81197.31	-2980.54	69397.65	72244.13	-2846.49	61559.21	64275.24	-2716.03
62	79329.20	82009.28	-2680.08	70289.13	72843.76	-2554.62	62266.49	64699.46	-2432.98
63	80445.98	82829.37	-2383.39	71181.23	73448.36	-2267.13	62971.66	65126.48	-2154.81
64	81566.58	83657.66	-2091.09	72073.41	74057.98	-1984.57	63674.23	65556.31	-1882.09
65	82690.40	84494.24	-1803.84	72965.08	74672.66	-1707.58	64373.62	65988.98	-1615.36
66	83816.85	85339.18	-1522.33	73855.66	75292.45	-1436.78	65069.28	66424.51	-1355.23
67	84945.28	86192.57	-1247.29	74744.52	75917.37	-1172.85	65760.61	66862.91	-1102.31
68	86075.02	87054.50	-979.48	75630.99	76547.49	-916.50	66446.97	67304.21	-857.24
69	87205.36	87925.04	-719.69	76514.39	77182.83	-668.44	67127.73	67748.42	-620.69
70	88335.53	88804.30	-468.76	77393.99	77823.45	-429.46	67802.19	68195.56	-393.37
71	89464.76	89692.34	-227.58	78269.02	78469.38	-200.36	68469.65	68645.65	-176.00
72	90592.20	90589.26	2.94	79138.70	79120.68	18.02	69129.36	69098.71	30.65
73	91716.98	91495.15	221.83	80002.20	79777.38	224.81	69780.55	69554.76	225.79
74	92838.18	92410.10	428.07	80858.62	80439.53	419.09	70422.41	70013.82	408.59
75	93954.82	93334.21	620.61	81707.08	81107.18	599.90	71054.11	70475.91	578.20
76	95065.89	94267.55	798.35	82546.60	81780.37	766.23	71674.76	70941.05	733.70
77	96170.33	95210.22	960.11	83376.20	82459.15	917.05	72283.44	71409.26	874.18
78	97267.01	96162.32	1104.69	84194.83	83143.56	1051.27	72879.22	71880.56	998.65
79	98354.76	97123.95	1230.81	85001.40	83833.65	1167.75	73461.09	72354.98	1106.11
80	99432.34	98095.19	1337.16	85794.78	84529.47	1265.31	74028.03	72832.52	1195.51
81	100498.48	99076.14	1422.34	86573.77	85231.06	1342.71	74578.95	73313.21	1265.74
82	101551.81	100066.90	1484.91	87337.15	85938.48	1398.67	75112.76	73797.08	1315.67
83	102590.92	101067.57	1523.35	88083.61	86651.77	1431.84	75628.27	74284.14	1344.13
84	103614.34	102078.24	1536.09	88811.82	87370.98	1440.83	76124.28	74774.42	1349.87
85	104620.51	103099.03	1521.49	89520.35	88096.16	1424.19	76599.54	75267.93	1331.62
86	105607.83	104130.02	1477.81	90207.76	88827.36	1380.41	77052.75	75764.70	1288.05
87	106574.59	105171.32	1403.28	90872.52	89564.62	1307.89	77482.53	76264.74	1217.78
88	107519.04	106223.03	1296.01	91513.03	90308.01	1205.02	77887.48	76768.09	1119.39
89	108439.32	107285.26	1154.06	92127.64	91057.57	1070.07	78266.14	77274.76	991.38
90	109333.50	108358.11	975.39	92714.62	91813.35	901.27	78616.98	77784.77	832.20
91	110199.59	109441.69	757.89	93272.18	92575.40	696.79	78938.42	78298.15	640.26
92	111035.46	110536.11	499.35	93798.46	93343.77	454.69	79228.82	78814.92	413.90
93	111838.95	111641.47	197.47	94291.50	94118.52	172.97	79486.47	79335.10	151.37
94	112607.74	112757.88	-150.14	94749.29	94899.71	-150.42	79709.62	79858.71	-149.10
95	113339.48	113885.46	-545.99	95169.72	95687.38	-517.66	79896.41	80385.78	-489.37
96	114031.66	115024.32	-992.66	95550.60	96481.58	-930.98	80044.95	80916.32	-871.38
97	114681.70	116174.56	-1492.86	95889.66	97282.38	-1392.72	80153.26	81450.37	-1297.12
98	115980.37	117336.31	-1355.94	96826.83	98089.82	-1262.99	80813.44	81987.94	-1174.50
99	117286.16	118509.67	-1223.50	97766.18	98903.97	-1137.78	81472.68	82529.07	-1056.39

x	i = 0.0100			i = 0.0083			i = 0.066		
	A_x	A^*_x	ϵ_x	A_x	A^*_x	ϵ_x	A_x	A^*_x	ϵ_x
100	118599.05	119694.77	-1095.71	98707.65	99724.87	-1017.22	82130.87	83073.76	-942.89
101	119918.99	120891.71	-972.73	99651.16	100552.59	-901.43	82787.94	83622.04	-834.11
102	121245.94	122100.63	-854.69	100596.64	101387.17	-790.53	83443.80	84173.95	-730.15
103	122579.88	123321.64	-741.76	101544.03	102228.69	-684.66	84098.41	84729.50	-631.09
104	123920.79	124554.85	-634.06	102493.27	103077.18	-583.91	84751.68	85288.71	-537.03
105	125268.66	125800.40	-531.74	103444.33	103932.72	-488.40	85403.56	85851.62	-448.06
106	126623.49	127058.40	-434.91	104397.15	104795.37	-398.22	86054.01	86418.24	-364.22
107	127985.29	128328.99	-343.69	105351.72	105665.17	-313.45	86702.99	86988.60	-285.61
108	129354.08	129612.28	-258.19	106308.00	106542.19	-234.19	87350.47	87562.72	-212.26
109	130729.89	130908.40	-178.51	107266.00	107426.49	-160.49	87996.42	88140.64	-144.22
110	132112.76	132217.48	-104.72	108225.71	108318.13	-92.42	88640.84	88722.37	-81.53
111	133502.76	133539.66	-36.90	109187.15	109217.17	-30.02	89283.72	89307.93	-24.21
112	134899.95	134875.05	24.89	110150.35	110123.67	26.68	89925.09	89897.37	27.72
113	136304.42	136223.80	80.61	111115.35	111037.70	77.65	90564.96	90490.69	74.27
114	137716.28	137586.04	130.24	112082.21	111959.31	122.90	91203.39	91087.93	115.46
115	139135.66	138961.90	173.76	113051.00	112888.57	162.42	91840.41	91689.11	151.31
116	140562.70	140351.52	211.18	114021.81	113825.55	196.26	92476.12	92294.26	181.86
117	141997.57	141755.04	242.53	114994.76	114770.30	224.47	93110.59	92903.40	207.19
118	143440.46	143172.59	267.87	115969.99	115722.89	247.09	93743.93	93516.56	227.37
119	144891.58	144604.31	287.27	116947.63	116683.39	264.24	94376.27	94133.77	242.50
120	146351.18	146050.35	300.83	117927.88	117651.86	276.02	95007.76	94755.05	252.71

```
        INTEGER X
        REAL*8 A(120,3),AS(120,3),E(120,3),I(3),AN(3),SN(3),R(3),V(3)
        WRITE(3,5)
      5 FORMAT(15(/))
        I(1) = 0.0100
        I(2) = 0.0083
        I(3) = 0.0066
C
        DO 10 J=1,3
        R(J) = 1. + I(J)
        V(J) = 1./R(J)
       AN(J) = (1. - (V(J)**48))/I(J)
         SN(J) = (((1. + I(J))**48)-1)/I(J)
       WRITE(3,15) J,I(J),R(J),V(J),SN(J),AN(J)
     10 CONTINUE
     15 FORMAT(5X,I3,5F15.6,/)
C
       X=0
       A0 = 1000000.00
       WRITE(3,5)
       WRITE(3,20) X,A0,A0,A0
     20 FORMAT(10X,I5,3(2X,F10.2,20X))
C
       DO 100 X=1,47
       DO 30  J=1,3
       A(X,J) = (A0/AN(J)) * ((1. + 1./AN(J))**(X-1))
     30 CONTINUE
C
       WRITE(3,20) X, (A(X,J),J=1,3)
    100 CONTINUE
C
       X=48
       DO 110 J=1,3
       AS(X,J) = A0 * (I(J)/(R(J) - 48/SN(J))) * (R(J)**X)
        A(X,J) = (A0/AN(J)) * ((1. + 1./AN(J)) ** (X-1))
        E(X,J) = A(X,J) - AS(X,J)
    110 CONTINUE
        WRITE(3,120) X,(A(X,J),AS(X,J),E(X,J),J=1,3)
    120 FORMAT(10X,I5,3(2X,3F10.2))
C
        X=49
        DO 130 J=1,3
        A(X,J) = (1. + 1./AN(J)) * A(X-1,J) - A0/AN(J)
        AS(X,J) = A0 * (I(J)/(R(J) - 48/SN(J))) * (R(J)**X)
        E(X,J) = A(X,J) - AS(X,J)
    130 CONTINUE
        WRITE(3,120) X, (A(X,J),AS(X,J),E(X,J),J=1,3)
        WRITE(3,5)
C
        DO 200 X=50,120
        DO 140 J=1,3
        AS(X,J) = A0 * (I(J)/(R(J) - 48/SN(J))) * (R(J)**X)
         A(X,J) = (1. + 1./AN(J)) * A(X-1,J) - A(X-49,J)/AN(J)
         E(X,J)  = A(X,J) - AS(X,J)
    140 CONTINUE
        WRITE(3,120) X, (A(X,J),AS(X,J),E(X,J),J=1,3)
        IF(X.EQ. 99) WRITE(3,5)
    200 CONTINUE
        STOP
        END
```

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