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CHI-SQUARE TESTS FOR GENERAL MODELS UNDER PROGRESSIVE CENSORING WITH BATCH ARRIVALS

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Hiranmay Majundar and Pranab Kumar Sen

Department of Biostatistics
University of North Carolina at Chapel Hill

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ABSTRACT

For some batch-arrival models relating to categorical data under time-sequential studies, suitably progressively censored tests based on chi-square statistics are proposed and studied. The necessary (asymptotic) distribution theory is considered for the null as well as local alternative hypotheses situations. To facilitate comparisons of the different proposed tests, a numerical illustration is presented at the end.

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1. INTRODUCTION

Chi-square statistics are widely used for testing of hypotheses for categorical data. In many situations relating to clinical trials and life-testing problems, the response categories are (ordered and) sequential in time, so that a complete collection of data may involve a considerable period of time (and hence, cost). For this reason, progressive censoring schemes (PCS) are often advocated with a view to terminating the experimentation at the earliest possible stage depending on the accumulated evidence at the successive stages. Further, in many such problems, not all the subjects enter into the study at a common point of time and, naturally, a batch-arrival model (BAM) seems to be more appropriate. Under a general model on the probability structure and assuming the number of batches to be fixed in advance, the current investigation provides suitable progressively censored tests based on chi-square statistics. Among other generalities, this includes as a special case, a time-sequential test essentially related to Mantel (1966) for $2 \times (k+1)$, $k \geq 2$ contingency tables.

The basic problem is formulated in Section 2. The proposed tests are presented in Section 3. Sections 4 and 5 deal with the distribution theory of the test statistics under suitable null and (local) alternative hypothesis. In this context, it is shown that one of the proposed tests has the nice property that it has the same level of significance and power of the overall test but can lead to rejection at an early stage resulting in lesser expected cost and time of experimentation. Section 6 is devoted to various extensions to multi-response models and comparisons with some other tests. The final section gives a numerical illustration.

2. FORMULATION OF THE PROBLEM

In the context of clinical trials or life testing, consider a typical truncated experimentation plan for a pre-determined amount of time T and suppose that the response (failures) are recorded in k (ordered) time intervals $I_1,\ldots,I_k$ and the surviving (beyond time T) subjects constitute the cell I^*_{k+1}. Thus, we have a set of $k+1$ ordered categories: $I_c = \{t: T_{c-1} < t \le T_c\}$, $1 \le c \le k$ and $I^*_{k+1} = \{t: t > T_k\}$ where $T_0 < T_1 < \ldots < T_k = T < \infty$. Conventionally, for every $c < k$, we let

$$I^*_{c+1} = I_{c+1} \cup I_{c+2} \cup \ldots \cup I_k \cup I^*_{k+1} . \tag{2.1}$$

We conceive of a BAM where all the subjects may not enter into the study at the same time. Instead, there are $\ell(\ge 1)$ batches, the j-th batch enters into the study at time-point T_{h_j} for some $0 \le h_j \le k-1$ so that the observable categories for this batch are $I_1,\ldots,I_{k-h_j}$ and $I^*_{k-h_j+1}$, for $1 \le j \le \ell$; also, $0 = h_1 < \ldots < h_\ell \le k-1$. Note that in this design, it is not necessary to fix in advance the individual batch sample sizes; rather the j-th batch sample size may be decided at the entry time T_{h_j}, for $j = 1,\ldots,\ell$. Further, keeping in mind the usual comparative experiments involving one or more factor (or treament etc.), we conceive of the $r(\ge 1)$ sample situtation, where, for each batch, the r parallel samples relate to the same categories. For a better understanding of the model, we refer to the numerical illustration in Section 7.

For the i-th sample in the j-th batch, the number of subjects entering into the study is denoted by n_{ij}, $j = 1,\ldots,\ell$, $1 \le i \le r$. Let $n_{ij,t}$ be the number of observations (among the n_{ij} subjects) belonging to I_t,

$t = 1,\ldots,k-h_j$ and $n^*_{ij,k-h_j+1}$ be the number of censored cases, $1 \le j \le \ell$, $1 \le i \le r$; on letting $k_j = k-h_j$, we denote the corresponding cell probabilities by $\Pi_{ij,t}$, $1 \le t \le k_j$ and Π^*_{ij,k_j+1}, $1 \le j \le \ell$, $1 \le i \le r$. Further,

$$\Pi_{ij} = \sum_{t=1}^{k_j} \Pi_{ij,t} + \Pi^*_{ij,k_j+1} = 1 , \quad \forall\ 1 \le j \le \ell,\ 1 \le i \le r ; \tag{2.2}$$

$$n_{ij} = \sum_{t=1}^{k_j} n_{ij,t} + n^*_{ij,k_j+1} , \quad \forall\ 1 \le j \le \ell,\ 1 \le i \le r . \tag{2.3}$$

The joint distribution of the observed cell frequencies is given by the product-multinomial:

$$\phi = \prod_{i=1}^{r} \prod_{j=1}^{\ell} \left\{ \frac{n_{ij}!}{\prod_{t=1}^{k_j} n_{ij,t}!\,(n^*_{ij,k_j+1})!} \left(\prod_{t=1}^{k_j} \Pi_{ij,t}^{n_{ij,t}} \right) \left(\Pi^*_{ij,k_j+1} \right)^{n^*_{ij,k_j+1}} \right\} \tag{2.4}$$

For this model, one usually expresses

$$\Pi_{ij,t} = \Pi_{ij,t}(\underset{\sim}{\theta}) ; \quad \underset{\sim}{\theta}' = (\theta_1,\ldots,\theta_s), \quad \forall\ 1 \le i \le r,\ 1 \le j \le \ell,\ 1 \le t \le k_j \tag{2.5}$$

and proceeds to test a null hypothesis

$$H_0 : F_m(\underset{\sim}{\theta}) = 0 , \quad m = 1,\ldots,q(\le s) , \tag{2.6}$$

where the F_m are suitable (linearly independent) functions of $\underset{\sim}{\theta}$. Let the test statistic be

$$\hat{\chi}^2 = \sum_{i=1}^{r} \sum_{j=1}^{\ell} \left\{ \sum_{t=1}^{k_j} \frac{(n_{ij,t} - n_{ij}\hat{\Pi}_{ij,t})^2}{n_{ij,t}\hat{\Pi}_{ij,t}} + \frac{(n^*_{ij,k_j+1} - n_{ij}\hat{\Pi}^*_{ij,k_j+1})^2}{n_{ij}\hat{\Pi}^*_{ij,k_j+1}} \right\} \tag{2.7}$$

where $\hat{\Pi}_{ij,t} = \Pi_{ij,t}(\hat{\underset{\sim}{\theta}})$, $\hat{\Pi}^*_{ij,k_j+1} = \Pi^*_{ij,k_j+1}(\hat{\underset{\sim}{\theta}})$ and $\hat{\underset{\sim}{\theta}}$ is obtained by minimizing Pearson's

$$(2.8)\quad \chi^2 = \sum_{i=1}^{r} \sum_{j=1}^{\ell} \left\{ \sum_{t=1}^{k_j} \frac{(n_{ij,t} - n_{ij}\Pi_{ij,t}(\underset{\sim}{\theta}))^2}{n_{ij}\Pi_{ij,t}(\theta)} + \frac{(n^*_{ij,k_j+1} - n_{ij}\Pi^*_{ij,k_j+1}(\underset{\sim}{\theta}))^2}{n_{ij}\Pi^*_{ij,k_j+1}(\underset{\sim}{\theta})} \right\}$$

with respect to $\underset{\sim}{\theta}$, subject to the constraints in (2.6). Actually, for large sample sizes, any BAN estimator of $\underset{\sim}{\theta}$ can be used, see Neyman (1949). For large sample sizes, under H_0 in (2.5), under suitable regularity conditions given by Cramér (1946, pp. 426-27) and subsequently modified by Birch (1964),

$$(2.9)\qquad \hat{\chi}^2 \overset{\mathcal{D}}{\to} \chi^2_{r(k_1+k_2+\ldots+k_\ell)-s+q}\ ,$$

and this provides a large sample test of H_0, where we reject H_0 for large values of $\hat{\chi}^2$.

The test-statistic $\hat{\chi}^2$ is generally used when we wait until the end of the experiment as envisaged in the design. However, as mentioned earlier, we like to adopt a PCS where we monitor the experiment from the beginning with a view to stopping experimentation at an early stage (i.e., at some time-point T_c, $c \le k$) if the evidence accumulated upto that stage is sufficient to reject H_0. Basically, without affecting the risk of making incorrect decisions, such a PCS procedure can be constructed and will result in a reduction in the time of experimentation, and hence, cost too. This is elaborated in the next section.

3. CHI-SQUARE TYPE TESTS FOR GENERAL MODEL UNDER PCS

Let use examine the structure of the categorical data at the completion of time T_c. Let $J_c = \{j: h_j \le c\}$. Then, for $j \in J_c$, writing $c_j = c-h_j$, the j-th batch (n_{ij}) relates to the categories $I_1,\ldots,I_{c_j}$,

$I^*_{c_j+1}$ with respective frequencies $n_{ij,1},\ldots,n_{ij,c_j},n^*_{ij,c_j}$, for $1 \le i \le r$, so that the joint probability function is

$$(3.1)\quad \phi_c = \prod_{i=1}^{n} \prod_{j\in J_c} \left\{ \frac{n_{ij}!}{\prod_{t=1}^{c_j} n_{ij,t}!\,(n^*_{ij,c_j+1})!} \prod_{t=1}^{c_j} \Pi_{ij,t}^{n_{ij,t}} \left(\Pi^*_{ij,c_j+1}\right)^{n^*_{ij,c_j+1}} \right\}$$

consider then the related minimum chi-square statistic

$$(3.2)\quad \hat{\chi}^2_c = \sum_{i=1}^{r} \sum_{j\in J_c} \left\{ \sum_{t=1}^{c_j} \frac{(n_{ij,t}-n_{ij}\tilde{\Pi}_{ij,t})^2}{n_{ij}\tilde{\Pi}_{ij,t}} + \frac{(n^*_{ij,c_j+1}-n_{ij}\tilde{\Pi}^*_{ij,c_j+1})^2}{n_{ij}\tilde{\Pi}^*_{ij,c_j+1}} \right\}$$

where $\tilde{\Pi}_{ij,t}$ and $\tilde{\Pi}^*_{ij,c_j+1}$ are the estimated probabilities obtained by substituting $\underset{\sim}{\theta} = \underset{\sim}{\tilde{\theta}}_c$ in (2.5) and $\underset{\sim}{\tilde{\theta}}_c$ is the minimum chi-square estimator of $\underset{\sim}{\theta}$ based on (3.1). Here also, use of any BAN estimator (based on (3.1)) is permissible for large n_{ij}. Note that $\phi = \phi_k$, $\hat{\chi}^2 = \hat{\chi}^2_k$ and the earliest stopping time-point T_b for constructing $\hat{\chi}^2_b$ involves $b+1$ cells $(I_1,\ldots,I_b,I^*_{b+1})$ for which $\sum_{j\in J_b} (b-h_j) > s-q$. Further note that ϕ_c involves only a subset of the probabilities $\{\Pi_{ij,t}\}$, and hence, the null hypothesis H_0 in (2.5) may involve only a subset $q(c)$ of restraints on the $\Pi_{ij,t}$ at the c-th stage, where $q(c)$ is $\nearrow$ in c. This is particularly true for contingency tables, where as c increases, we deal with an increasing number of marginal probabilities associated with the categories. Thus we conceive of (2.4) and set

$$(3.3)\qquad H_0 \equiv \bigcap_{b\le c\le k} H_{0,c}\,,\quad H_{0c} \text{ is monotonic ;}$$

$$(3.4)\qquad H_{0,c}: F^{(c)}_m(\underset{\sim}{\theta}) = 0\,,\quad \forall\ 1 \le m \le q(c)\ ;\quad q(c) \nearrow \text{ in } c\ .$$

Naturally, if H_{0c} is not true for some $c(\le k)$, then H_0 is also not true.

3.1. Type A PCS test for H_0. Consider the set of time-points $\{T_c: b \le c \le k\}$. At T_c, compute $\hat{\chi}_c^2$. Continue experimentation so long as $\hat{\chi}_c^2 \le \omega_\alpha$, $c \le k$. If, for the first time, for $c = C$, $\hat{\chi}_C^2 > \omega_\alpha$, stop experimentation at time T_C along with rejection of H_0. If no such $C(\le k)$ exists, accept H_0 at time T_k. Here we need to choose ω_α in such a way that the overall level of significance of the test is α: $0 < \alpha < 1$. Note that in this setup, both C and T_C are stochastic variables. Further, the tests proposed here depends only on the set of frequencies of the uncensored cells. However, parametric considerations may enter (excepting in some cases like contingence table situations) through the dependence of $\Pi_{ij,t}$ on $\underset{\sim}{\theta}$.

3.2. Type B PCS test for H_0. Let us define

$$D_c = \max\{\hat{\chi}_c^2 - \hat{\chi}_{c-1}^2, 0\}, \quad b < c \le k, \tag{3.5}$$
$$= \hat{\chi}_c^2 \quad \text{if} \quad c = b.$$

Then continue experimentation so long as $D_c \le \Delta_c$, $c \ge b$. If for the first time, for $c = C$, $D_C > \Delta_C$, stop experimentation at time T_C along with the rejection of H_0. If no such $C(\le k)$ exists, accept H_0. Here also, $\{\Delta_b, \ldots, \Delta_k\}$ are positive constants such that

$$P\{D_c \le \Delta_c, \ \forall\ b \le c \le k | H_0\} = 1-\alpha, \tag{3.6}$$

where α = level of significance.

For both the tests, as we shall see in the next section, the design must be compatiable in the sense that (i) $q(c)$ should be $\nearrow$ in c and (ii) $\sum_{J_c} (c-h_j) - s + q(c)$ should be $\nearrow$ in c: $b \le c \le k$.

3.3. Two simultaneous tests. We have considered earlier the situation when the number of batches as well as the time-points T_{h_j}, $0 \le h_j \le k-1$, $j = 1,\ldots,\ell$ at which the batches entering into the study are predetermined in order that we know in advance the degrees of freedom of the terminal statistic $\hat{\chi}^2$ relating to the overall test computed over all the batches and all the samples. Corresponding to Type A and Type B tests, we may devise two simultaneous test procedures across the independent batches which will be termed as Type C and Type D tests respectively. As we shall later show these tests will be particularly useful when the number of batches is large and the experiment is continued over a long period of time. In these latter tests, the terminal statistics are based on the respective batches only. Consequently, we may relax, in this case, the condition of batch-arrivals at pre-determined time-points; rather they may be conveniently decided as the experimentation progresses. We now proceed to describe these two tests.

3.3.1. Type C PCS test for H_0. As in Type A test but separately for the relevant batches, compute $\{{}_j\hat{\chi}_c^2\}_{j=1}^{\ell_c}$, at time-point T_c, where ℓ_c is the number of batches involved at the c-th stage of censoring. If the earliest time-point for constructing this statistic is T_{b_0} for the first batch, then these are $T_{b_0+h_2}, T_{b_0+h_3}, \ldots, T_{b_0+h_\ell}$ for the subsequent batches respectively. The dependence of ℓ_c on c comes due to the fact that

$$(3.7) \quad \ell_c = \sum_{j=1}^{\ell} I(b_0 + h_j \ge c), \quad \text{where} \quad h_1 = 0 < h_2 < \ldots < h_\ell \le k-b_0$$

and $I(x)$ stands for the indicator function for the event. At the time-point T_{b_0}, the sequence $\{{}_j\hat{\chi}_c^2\}$ consists of only single element ${}_1\hat{\chi}_{b_0}^2$.

This $'b_0'$ may be different from 'b' defined in sections 3.1 and 3.2. Continue experimentation so long as $\{{}_j\hat{\chi}^2_c \le {}_j\omega_{\alpha''}\}$, $\forall\ j = 1,\ldots,\ell_c$. Stop experimentation along with the rejection of H_0, if for some j, ${}_j\hat{\chi}^2_c > {}_j\omega_{\alpha''}$. If no such $c(b_0 \le c \le k)$ exists, accept H_0 at time $T_k = T$. Here we choose α'' in such a way that the overall level of significance for every $j = 1,\ldots,\ell$ is α'': $0 < \alpha'' < 1$ and overall error rate of the experiment is α: $0 < \alpha < 1$ so that $1-\alpha = (1-\alpha'')^{\ell}$.

3.3.2. Type D PCS test for H_0. At T_c, for each $j = 1,\ldots,\ell_c$, compute

(3.8) $$ {}_jD_c = \max\{{}_j\hat{\chi}^2_c - {}_j\hat{\chi}^2_{c-1}, 0\},\ b_0 + h_j < c \le k;\ j = 1,\ldots,\ell; $$

$$ 0 = h_1 < h_2 \ \ldots\ h_\ell \le k - b_0 $$

$$ = {}_j\hat{\chi}^2_c \quad \text{if} \quad c = b_0 + h_j,\ j = 1,\ldots,\ell . $$

Then continue experimentation so long as ${}_jD_c \le {}_j\Delta_c$, $\forall_j$; $c \ge b_0 + h_j$. If for the first time for $c = C$, ${}_jD_C > {}_j\Delta_C$, for some j, stop experimentation at time-point T_C along with the rejection of H_0. If no such $C(\le k)$ exists, accept H_0 at $T_k = T$. Here $\{{}_j\Delta_{b_0+h_j},\ldots,{}_j\Delta_k\}$ form a double sequence of positive constants such that

(3.9) $$ P\{{}_jD_c \le {}_j\Delta_c,\ \forall\ b_0 + h_j \le c \le k \mid {}_jH_0\} = 1-\alpha'' $$

where α'' is the level of significance for an individual batch and α, the overall error rate of the experiment satisfies $1-\alpha = (1-\alpha'')^{\ell}$.

3.3. Relevance to the Mantel procedure. In further extension of the methods of Mantel-Haenszel (1959) and Mantel (1963), Mantel (1966) considered an overall comparison of two sets of life tables in their entirety.

The method is equivalent to decomposing a $2 \times (k+1)$ contingency table into k correlated 2×2 contingency tables and combining the result of each into a summary chi-square with one degree of freedom. By bringing a conditionality argument to bear, he showed how this statistic is the square of the sum of k orthogonal random variables. He also proposed a time-sequential procedure where the summary chi-square was computed after each failure and the test statistic was the maximum of these chi-squares over the course of the study. Mantel, however, did not mention that it is possible to develop another time-sequential test on the basis of the distribution of the maximum of the set of k conditionally orthogonal chi-squares each with one degree of freedom from the k 2×2 contingency tables, along the lines of section 3.2. Structurally, Mantel's procedure is akin to the Type B PCS test.

4. ASYMPTOTIC DISTRIBUTION THEORY UNDER H_0

First, through the following two lemmas, we show that

$$\hat{\chi}_c^2 \text{ is non-decreasing in } c: b \le c \le k . \tag{4.1}$$

Lemma 4.1. *Let* $\{m_{ij,t};\ 1 \le t \le k_j,\ m^*_{ij,k_j+1},\ 1 \le j \le \ell,\ 1 \le i \le r\}$ *be a set of positive numbers such that* $\sum_{t=1}^{k_j} m_{ij,t} + m^*_{ij,k_j+1} = n_{ij},\ \forall\ 1 \le i \le r,\ 1 \le j \le \ell,$ *and let* m^*_{ij,c_j+1} *be defined as in before. Let then*

$$\chi_c^2(\underset{\sim}{m}) = \sum_{i=1}^{r} \sum_{j \in J_c} \left\{ \sum_{t=1}^{c_j} \frac{(n_{ij,t} - m_{ij,t})^2}{m_{ij,t}} + \frac{(n^*_{ij,c_j+1} - m^*_{ij,c_j+1})^2}{m^*_{ij,c_j+1}} \right\},\ b \le c \le k . \tag{4.2}$$

Then, $\chi_c^2(\underset{\sim}{m})$ *is non-decreasing in* $c: b \le c \le k$.

Proof: By definition,

$$(4.3)\quad \chi^2_{c+1}(\underset{\sim}{m}) = \sum_{i=1}^{r} \sum_{j\in J_{c+1}} \left\{ \sum_{t=1}^{c_j+1} \frac{(n_{ij,t}-m_{ij,t})^2}{m_{ij,t}} + \frac{(n^*_{ij,c_j+2}-m^*_{ij,c_j+2})^2}{m^*_{ij,c_j+2}} \right\}$$

$$\geq \sum_{i=1}^{r} \sum_{j\in J_{c}} \left\{ \sum_{t=1}^{c_j+1} \frac{(n_{ij,t}-m_{ij,t})^2}{m_{ij,t}} + \frac{(n^*_{ij,c_j+2}-m^*_{ij,c_j+2})^2}{m^*_{ij,c_j+2}} \right\}$$

$$= \sum_{i=1}^{r} \sum_{j\in J_{c}} \left\{ \sum_{t=1}^{c_j} \frac{(n_{ij,t}-m_{ij,t})^2}{m_{ij,t}} + \frac{(n_{ij,c_j+1}-m_{ij,c_j+1})^2}{m_{ij,c_j+1}} + \frac{(n^*_{ij,c_j+2}-m^*_{ij,c_j+2})^2}{m^*_{ij,c_j+2}} \right\} .$$

We now write $n_{ij,c_j+1} - m_{ij,c_j+1} = a_{ij}$, $n^*_{ij,c_j+2} - m^*_{ij,c_j+2} = d_{ij}$, $m_{ij,c_j+1} = e_{ij}$ and $m^*_{ij,c_j+2} = f_{ij}$. Then

$$(4.4)\quad \chi^2_{c+1}(\underset{\sim}{m}) - \chi^2_{c}(\underset{\sim}{m}) \geq \sum_{i=1}^{r} \sum_{j\in J_{c}} \left\{ \frac{(n_{ij,c_j+1}-m_{ij,c_j+1})^2}{m_{ij,c_j+1}} + \frac{(n^*_{ij,c_j+2}-m^*_{ij,c_j+2})^2}{m^*_{ij,c_j+2}} - \frac{(n^*_{ij,c_j+1}-m^*_{ij,c_j+1})^2}{m^*_{ij,c_j+1}} \right\}$$

$$= \sum_{i=1}^{r} \sum_{j\in J_{c}} \left\{ \frac{a^2_{ij}}{e_{ij}} + \frac{d^2_{ij}}{f_{ij}} - \frac{(a_{ij}+d_{ij})^2}{(e_{ij}+f_{ij})} \right\} .$$

Now,

$$(4.5)\qquad \frac{a_{ij}^2}{e_{ij}} + \frac{d_{ij}^2}{f_{ij}} - \frac{(a_{ij}+d_{ij})^2}{(e_{ij}+f_{ij})}$$

$$= \frac{a_{ij}^2(e_{ij}\cdot f_{ij}+f_{ij}^2)+d_{ij}^2(e_{ij}\cdot f_{ij}+e_{ij}^2)-e_{ij}f_{ij}(a_{ij}+d_{ij})^2}{e_{ij}f_{ij}(e_{ij}+f_{ij})}$$

$$= \frac{(a_{ij}^2 f_{ij}^2+d_{ij}^2 e_{ij}^2-2a_{ij}d_{ij}e_{ij}f_{ij})}{e_{ij}f_{ij}(e_{ij}+f_{ij})}$$

$$= \frac{(a_{ij}f_{ij}-d_{ij}e_{ij})^2}{e_{ij}f_{ij}(e_{ij}+f_{ij})} \geq 0, \ \forall \ \text{real} \ a_{ij} \ \text{and} \ d_{ij},$$

since e_{ij} and f_{ij} are positive. □

As in after (3.2), let $\underset{\sim}{\theta}_c$ be the minimum chi-square estimator of $\underset{\sim}{\theta}$ based on (3.1), so that by (3.2),

$$(4.6)\qquad \hat{\chi}_c^2 = \chi_c^2(\underset{\sim}{\hat{m}}_c);\ \underset{\sim}{\hat{m}}_c = (\hat{m}_{ij,t} = n_{ij}\pi_{ij,t}(\underset{\sim}{\tilde{\theta}}_c),\ \forall\ i,j,t)\ .$$

<u>Lemma 4.2</u>. $\chi_c^2(\underset{\sim}{m}) \geq \chi_c^2(\underset{\sim}{\hat{m}}_c),\ \forall\ \underset{\sim}{m} \neq \underset{\sim}{\hat{m}}_c,\ b \leq c \leq k.$

The proof directly follows from the fact that $\underset{\sim}{\tilde{\theta}}_c$, being the minimum chi-square estimator of $\underset{\sim}{\theta}$ based on ϕ_c, $\underset{\sim}{\hat{m}}_c = \underset{\sim}{m}(\underset{\sim}{\tilde{\theta}}_c)$ leads to the minimum (over all $\underset{\sim}{m}$) value of $\chi_c^2(\underset{\sim}{m})$.

From the preceeding two lemmas, we immediately conclude that

$$(4.7)\qquad \hat{\chi}_c^2 = \chi_c^2(\underset{\sim}{\hat{m}}_c) \leq \chi_c^2(\underset{\sim}{\hat{m}}_{c+1}) \leq \hat{\chi}_{c+1}^2(\underset{\sim}{\hat{m}}_{c+1}) = \hat{\chi}_{c+1}^2,\ \forall\ c\ .$$

By the results of Neyman (1949), (4.7) holds (upto the order of $o_p(1)$) for any sequence of BAN estimators.

4.1. <u>Asymptotic null distribution of Type A test statistic</u>. Let

$$(4.8)\qquad f_c = r\left(\sum_{j\in J_c} c_j\right) - s + q(c),\ b \leq c \leq k\ ,$$

so that by the compatibility condition of the design f_c is $\nearrow$ in c. Further we let

$$d_c = f_c - f_{c-1} \quad \text{for} \quad b < c \le k \quad \text{and} \quad d_b = f_b \ . \tag{4.9}$$

Finally, let $\chi^2_{p,\gamma}$ be the upper $100\gamma\%$ point of the chi-square distribution with p degrees of freedom (d.f.). It follows then from the basic results of Cramér (1946, pp. 424-34) that under the regularity conditions stated in Birch (1964), for every $c: b \le c \le k$,

$$\hat{\chi}^2_c \xrightarrow{\mathcal{D}} \chi^2_{f_c} \quad \text{when} \quad H_{0,c} \quad \text{holds.} \tag{4.10}$$

However, the $\hat{\chi}^2_c$, $b \le c \le k$ are obviously (even asymptotically) not independent. But by (4.1), and (4.10),

$$\begin{aligned} &P\{\hat{\chi}^2_c > \omega_\alpha \quad \text{for some} \quad c: b \le c \le k \mid H_{0,c}\} \\ &= P\{\hat{\chi}^2_k > \omega_\alpha \mid H_0\} \to \alpha, \quad \text{if} \quad \omega_\alpha = \chi^2_{f_k,\alpha} \ . \end{aligned} \tag{4.11}$$

Note that $\omega_\alpha = \chi^2_{f_k,\alpha}$ is the critical value of the overall test (based on completion of experiment upto the time $T_k = T$). Hence the Type A test consists in choosing the same critical value of the overall test but allows the possibility of an early rejection without any increased risk of making incorrect decision.

4.2. Asymptotic null distribution of the Type B test statistic. By (4.1), (4.10) and the classical Cochran theorem [viz. Searle (1971, p. 64)], under the same (Cramér-Birch) regularity conditions, under H_0

$$D_c \xrightarrow{\mathcal{D}} \chi^2_{d_c} \quad \text{for} \quad b \le c \le k \ . \tag{4.12}$$

Again, $\hat{\chi}^2_k = D_b + \ldots + D_k$, $D_c \ge 0$, $\forall$ c and $f_k = d_b + \ldots + d_k$, $d_c > 0, \forall$ c, so that $\{D_c: b \leq c \leq k\}$ forms an asymptotically independent set of chi-square statistics, see Searle (1971, pp. 60-61). Hence, asymptotically,

$$\text{(4.13)} \quad P\{D_c \leq \Delta_c,\ \forall\ b \leq c \leq k | H_0\} \to \prod_{c=b}^{k} P\{D_c \leq \Delta_c | H_{0,c}\} \to \prod_{c=b}^{k} P\{\chi^2_{d_c} \leq \Delta_c\} ,$$

and thus, if we let

$$\text{(4.14)} \qquad \Delta_c = \chi^2_{d_c,\alpha'},\ b \leq c \leq k;\ 1-\alpha = (1-\alpha')^{k-b+1} ,$$

we obtain from (4.13) and (4.14) that

$$\text{(4.15)} \qquad P\{D_c \leq \chi^2_{d_c,\alpha'},\ \forall\ b \leq c \leq k | H_0\} \to 1-\alpha .$$

As a result, the Type B test consists in comparing the successive differences $\{D_c: b \leq c \leq k\}$ with the corresponding critical values $\{\chi^2_{d_c,\alpha'}: b \leq c \leq k\}$ and rejecting the H_0 at the earliest possible value of c, if there is any at all.

4.3. Asymptotic null distribution of Type C test statistic. Let at the c-th censoring stage, ${}_jq(c)$ be the number of restriction on the model for the j-th batch. Because of the compatibility condition ${}_jq(c)$ is $\nearrow$ in c. We also define

$$\text{(4.16)} \qquad {}_jf_c = r(c - b_0 - h_j) - s + {}_jq(c),\ b_0 + h_j \leq c \leq k .$$

Due to the compatibility condition of the design ${}_jf_c$ also is $\nearrow$ in c. Further we let

$$\text{(4.17)} \qquad {}_jd_c = {}_jf_c - {}_jf_{c-1},\ b_0 + h_j < c \leq k;\ j = 1,\ldots,\ell .$$

$$\qquad {}_jd_c = {}_jf_c,\ c = b_0 + h_j$$

Leaving the details, we proceed as in section 4.1 and obtain

(4.18) $$P\{{}_j\chi_c^2 > {}_j\omega_{\alpha''}, \text{ for some } c: b_0 + h_j \le c \le k / {}_jH_{0,c}\}$$
$$= P\{{}_j\hat{\chi}_k^2 > {}_j\omega_{\alpha''} | {}_jH_0\} \to \alpha'' \text{ if } {}_j\omega_{\alpha''} = \hat{\chi}^2_{{}_jf_k,\alpha''} .$$

where we conceive of H_0 as

(4.19) $$H_0 \equiv \bigcap_{j=1}^{\ell} \bigcap_{c=b_0+h_j}^{k} {}_jH_{0,c}, \; {}_jH_{0,c} \text{ is monotonic}$$

(4.20) $${}_jH_0 = \bigcap_{c=b_0+h_j}^{k} {}_jH_{0,c}$$

(4.21) $${}_jH_{0,c}: {}_jF_m^{(c)}(\underset{\sim}{\theta}) = 0, \; \forall \; 1 \le m \le {}_jq(c) .$$

Consequently,

(4.22) $$P\{{}_j\hat{\chi}_c^2 \le {}_j\omega_{\alpha''}, \forall \; c: b_0 + h_j \le c \le k | {}_jH_0\}$$
$$= P\{{}_j\hat{\chi}_k^2 \le {}_j\omega_{\alpha''} | {}_jH_0\} \to 1-\alpha'' \text{ and}$$

(4.23) $$P\{{}_j\hat{\chi}_c^2 \le {}_j\omega_{\alpha}, \forall \; c: b_0 + h_j \le c \le k \text{ and } \forall \; j = 1,\dots,\ell | H_0\}$$
$$\to (1-\alpha'')^{\ell} , \text{ because the batches are independent .}$$

Therefore, the overall level of significance α is

(4.24) $$P\{{}_j\hat{\chi}_c^2 > {}_j\omega_{\alpha''} \text{ for some } c: b_0 + h_j \le c \le k$$
$$\text{and some } j = 1,\dots,\ell | {}_jH_{0,c}\} \to 1 - (1-\alpha'')^{\ell} = \alpha .$$

As such, Type C test consists in using the same critical value at all stages of censoring for the same batch, but different critical values for the different batches.

4.4. Asymptotic null distribution of Type D test statistic. By similar arguments as in section 4.2, under H_0,

$$ {}_jD_c \overset{\mathcal{D}}{\rightarrow} \chi^2_{{}_jd_c}, \quad b_0 + h_j \le c \le k \;. \tag{4.25}$$

Also,

$$ {}_j\chi^2_k = \sum_{c=b_0+h_j}^{k} {}_jD_c, \quad {}_jD_c \ge 0, \ \forall\ c \quad \text{and} \tag{4.26}$$

$$ {}_jf_k = \sum_{c=b_0+h_j}^{k} {}_jd_c, \quad {}_jd_c > 0, \ \forall\ c \;,$$

so that $\{{}_jD_c : b_0 + h_j \le c \le k\}$ forms an asymptotically independent set of chi-square statistics. Asymptotically, then

$$\begin{aligned} &P\{{}_jD_c \le {}_j\Delta_c, \ \forall\ b_0 + h_j \le c \le k \,|\, {}_jH_0\} \\ &\rightarrow \prod_{c=b_0+h_j}^{k} P\{{}_jD_c \le {}_j\Delta_c \,|\, {}_jH_{0,c}\} \\ &\rightarrow \prod_{c=b_0+h_j}^{k} P\{\chi^2_{{}_jd_c} \le {}_j\Delta_c\} \;. \end{aligned} \tag{4.27}$$

If we let ${}_j\Delta_c = \chi^2_{{}_jd_c, {}_j\alpha'''}$, $b_0 + h_j \le c \le k$; $(1-\alpha'') = (1-\alpha''')^{(k-b_0-h_j+1)}$ we obtain

$$ P\{{}_jD_c \le \chi^2_{{}_jd_c, \alpha'''}, \ \forall\ b_0 + h_j \le c \le k \,|\, {}_jH_0\} \rightarrow 1-\alpha'' \tag{4.28}$$

Further if $1-\alpha = (1-\alpha'')^{\ell}$

$$\begin{aligned} &P\{{}_jD_c \le \chi^2_{{}_jd_c, \alpha'''}, \ \forall\ b_0 + h_j \le c \le k, \ \forall\ j = 1,\ldots,\ell \,|\, H_0\} \\ &\rightarrow (1-\alpha'')^{\ell} = 1-\alpha \;. \end{aligned} \tag{4.29}$$

Consequently, the Type D test amounts to comparing successive differences

$\{{}_jD_c\colon b_0 + h_j \le c \le k;\ j = 1,\dots,\ell\}$ with the respective critical values $\{\chi^2_{{}_jd_c,\alpha'''},\ b_0 + h_j \le c \le k;\ j = 1,\dots,\ell\}$ for an early rejection.

4.5. Restrictions on the probability structure with BAM. With BAM, we test the null hypothesis of homogeneity of the r samples, the batch arrivals being assumed to have uniform pattern for every sample. It is not unrealistic to further assume that the batches within the same sample have the same underlying distribution. As such, we may impose additional restrictions on the probability structure of the model as

$$\Pi_{ij,t} = \Pi_{ij',t},\ j < j';\ t = 1,\dots,k_{j'},\ \forall\ i\ , \tag{4.30}$$

apart from the restraints on $\Pi_{ij,t}$ under H_0.

4.6. Some special cases. With single batch in one sample situation the sequence of D_c statistics will uniformly have one degree of freedom, assuming general model. When the model is unspecified i.e., when parametric dependence is not assumed in the model, we encounter the case of contingency table (with single batch). With r samples D_c will uniformly have $r-1$ degrees of freedom, for every c, in a contingency table situation. In the two-sample problem of Mantel (1966), D_c will always have one degree of freedom.

5. ASYMPTOTIC DISTRIBUTION THEORY OF THE PROPOSED TEST STATISTICS UNDER LOCAL ATLTERNATIVES AND POWER CONSIDERATIONS.

5.1. Consistency of the test-statistics. We have under $H_{0,c}\colon F_m^{(c)}(\underset{\sim}{\theta}) = 0$, $\forall\ 1 \le m \le q(c)$ and under the fixed alternatives $H_{1,c}\colon F_m^{(c)}(\underset{\sim}{\theta}) \ne 0$, for at least one m. Let $\Pi_{ij,t} = \Pi^0_{ij,t}$ under $H_{0,c}$ and $\Pi_{ij,t} = \Pi^1_{ij,t}$ under

$H_{1,c}$, $i = 1,\ldots,r$; $j = 1,\ldots,\ell$; $t = 1,\ldots,c_j$. Then under the Cramér regularity conditions modified by Birch (1964) $\Pi_{ij,t}(\tilde{\underset{\sim}{\theta}}_c) \xrightarrow{p} \Pi^0_{ij,t}$ when $H_{0,c}$ is true and $\Pi_{ij,t}(\hat{\underset{\sim}{\theta}}_c) \xrightarrow{p} \Pi^1_{ij,t}$ when $H_{1,c}$ is true. Therefore, as n gets large, under $H_{1,c}$, $n^{-1}\hat{\chi}^2_c \xrightarrow{p} E_c$, where $E_c(>0)$ is a constant depending on c and n is the combined sample size. As $\chi^2_{r(k_1+\ldots+k_\ell)-s+q,\alpha}$ is $0(1)$, the probability of $\hat{\chi}^2_c$ exceeding $\chi^2_{r(k_1+\ldots k_\ell)-s+q,\alpha}$ tends to one as $n \to \infty$. By similar reasoning, the powers of the other three tests viz. Type B, Type C, and Type D can be shown to go to one as $n \to \infty$. Consequently, in order to compare the asymptotic performance of the four statistics, we consider only local alternatives.

5.2. <u>Asymptotic distribution theory of</u> $\hat{\chi}^2_c$. Let under $H_{0,c}$ and local alternative $H^{(n)}_{1,c}$

(5.1)
$$H_{0,c}:\ F^{(c)}_m(\underset{\sim}{\theta}) = 0,\ \forall\ 1 \le m \le q(c)$$
$$H^{(n)}_{1,c}:\ F^{(c)}_m(\underset{\sim}{\theta}) = n^{-\frac{1}{2}}e_m,\ \forall\ 1 \le m \le q(c)$$

where

(5.2) $e_m \neq 0$ for at least one m; $n = \sum_{i=1}^{r}\sum_{j=1}^{\ell}\left\{\sum_{t=1}^{c_j} n_{ij,t} + n^*_{ij,c_j+1}\right\}$;

$\lim_{n\to\infty} \frac{n_{ij}}{n} = Q_{ij}$, $0 < Q_{ij} < 1$.

For convenience we reparaphrase $H_{0,c}$ and $H^{(n)}_{1,c}$ as

(5.3)
$$H_{0,c}:\ \Pi^0_{ij,t} = \Pi_{ij,t}(\underset{\sim}{\theta}^0);\ \underset{\sim}{\theta}^{0'} = (\theta^0_1,\ldots,\theta^0_s)$$
$$F^{(c)}_m(\underset{\sim}{\theta}^0) = 0,\ \forall\ 1 \le m \le q(c)\ .$$

or

$$\Pi^{0,c}_{ij,t} = \Pi_{ij,t}(\underset{\sim}{\theta}^0_c), \quad \underset{\sim}{\theta}^{0'}_c = (\theta^0_{1c},\ldots,\theta^0_{s-q(c),c})$$

$$H^{(n)}_{1,c}: \ \Pi^{(n)}_{ij,t} = \Pi^0_{ij,t} + n^{-\frac{1}{2}}\delta_{ij,t}$$

$$F^{(c)}_m(\underset{\sim}{\theta}^0) = 0, \ \forall\ 1 \le m \le q(c), \ i = 1,\ldots,r\ ;$$

$$j = 1,\ldots,\ell;\ t = 1,\ldots,c_j\ .$$

Note that for $t = c_j + 1$

$$\Pi^{*(n)}_{ij,c_j+1} = \Pi^{*0}_{ij,c_j+1} + n^{-\frac{1}{2}}\delta_{ij,c_j+1} \tag{5.4}$$

$$\delta_{ij,c_j+1} = -\sum_{t=1}^{c_j}\delta_{ij,t}, \quad \text{since} \quad \sum_{t=1}^{c_j}\delta_{ij,t} + \delta^*_{ij,c_j+1} = 0,\ \forall j$$

We are assuming that at least one δ_{ijt} is non-zero. Then by theorem 3.1 of Diamond (1963) under the regularity conditions (a), (b), (c), (d) of Mitra (1958) and (e) and (f) of Diamond (1963), under $H^{(n)}_{1,c}$

$$\hat{\chi}^2_c \overset{\mathcal{D}}{\sim} \chi^2_{f_c, G_c}, \quad c = b,\ldots,k; \quad \text{where} \tag{5.5}$$

$$G_c = \underset{\sim}{\delta}'_c[I - \underset{\sim}{B}(\underset{\sim}{B}'\underset{\sim}{B})^{-1}\underset{\sim}{B}']\underset{\sim}{\delta}_c \tag{5.6}$$

$$\underset{\sim}{\delta}_c(R_c \times 1) = \{(Q_{ij}/\Pi^0_{ij,t})^{\frac{1}{2}}\delta_{ij,t}\}$$

$$R_c = r\left(\sum_{j\in J_c}(c_j+1)\right)$$

$$B(R_c \times (s-q(c))) = \left\{(Q_{ij}/\Pi^0_{ij,t})^{\frac{1}{2}}\left(\frac{\partial\Pi_{ij,t}(\underset{\sim}{\theta}_c)}{\partial\theta_c}\right)\underset{\sim}{\theta}_c = \underset{\sim}{\theta}^0_c\right\}\ .$$

In the above expressions for $\underset{\sim}{\delta}_c$ and $\underset{\sim}{B}$, $\Pi^*_{ij,c_j+1} = \Pi_{ij,c_j+1}$ and $\delta^*_{ij,c_j+1} = \delta_{ij,c_j+1}$. These have not been distinguished in order to avoid clumsiness.

Let A^* be the event that $\hat{\chi}^2_c > \omega_\alpha$ for some c; $b \le c \le k$. Then the power $P(A^*)$ is given by

$$(5.7)\qquad P(A^*) = P\{\hat{\chi}_c^2 > \omega_\alpha \ \text{ for some } \ c:\ b \le c \le k | H_{1,c}^{(n)}\}$$

$$= P\{\hat{\chi}_k^2 > \omega_\alpha | H_1^{(n)}\} \to 1-\beta \ ,$$

where β is the error of type II and $H_1^{(n)} = \bigcap_{c=b}^{k} H_{1,c}^{(n)}$. As such, by this testing procedure both the type I and type II errors remain the same as those of the overall test. Now, A^* is the union of the following disjoint events:

$$(5.8)\qquad A_b \equiv \{\hat{\chi}_b^2 > \omega_\alpha\}$$

$$A_{b+1} \equiv \{\hat{\chi}_b^2 \le \omega_\alpha,\ \hat{\chi}_{b+1}^2 > \omega_\alpha\}$$

$$\cdots$$

$$A_c \equiv \{\hat{\chi}_b^2 \le \omega_\alpha,\ \hat{\chi}_{b+1}^2 \le \omega_\alpha, \ldots, \hat{\chi}_{c-1}^2 \le \omega_\alpha,\ \hat{\chi}_c^2 > \omega_\alpha\}$$

$$\equiv \{\hat{\chi}_{c-1}^2 \le \omega_\alpha,\ \hat{\chi}_c^2 > \omega_\alpha\}$$

$$\cdots$$

$$A_k \equiv \{\hat{\chi}_{k-1}^2 \le \omega_\alpha,\ \hat{\chi}_1^2 > \omega_\alpha\} \ .$$

Let T_C be the time-point at which the experiment is terminated. The event $T_C = T_c$ is the event A_c. Therefore, the expected stopping time with this test is

$$(5.9)\qquad \tau_A = E(T_C) = \sum_{c=b}^{k} T_c P(A_c) + T(1-P(A^*)) \ .$$

The probabilities of the constituent (disjoint—) events of A are given by:

(5.10) $$P(A_b) \to P\{\hat{\chi}_b^2 > \omega_\alpha\}$$

$$= \sum_{u=0}^{\infty} \int_{\omega_\alpha}^{\infty} \frac{e^{-\lambda b}(\lambda_b)^u}{u!} \zeta_{2u+d_b}(x)dx, \quad \text{where}$$

$\lambda_b = \frac{1}{2} G_b$ and $\zeta_{2u+d_b}(x)$ is the density of the central chi-square distribution with $2u + d_b$ d.f.

For $c > b$,

(5.11) $$P(A_c) = P\{\hat{\chi}_{c-1}^2 \le \omega_\alpha, \ \hat{\chi}_c^2 > \omega_\alpha\} .$$

By Searle (1971, pp. 60-61), $\hat{\chi}_{c-1}^2$ which is asymptotically distributed as $\chi^2_{f_{c-1}, G_{c-1}}$ is asymptotically independent of $D_c = \hat{\chi}_c^2 - \hat{\chi}_{c-1}^2$ which is asymptotically distributed as $\chi^2_{d_c, (G_c - G_{c-1})} = \chi^2_{d_c, G_c^*}$. If now, $\hat{\chi}_{c-1}^2 \le x \le \omega_\alpha$, then $D_c > \omega_\alpha - x \Rightarrow \hat{\chi}_c^2 > \omega_\alpha$. As such,

(5.12) $$P(A_c) \to \sum_{u=0}^{\infty} \sum_{v=0}^{\infty} \int_0^{\omega_\alpha} \left[\frac{e^{-\lambda_{c-1}}(\lambda_{c-1})^u}{u!} \zeta_{2u+f_{c-1}}(x)dx \int_{\omega_\alpha - x}^{\infty} \frac{e^{-\lambda_c^*}(\lambda_c^*)^v}{v!} \zeta_{2v+d_c}(z)dz \right] \quad \text{where}$$

$$\lambda_c = \tfrac{1}{2} G_c \quad \text{and} \quad \lambda_c^* = \lambda_c - \lambda_{c-1} .$$

5.3. <u>Asymptotic distribution theory of</u> D_c. It has already been proved in section 5.2 that under local alternatives $H_1^{(n)}$

(5.13) $$D_c \overset{\mathcal{D}}{\sim} \chi^2_{d_c, G_c^*}, \quad c = b, \ldots, k .$$

Letting $\chi^2_{d_c, \alpha'} = y_c$, we have

$$(5.14)\quad 1\text{-power} = P\{D_c \le y_c,\ \forall\ b \le c \le k \,|\, H_1^{(n)}\}$$
$$\to \prod_{c=b}^{k} P\{D_c \le y_c \,|\, H_{1,c}^{(n)}\} \to \prod_{p=0}^{k-b} \left(\sum_{u_p=0}^{\infty} \int_0^{y_{p+b}} \frac{e^{-\lambda^*_{p+b}} (\lambda^*_{p+b})^{u_p}}{u_p!} \zeta_{2u_p+d_{p+b}}(x)\,dx \right) .$$

Let B_c be the event that the experiment terminates at the censoring stage c. Then

$$(5.15)\qquad \begin{aligned} B_b &\equiv \{D_b > y_b\} \\ B_{b+1} &\equiv \{D_b \le y_b,\ D_{b+1} > y_{b+1}\} \\ &\cdots \\ B_c &\equiv \{D_b \le y_b,\ D_{b+1} \le y_{b+1}, \ldots, D_{c-1} \le y_{c-1},\ D_c > y_c\} \\ &\cdots \\ B_k &\equiv \{D_b \le y_b, \ldots, D_{k-1} \le y_{k-1},\ D_k > y_k\} . \end{aligned}$$

These are the mutually exclusive subevents of B^* which is the event that $D_c > y_c$, for some c. We compute respective probabilities as

$$(5.16)\qquad P(B_b) = \sum_{u=0}^{\infty} \int_{y_b}^{\infty} \frac{e^{-\lambda^*_b} (\lambda^*_b)^u}{u!} \zeta_{2u+d_b}(x)\,dx$$
$$P(B_c) = \prod_{p=0}^{c-1-b} \left(\sum_{u_p=0}^{\infty} \int_0^{y_{p+b}} \frac{e^{-\lambda^*_{p+b}} (\lambda^*_{p+b})^{u_p}}{u_p!} \zeta_{2u_p+d_{p+b}}(x)\,dx \right)$$
$$\sum_{u=0}^{\infty} \int_{y_c}^{\infty} \frac{e^{-\lambda^*_c} (\lambda^*_c)^u}{u!} \zeta_{2u+d_c}(x)\,dx$$

From the above we obtain $E(T_C) = \tau_B = \sum_{c=b}^{k} T_c P(B_c) + T(1-P(B^*))$.

5.4. Asymptotic performance properties of Type C test. Define similarly non-centrality parameters as in section 5.2 but for individual batches and index them by j. As for example, we denote by ${}_jG_c$, the non-centrality parameter associated with the asymptotic chi-square distribution of the test statistic ${}_j\hat{\chi}^2_c$ at the time-point T_c for the j-th batch and ${}_j\lambda_c = \frac{1}{2}\,{}_jG_c$ under local alternatives ${}_jH^{(n)}_{1,c}$. Note that both ${}_jH_{0,c}$ and ${}_jH^{(n)}_{1,c}$ are monotonic and as in (4.19) and (4.20), we conceive of ${}_jH^{(n)}_1$ and $H^{(n)}_1$ as

$$(5.17)\qquad {}_jH^{(n)}_1 \equiv \bigcap_{c=b_0+h_j}^{k} {}_jH^{(n)}_{1,c};\; H^{(n)}_1 = \bigcap_{j=1}^{\ell} {}_jH^{(n)}_1 .$$

The power of the test is expressed as

$$(5.18)\quad 1\text{-power} = P\{{}_j\hat{\chi}^2_c \le {}_j\omega_{\alpha''},\ \forall\ c:\ b_0+h_j \le c \le k \text{ and } \forall\ j=1,\ldots,\ell \mid H^{(n)}_1\}$$

$$= \prod_{j=1}^{\ell} P\{{}_j\hat{\chi}^2_k \le {}_j\omega_{\alpha''} \mid {}_jH^{(n)}_1\}$$

$$= \prod_{j=1}^{\ell}\left(\sum_{u=0}^{\infty}\int_0^{{}_j\omega_{\alpha''}} \frac{e^{-{}_j\lambda_k}({}_j\lambda_k)^u}{u!}\,\zeta_{2u+{}_jf_k}(x)\,dx\right)$$

The event E^* that ${}_j\hat{\chi}^2_c > {}_j\omega_{\alpha''}$, for some $c:\ b_0+h_j \le c \le k$ and $j=1,\ldots,\ell$ consists of the following mutually exclusive events:

$$(5.19)\quad E_{b_0} \equiv \{{}_1\hat{\chi}^2_{b0} > {}_1\omega_{\alpha''}\}$$

$$\cdots$$

$$E_c \equiv \{{}_j\hat{\chi}^2_{c-1} \le {}_j\omega_{\alpha''},\ \text{for all } 1 \le j \le \ell_c \text{ and } {}_j\hat{\chi}^2_c > {}_j\omega_{\alpha''},\ \text{for some}$$

$$j=1,\ldots,\ell_c\},\ b_0+h_j \le c \le k$$

$$\cdots$$

Note that there are only ℓ_c j's which satisfy: $b_0+h_j \le c \le k$. The corresponding probabilities are given by

$$(5.20)\quad P(E_{b_0}) = \sum_{u=0}^{\infty}\int_{{}_1\omega_{\alpha''}}^{\infty} \frac{e^{-{}_1\lambda_{b_0}}({}_1\lambda_{b_0})^u}{u!}\,\zeta_{2u+{}_1d_{b_0}}(x)\,dx$$

$$\cdots$$

$$P(E_c) = \prod_{j=1}^{\ell_c}\left(\sum_{u_j=0}^{\infty}\int_0^{{}_j\omega_{\alpha''}}\left[\frac{e^{-{}_j\lambda_{c-1}}({}_j\lambda_{c-1})^{u_j}}{u_j!}\,\zeta_{2u_j+{}_jf_{c-1}}(x)\,dx\right.\right.$$

$$\left.\left.\left\{1-\prod_{j=1}^{\ell_c}\sum_{v_j=0}^{\infty}\int_0^{{}_j\omega_{\alpha''}-x}\frac{e^{-{}_j\lambda^*_c}({}_j\lambda^*_c)^{v_j}}{v_j!}\,\zeta_{2v_j+{}_jd_c}(z)\,dz\right\}\right]\right)$$

$$\cdots$$

Expected stopping time τ_C is then

(5.21) $$E(T_C) = \tau_C = \sum_{c=b_0}^{k} T_c\, P(E_c) + T(1-P(E^*))$$

5.5. Asymptotic performance properties of Type D test. Let

(5.22) $$\chi^2_{{}_jd_c,\alpha'''} = {}_jy_c\ .$$

Then

(5.23) $$1\text{-power} = P\{{}_jD_c \le {}_jy_c,\ \forall\ j = 1,\dots,\ell \text{ and } \forall\ c\colon b_0 + h_j \le c \le k \,|\, H_1^{(n)}\}$$

$$\to \prod_{j=1}^{\ell} \prod_{c=b_0+h_j}^{k} P\{{}_jD_c \le {}_jy_c \,|\, {}_jH_{1,c}^{(n)}\}$$

$$\to \prod_{j=1}^{\ell} \prod_{c=b_0+h_j}^{k} \left(\sum_{{}_ju_c=0}^{\infty} \int_0^{{}_jy_c} \frac{e^{-{}_j\lambda_c} ({}_j\lambda_c)^{{}_ju_c}}{{}_ju_c!}\, \zeta_{2\,{}_ju_c + {}_jd_c}(x)\,dx \right) .$$

The event that the experiment terminates at some time-point T_c consists of the mutually exclusive events

(5.24) $$F_{b_0} \equiv \{{}_1D_{b_0} > {}_1y_{b_0}\}$$

$$F_c \equiv \{{}_jD_{c-1} \le {}_jy_{c-1} \text{ for all } 1 \le j \le \ell_c,$$

$${}_jD_c \le {}_jy_c, \text{ for some } j = 1,\dots,\ell_c\} ,$$

$$b_0 + h_j \le c \le k\ .$$

The expressions for the associated probabilities are given by

$$(5.25)\quad P(F_{b_0}) \to \sum_{u=0}^{\infty} \int_{{}_1y_{b_0}}^{\infty} \frac{e^{-{}_1\lambda^*_{b_0}} ({}_1\lambda^*_{b_0})^u}{u!} \zeta_{2u+{}_1d_{b_0}}(x)\,dx$$

$$\cdots$$

$$P(F_c) = \prod_{j=1}^{\ell_{c-1}} \prod_{v=0}^{c-1-b_0-h_j} \left(\sum_{u_v=0}^{\infty} \int_0^{{}_jy_{v+b_0+h_j}} \cdot \frac{e^{-{}_j\lambda^*_{v+b_0+h_j}} ({}_j\lambda^*_{v+b_0+h_j})^{u_v}}{u_v!} \zeta_{2u_v+{}_jd_{v+b_0+h_j}}(x)\,dx \right)$$

$$\left(1 - \prod_{j=1}^{\ell_c} \left(\sum_{u_j=0}^{\infty} \int_0^{{}_jy_c} \frac{e^{-{}_j\lambda^*_c} ({}_j\lambda^*_c)^{u_j}}{u_j!} \zeta_{2u_j+{}_jd_c}(x)\,dx \right)\right).$$

The expected stopping time τ_D can now be computed as before.

6. CONCLUDING REMARKS

For grouped data, Ghosh (1973) has extended the results of Sen (1967) in deriving a class of conditionally distribution-free rank order tests having some asymptotically optimal properties. It is possible to extend these test procedures to our batch arrival model under PCS. However, this will need certain weak convergence results to multi-dimensional Brownian motion processes as well as a (crude) inequality involving the actual and an upper bound for a scalar constant appearing in the test statistic. The details will be provided in a separate communication.

The proposed test procedure can also be extended to the multi-dimensional categorical data situation, where the main response is time-sequential and at each time-point of censoring, responses with regard to the other categories are complete. Following Bahadur (1961), if it is possible to derive an approximate lower dimensional representation of the joint probability function, we can start censoring at an earlier stage for purposes of tests under PCS.

7. NUMERICAL ILLUSTRATION

7.1. Description of the model and underlying distribution. For purposes of numerical illustration, we have considered a BAM with two samples and two batches. For the sake of clarification, the two samples may be conceived of as control and treatment groups in a contraceptive effectiveness study with the subjects entering into the experiment in batches. The response categories are conception times recorded in one-month intervals starting from the time-points of the individuals' entry into the trial. From time and cost considerations, the study may not be continued beyond a time-point T. Allowing for seasonal variation in reproductive behavior, the batches within the same sample may be assumed to differ in respect of conception time distribution. Keeping this in view, two samples of 700 each were generated from exponential distributions with mean survival times (i) $\lambda_1 = \lambda_2 = 16.0$ months, (ii) $\lambda_1 = 16.0$ months, $\lambda_2 = 18.0$ months and (iii) $\lambda_1 = 16.0$ months, $\lambda_2 = 20.0$ months. The size of the first batch within each sample was 500; the remaining 200 individuals belonged to the second batch. It was assumed that the experiment was observed for a period of 24 months since inception. With the further assumption that the second batch entered into the study one year after the first, the failures in the two batches were recorded in 24 month-intervals (categories) for the first batch and in 12 month-intervals for the second batch. The last categories of the two batches (25^{th} and 13^{th} respectively) included the censored individuals.

7.2. Computation of the statistics. In order to derive the test statistics A, B, C, and D, we started with unspecified model, allowing for inter-batch difference in probability structure. Under the null hypothesis of no difference between the survival time distribution of the two samples, $\hat{\chi}_c^2$ and ${}_j\hat{\chi}_c^2$ were computed by plugging in the maximum likelihood estimators $\hat{\Pi}_{ij,t}$ derived separately from the two batches, in (3.2). It is important to note that in this unspecified model situation, both $\{\hat{\chi}_c^2\}$ and $\{{}_j\hat{\chi}_c^2\}$ form strictly non-decreasing sequence in c by virtue of Lemma 4.1. The asymptotic powers and mean stopping times for the four tests were derived empirically by repeating the experiment 500 times. The results are summarized in the next section.

7.3. Empirical powers and mean stopping times. By the formulae of section 4, we can at once derive the degrees of freedom associated with the various tests. The degrees of freedom of the overall test, $\hat{\chi}_{24}^2$ are 36. We similarly obtain ${}_1f_{24} = 24$, ${}_2f_{12} = 12$; $d_c = 1$, $c = 1,\ldots,12$; $d_c = 2$, $c = 12,\ldots,24$; ${}_jd_c = 1$, $\forall\ c$ and $\forall\ j$. Next, the test statistics were compared against respective percentile points of the chi-square distributions (central), in order to compute empirical powers and empirical mean stopping times associated with the four tests. The results from the 500 repititions are demonstrated in the table below.

TABLE 1

Tests and Significance Levels

CHARACTERISTICS*	A		B		C		D	
	α=.10	α=.05	α=.10	α=.05	α=.10	α=.05	α=.10	α=.05
Case 1								
Power	.09	.05	.09	.03	.10	.05	.08	.03
Mean Stopping time	23.87	23.97	22.90	23.56	23.79	23.91	23.27	23.73
Case 2								
Power	.18	.10	.12	.08	.17	.09	.12	.07
Mean Stopping time	23.66	23.84	22.36	22.89	23.63	23.82	22.65	23.21
Case 3								
Power	.48	.35	.28	.18	.41	.28	.24	.14
Mean Stopping time	22.56	23.09	20.14	21.47	22.38	23.01	21.01	22.27

*Case 1: $\lambda_1 = 16.0$, $\lambda_2 = 16.0$

Case 2: $\lambda_1 = 16.0$, $\lambda_2 = 18.0$

Case 3: $\lambda_1 = 16.0$, $\lambda_2 = 20.0$

From the above the following salient points emerge:

(i) All the tests are consistent

(ii) Type A is the most powerful test; next comes Type C; D is the least powerful of all.

(iii) If we assume that each extra time interval costs s unit of cost more, then the expected cost for each test (apart from overhead cost) is $s \times$ expected stopping time. From this consideration, Type B is the most economical test; D, C, A come in this order so far as this index of efficieny is concerned.

(iv) The efficacy of the PCS test procedures becomes more and more pronounced as the difference between the two samples gets larger and larger.

7.4. Additional comments. Note that we could have derived an overall chi-square statistic with 48 degrees of freedom for the Type A test, by imposing the restriction on the model that the two batches within the same sample have the same probability structure. Type C and Type D tests are more flexible in the sense that we need not know beforehand the time-points of entry of different batches as in Type A and Type B tests. On the other hand, Type C and Type D tests do not utilize this inter-block structure. Among all the tests, Type A test only preserves the power of the overall test. In all the other tests power is affected, in fact reduced, since we increase the critical level of the overall test.

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