SOME PROPERTIES AND GENERALIZATIONS OF MULTIVARIATE MORGENSTERN DISTRIBUTIONS*

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May, 1976

Institute of Statistics Mimeo Series #1069

* This research was supported by the Air Force Office of Scientific Research under Grant AFOSR-75-2796.

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ABSTRACT

The admissible values of the coefficient in a bivariate Morgenstern distribution are found. For multivariate Morgenstern distributions necessary and sufficient conditions are given for its coefficients, and its conditional distributions are found and shown to belong to a family of distributions further extending the multivariate Morgenstern family.

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1. INTRODUCTION

Morgenstern [1] introduced a family of bivariate distributions $H(x_1,x_2)$, with marginals $F_1(x_1)$ and $F_2(x_2)$, of the form

$$H(x_1,x_2) = F_1(x_1)F_2(x_2)\{1 + \alpha[1-F_1(x_1)][1-F_2(x_2)]\} \qquad (1)$$

where α is a real constant. In Section 2 we find all values of α for which H as defined by (1) is a bivariate distribution, assuming of course that F_1 and F_2 are univariate distributions.

Johnson and Kotz [2] introduced a multivariate Morgenstern family of distributions $H(x_1,x_2,\ldots,x_n)$, with marginals $F_1(x_1),F_2(x_2),\ldots,F_n(x_n)$, by

$$\begin{aligned} H(x_1,x_2,\ldots,x_n) = F_1(x_1)F_2(x_2) \ldots F_n(x_n)\Big\{1 &+ \sum_{j_1<j_2}^{n-1}\sum^{n} \alpha_{j_1j_2}[1-F_{j_1}(x_{j_1})][1-F_{j_2}(x_{j_2})] \\ &+ \sum_{j_1<}^{n-2}\sum_{j_2<j_3}^{n-1}\sum^{n} \alpha_{j_1j_2j_3}[1-F_{j_1}(x_{j_1})][1-F_{j_2}(x_{j_2})][1-F_{j_3}(x_{j_3})] + \ldots \\ &+ \alpha_{12\ldots n}[1-F_1(x_1)][1-F_2(x_2)] \ldots [1-F_n(x_n)]\Big\} \qquad (2) \end{aligned}$$

where the coefficients α are real constants. In Section 2 we also give necessary and sufficient conditions on the coefficients α so that (2) defines an n-dimensional distribution, assuming again that $F_1,\ldots,F_n$ are univariate distributions.

In Section 3 we introduce a family of distributions closely related to the multivariate Morgenstern family, by inserting within the brackets on the right hand side of (2) the term $\sum_{j=1}^{n}\alpha_j[1-F_j(x_j)]$, and we show in Section 4 that if $X_1,\ldots,X_n$ have an n-dimensional Morgenstern distribution then the conditional distribution of $X_1,\ldots,X_k$ given $X_{k+1} = x_{k+1}$, $\ldots,X_n = x_n$ $(k = 1,\ldots,n-1)$ belongs to this family.

2. THE VALUES OF THE COEFFICIENTS α.

For a distribution function $F_i(x)$ let E_i be the set of all values of $F_i(x)$ with the exception of 0 and 1, i.e., E_i is the subset of (0,1) defined by

$$E_i = \{F_i(x),\ -\infty<x<+\infty\} - \{0,1\},$$

and let

$$m_i = \text{g.l.b. } E_i\ (= \inf E_i), \qquad M_i = \text{l.u.b. } E_i\ (= \sup E_i).$$

Then clearly $0 \le m_i \le M_i \le 1$ and we have the following

THEOREM 1. $H(x_1,x_2)$ *defined by (1) is a bivariate distribution if and only if*

$$\alpha_{min} = -\min\left\{\frac{1}{M_1M_2},\ \frac{1}{(1-m_1)(1-m_2)}\right\} \le \alpha \le \min\left\{\frac{1}{M_1(1-m_2)},\ \frac{1}{(1-m_1)M_2}\right\} = \alpha_{max}\ .$$

Proof. $H(x_1,x_2)$ is a bivariate distribution if and only if for all $x_1 < x_1'$ and $x_2 < x_2'$ we have

$$\Delta_{x_1,x_2}^{x_1',x_2'} H = H(x_1',x_2') - H(x_1,x_2') - H(x_1',x_2) + H(x_1,x_2) \ge 0.$$

But by (1),

$$\Delta_{x_1,x_2}^{x_1',x_2'} H = \Delta_{x_1}^{x_1'} F_1 \circ \Delta_{x_2}^{x_2'} F_2 + \alpha\, \Delta_{x_1}^{x_1'} \{F_1(1-F_1)\} \circ \Delta_{x_2}^{x_2'} \{F_2(1-F_2)\},$$

where $\Delta_x^{x'} F = F(x') - F(x)$ and

$$\Delta_x^{x'}\{F(1-F)\} = F(x')[1-F(x')] - F(x)[1-F(x)] = \Delta_x^{x'} F \circ [1-F(x)-F(x')],$$

and thus

$$\Delta_{x_1,x_2}^{x_1',x_2'} H = \Delta_{x_1}^{x_1'} F_1 \circ \Delta_{x_2}^{x_2'} F_2 \{1 + \alpha A_1(x_1,x_1')A_2(x_2,x_2')\},$$

where $A_i(x,x') = 1 - F_i(x) - F_i(x')$. Hence $H(x_1,x_2)$ is a bivariate distribution if and only if

$$1 + \alpha A_1(x_1,x_1')A_2(x_2,x_2') \geq 0 \qquad (3)$$

whenever $\Delta_{x_1}^{x_1'} F_1 > 0$ and $\Delta_{x_2}^{x_2'} F_2 > 0$.

We now show that whenever $\Delta_x^{x'} F > 0$ we have

$$-M \leq A(x,x') \leq 1 - m \qquad (4)$$

where the bounds are tight, but they are not necessarily achieved. It suffices to prove the left hand side inequality, the proof of the right hand side inequality being similar. It is clear that $A(x,x') = 1-F(x)-F(x')$, $x < x'$, is decreasing (i.e. non-increasing) in x and x'. Thus it suffices to show that as $x \uparrow +\infty$, $A(x,x') \downarrow -M$.

If $M = 1$ then for each x there exists $x' > x$ such that $\Delta_x^{x'} F > 0$ and thus as $x \uparrow +\infty$, $A(x,x') \downarrow 1 - 1 - 1 = -1 = -M$.

If $M < 1$ then for all

$$x < \sup\{u : F(u) \leq M\} = F^{-1}(M) < x'$$

we have $F(x) \leq M < 1 = F(x')$ and $\Delta_x^{x'} F > 0$, and thus as $x \uparrow F^{-1}(M)$, $A(x,x') \downarrow 1 - M - 1 = -M$.

Now (4) implies that whenever $\Delta_{x_1}^{x_1'} F_1 > 0$ and $\Delta_{x_2}^{x_2'} F_2 > 0$ we have

$$-\max\{M_1(1-m_2),\ (1-m_1)M_2\} \leq A_1(x_1,x_1')A_2(x_2,x_2') \leq \max\{M_1M_2,\ (1-m_1)(1-m_2)\}$$

where again the bounds are tight (but not necessarily achieved) and the result of the theorem follows immediately from (3). □

Notice that

$$\alpha_{min} \leq -1 \quad \text{and} \quad 1 \leq \alpha_{max}$$

and that in fact

$\alpha_{min} = -1$ if and only if $M_1 = M_2 = 1$ or $m_1 = m_2 = 0$

$\alpha_{max} = 1$ if and only if ($M_1=1$ and $m_2=0$) or ($m_1=0$ and $M_2=1$).

When the marginals are identical, $F_1 = F_2 = F$, then

$$\alpha_{min} = -\frac{1}{\{\max(M,1-m)\}^2} \leq \alpha \leq \frac{1}{M(1-m)} = \alpha_{max} .$$

As an example, when $F(x) = 0$ for $x < 0$, $= p$ for $0 \leq x < 1$, $= 1$ for $1 \leq x$, with $0 < p < 1$, then $M = m = p$ and the admissible values of α are

$$-\frac{1}{\{\max(p,1-p)\}^2} \leq \alpha \leq \frac{1}{p(1-p)} .$$

This example is considered by Johnson and Kotz [2].

If F has a density then $m = 0$ and $M = 1$. Thus if both F_1 and F_2 have densities then the admissible values of α are $-1 \leq \alpha \leq 1$, a result obtained by Johnson and Kotz [2]. It is clear however that we may have $m = 0$ and $M = 1$ even when F is not absolutely continuous. For instance, if F is a discrete distribution with (positive) jumps at the integers (or at x_n with $\inf x_n = -\infty$ and $\sup x_n = +\infty$) then $m = 0$ and $M = 1$. Thus for such marginal distributions the admissible values are again $-1 \le \alpha \le 1$.

As a final example consider a discrete distribution F with mass p_n at each x_n , $n = 1,2,\ldots$, where $x_n < x_{n+1}$. Then $m = p_1$ and $M = 1$. Thus if both marginals F_1, F_2 are of this type the admissible values of α are

$$-1 \le \alpha \le \frac{1}{\max(1-p_{1,1},\ 1-p_{2,1})} .$$

The same method can be used to obtain necessary and sufficient conditions on the coefficients α so that $H(x_1,\ldots,x_n)$ defined by (2) is an n-dimensional distribution. As in the proof of Theorem 1 we have

$$\Delta_{x_1,\ldots,x_n}^{x_1',\ldots,x_n'} H = \Delta_{x_1}^{x_1'} F_1 \ldots \Delta_{x_n}^{x_n'} F_n \Big\{1 + \sum_{j_1<j_2}^{n-1}\sum^{n} \alpha_{j_1j_2} A_{j_1}(x_{j_1},x_{j_1}')A_{j_2}(x_{j_2},x_{j_2}')$$

$$+ \ldots + \alpha_{1\ldots n} A_1(x_1,x_1') \ldots A_n(x_n,x_n')\Big\}.$$

Hence H is an n-dimensional distribution if and only if

$$1 + \sum_{j_1<j_2}^{n-1}\sum^{n} \alpha_{j_1j_2} A_{j_1}(x_{j_1},x_{j_1}')A_{j_2}(x_{j_2},x_{j_2}') + \ldots + \alpha_{1\ldots n} A_1(x_1,x_1') \ldots A_n(x_n,x_n') \ge 0$$

whenever $\Delta_{x_1}^{x_1'} F_1 > 0, \ldots ,\Delta_{x_n}^{x_n'} F_n > 0$. Since by (4), $-M_i \le A_i(x_i,x_i') \le 1 - m_i$ whenever $\Delta_x^{x'} F_i > 0$, it follows that the necessary and sufficient conditions on the α's are the following 2^n inequalities

$$1 + \sum_{j_1<j_2}^{n-1} \sum^{n} \varepsilon_{j_1}\varepsilon_{j_2}\alpha_{j_1j_2} + \cdots + \varepsilon_1\cdots\varepsilon_n\alpha_{1\ldots n} \geq 0 \qquad (5)$$

where for each $i = 1,\ldots,n$, $\varepsilon_i = -M_i$ or $1 - m_i$. These necessary and sufficient conditions were obtained by Johnson and Kotz [2] under the assumption that all marginal distributions F_i have densities, in which case of course $\varepsilon_i = \pm 1$ in (5).

If $H(x_1,\ldots,x_n)$ has the following (simplest possible symmetric) form

$$H(x_1,\ldots,x_n) = F_1(x_1) \ldots F_n(x_n)\{1 + \alpha[1-F_1(x_1)] \ldots [1-F_n(x_n)]\},$$

i.e., if $\alpha_{1\ldots n}$ is the only nonzero coefficient α, then the admissible values of α are

$$-\frac{1}{\max\{\gamma_1\cdots\gamma_n\}} \leq \alpha \leq \frac{1}{\max\{\delta_1\cdots\delta_n\}}$$

where the maxima are taken over all products with each $\gamma_i = M_i$ or $1 - m_i$ and an even number of γ_i's equal to M_i , and each $\delta_i = M_i$ or $1 - m_i$ and an odd number of δ_i's equal to M_i .

3. A RELATED FAMILY OF DISTRIBUTIONS

In this section we introduce and study a family of distributions which constitutes a natural generalization of the multivariate Morgenstern family. This family has some interest on its own but its raison d'être here is the fact (shown in Section 4) that it contains all conditional distributions of the multivariate Morgenstern family.

Let $F_1(x_1), \ldots, F_n(x_n)$ be univariate distributions and let us consider the family M_1 of multivariate distributions $H(x_1,\ldots,x_n)$ defined by

$$H(x_1,\ldots,x_n) = F_1(x_1) \ldots F_n(x_n)\Big\{1 + \sum_{j=1}^{n} \beta_j[1 - F_j(x_j)] + \sum_{j_1<j_2}^{n-1}\sum^{n} \beta_{j_1 j_2} [1-F_{j_1}(x_{j_1})][1-F_{j_2}(x_{j_2})] + \ldots + \beta_{1\ldots n} [1-F_1(x_1)] \ldots [1-F_n(x_n)]\Big\} \tag{6}$$

where the coefficients β are real constants. Notice that (6) differs from (2) only by the introduction of the first order terms: $\sum_{j=1}^{n} \beta_j\ [1-F_j(x_j)]$. As a result the marginal distributions $H_1(x_1),\ldots,H_n(x_n)$ of H defined by (6) are now given by

$$H_i(x_i) = F_i(x_i)\{1 + \beta_i[1-F_i(x_i)]\} \tag{7}$$

and are not equal to the original set of univariate distributions F_i unless all β_i equal zero. Clearly the family M_1 contains the family M of multivariate Morgenstern distributions and in fact a distribution H in M_1 (given by (6)) belongs to M if and only if its marginals are the distributions $F_1,\ldots,F_n$.

The method used in Section 2 can give us the necessary and sufficient conditions on the coefficients β for H defined by (6) to be an n-dimensional distribution. Let us first concentrate on the marginal distributions H_i. It is an immediate consequence of $\Delta_x^{x'} H_i = \Delta_x^{x'} F_i\{1+\beta_i A_i(x,x')\}$ and (4) that each H_i is a univariate distribution if and only if

$$-\frac{1}{1-m_i} \le \beta_i \le \frac{1}{M_i} . \qquad (8)$$

The necessary and sufficient conditions on the remaining coefficients β are of course the (corresponding to (5)) 2^n inequalities

$$1 + \sum_{j=1}^{n} \varepsilon_j \beta_j + \sum_{j_1<j_2}^{n-1\ n} \varepsilon_{j_1} \varepsilon_{j_2} \beta_{j_1 j_2} + \ldots + \varepsilon_1 \ldots \varepsilon_n \beta_{1\ldots n} \ge 0$$

where for each $i = 1,\ldots,n$, $\varepsilon_i = -M_i$ or $1 - m_i$. When $n = 2$ the admissible values of β_{12} are $b_{12} \le \beta_{12} \le B_{12}$ where

$$b_{12} = \max\left\{-\frac{1+\beta_1(1-m_1)+\beta_2(1-m_2)}{(1-m_1)(1-m_2)} ,\ -\frac{1-\beta_1 M_1-\beta_2 M_2}{M_1 M_2}\right\}$$

$$B_{12} = \min\left\{\frac{1+\beta_1(1-m_1)-\beta_2 M_2}{(1-m_1)M_2} ,\ \frac{1-\beta_1 M_1+\beta_2(1-m_2)}{M_1(1-m_2)}\right\} .$$

In particular, when $M_i = 1$ and $m_i = 0$, $i = 1,2$, we have by (8): $-1 \le \beta_i \le 1$ and the admissible values of β_{12} are

$$-1 + |\beta_1+\beta_2| \le \beta_{12} \le 1 - |\beta_1-\beta_2|$$

(it is easily seen that this interval of admissible values is always nonempty).

Let $X_1,\ldots,X_n$ be random variables with joint distribution in M_1. It is easily seen from (6) that the joint distribution of each subset of $X_1,\ldots,X_n$ is also in the family M_1. Also, from (6) and (7), it is easily seen that the random variables $X_1,\ldots,X_n$ are independent if and only if for $k = 2,\ldots,n$ and $j_1 < \ldots < j_k$ we have

$$\beta_{j_1\ldots j_k} = \beta_{j_1} \cdots \beta_{j_k} .$$

The full meaning of the coefficients β is given by the following relationship

$$E\{(X_{j_1}-\mu'_{j_1}) \ldots (X_{j_k}-\mu'_{j_k})\} = (-1)^k c_{j_1} \ldots c_{j_k} \beta_{j_1 \ldots j_k} \tag{9}$$

where

$$c_i = \int_{-\infty}^{\infty} F_i(x_i)\{1 - F_i(x_i)\}dx_i = -\int_{-\infty}^{\infty} x_i\{1 - 2F_i(x_i)\}dF_i(x_j) \tag{10}$$

$$\mu'_i = \int_{-\infty}^{\infty} x_i \, dF_i(x_i) \tag{11}$$

and where the following assumption is needed to guarantee that all integrals and expectations are finite

$$\int_{-\infty}^{\infty} |x_i| dF_i(x_i) < \infty,$$

$i = 1,\ldots,n$. The functionals c of F were first introduced by Johnson and Kotz [3] in evaluating the regression $E(X_1/X_2)$ when X_1, X_2 have a bivariate Morgenstern distribution. Notice from (7) that

$$dH_i(x_i) = dF_i(x_i)\{1 + \beta_i[1 - 2F_i(x_i)]\}$$

and thus by (10) and (11) we have

$$\mu_i = E(X_i) = \int_{-\infty}^{\infty} x_i\{1 + \beta_i[1 - 2F_i(x_i)]\}dF_i(x_i)$$

$$= \mu'_i - c_i\beta_i \,. \tag{12}$$

Thus μ'_i is the mean of X_i under F_i, while μ_i is the (true) mean of X_i under H_i. For the Morgenstern family we have from (12) $\mu_i = \mu'_i$ and (9) reads

$$E\{(X_{j_1}-\mu_{j_1}) \ldots (X_{j_k}-\mu_{j_k})\} = (-1)^k c_{j_1} \ldots c_{j_k} \alpha_{j_1 \ldots j_k} .$$

In order to prove (9) we simply notice that (6) implies

$$d^n H(x_1,\ldots,x_n) = dF_1(x_1) \ldots dF_n(x_n)\Big\{1 + \sum_{j=1}^{n} \beta_j[1 - 2F_j(x_j)] + \ldots + \beta_{1\ldots n}[1 - 2F_1(x_1)] \ldots [1 - 2F_n(x_n)]\Big\}$$

and that by (10) and (11) we have

$$\int_{-\infty}^{\infty} (x_i-\mu_i')dF_i = 0$$

$$\int_{-\infty}^{\infty} (x_i-\mu_i')[1 - 2F_i(x_i)]dF_i(x_i) = -c_i - \mu_i \{F_i(x_i)[1 - F_i(x_i)]\}_{-\infty}^{+\infty} = -c_i .$$

(9) now follows immediately. For $k = 2$, using (12) and some simple algebra, (9) can be written in the following form

$$\mathrm{Cov}(X_{j_1},X_{j_2}) = c_1c_2(\beta_{12} - \beta_1\beta_2).$$

From this it follows that if X_{j_1} and X_{j_2} are uncorrelated then they are independent.

As a final property of the family M_1 we mention that it is closed with respect to conditioning in the sense that all conditional distributions of an element of M_1 belong to M_1. This can be seen in a way identical to that by which Theorem 2 is established in Section 4.

We conclude this section with the following remark. The family M_1 consists of multivariate distributions $H(x_1,\ldots,x_n)$ of the form (6) with marginals $H_1,\ldots,H_n$ given by (7). At first glance it may seem that (7) is a restriction on the univariate marginals of distribution in the family M_1. However this is not the case, as every univariate distribution $H(x)$ can be written in the form

$$H(x) = F(x)\{1 + \beta[1 - F(x)]\} \tag{13}$$

for some univariate distribution $F(x)$ and some real number β. This representation is of course far from unique. In fact it can be easily seen that given H and given any real number β there is a univariate distribution $F(x)$ such that (13) holds. F depends of course on H and β but is not necessarily uniquely determined by them. For $\beta = 0$ we can clearly take $F \equiv H$. For $\beta \neq 0$, (13) can be written as

$$\beta F^2(x) - (1+\beta)F(x) + H(x) = 0$$

and the nondecreasing, right continuous root $r(x) = (2\beta)^{-1}\left(1 + \beta - \sqrt{(1+\beta)^2 - 4\beta H(x)}\right)$ is the obvious candidate for $F(x)$. When $-1 \leq \beta \leq 1$ this root has the proper limits at $\pm\infty$, but when $1 < \beta$ and $H(x) = 1$ we have a choice between 1 and $\frac{1}{\beta}$, and, similarly, when $\beta < -1$ and $H(x) = 0$ we have a choice between 0 and $1 + \frac{1}{\beta}$. Hence F may not be uniquely determined by H and β, but a solution can always be found as follows:

$$\text{for } \beta < -1 : F(x) = \begin{cases} r(x) & \text{for } 0 < H(x) \leq 1 \\ 0 & \text{for } H(x) = 0 \end{cases}$$

$$\text{for } -1 \leq \beta \leq 1 : F(x) = r(x) \quad \text{for all } x$$

$$\text{for } 1 < \beta : F(x) = \begin{cases} r(x) & \text{for } 0 \leq H(x) < 1 \\ 1 & \text{for } H(x) = 1. \end{cases}$$

4. THE CONDITIONAL DISTRIBUTIONS OF THE MULTIVARIATE MORGENSTERN FAMILY

In this section we compute the (regular) conditional distribution of $X_1, \ldots, X_k$ given $X_{k+1} = x_{k+1}, \ldots, X_n = x_n$, $(k = 1, \ldots, n-1)$ when $X_1, \ldots, X_n$ have a mlutivariate Morgenstern distribution and show that the former always belongs to the family M_1.

Let the random variables $X_1, \ldots, X_n$ have a distribution in M given by (2). For $j_1 < j_2 < \ldots < j_m$ we denote by $H_{j_1 j_2 \ldots j_m}$ the distribution of $X_{j_1}, X_{j_2}, \ldots, X_{j_m}$ (which also belongs to M). For convenience we will use the same symbol for a distribution as for its corresponding (Lebesgue-Stieltjes) probability measure. Also if the finite (possibly signed) measure λ is absolutely continuous with respect to the finite measure ν, $\lambda \ll \nu$, we will denote by $[\frac{d\lambda}{d\nu}]$ the corresponding Radon-Nikodym derivative. Notice that $F(1 - F) \ll F$ and

$$\left[\frac{d\{F(1-F)\}}{dF}\right](x) = 1 - 2F(x).$$

It then follows from (2) that $H_{1\ldots n} \ll F_1 \times \ldots \times F_n$ with

$$\left[\frac{dH_{1\ldots n}}{dF_1 \times \ldots \times dF_n}\right](x_1, \ldots, x_n) = 1 + \sum_{j_1<j_2}^{n-1} \sum^{n} \alpha_{j_1 j_2} [1 - 2F_{j_1}(x_{j_1})][1 - 2F_{j_2}(x_{j_2})]$$

$$+ \ldots + \alpha_{1\ldots n} [1 - 2F_1(x_1)] \ldots [1 - 2F_n(x_n)]. \quad (14)$$

For $j_1 < \ldots < j_m$ we will use the following notation

$$d_{j_1 \ldots j_m}(x_{j_1}, \ldots, x_{j_m}) = \left[\frac{dH_{j_1 \ldots j_m}}{dF_{j_1} \times \ldots \times dF_{j_m}}\right](x_{j_1}, \ldots, x_{j_m})$$

and of course we have an expression similar to (14) for $d_{j_1 \ldots j_m}$.

Let us recall that a regular conditional distribution $H_{1\ldots k/k+1\ldots n}(x_1, \ldots, x_k/x_{k+1}, \ldots, x_n)$ of $X_1, \ldots, X_k$ given $X_{k+1} = x_{k+1}, \ldots, X_n = x_n$ is a function from R^n to $[0,1]$ which is Borel measurable in $x_{k+1}, \ldots, x_n$ for each fixed $x_1, \ldots, x_k$; a multivariate distribution in $x_1, \ldots, x_k$ for each fixed $x_{k+1}, \ldots, x_k$; and which satisfies

$$H_{1\ldots n}(x_1, \ldots, x_n) = \int_{-\infty}^{x_{k+1}} \cdots \int_{-\infty}^{x_n} H_{1\ldots k/k+1\ldots n}(x_1, \ldots, x_k/u_{k+1}, \ldots, u_n) \, dH_{k+1\ldots n}(u_{k+1}, \ldots, u_n). \quad (15)$$

THEOREM 2. *With the above assumptions and notation the function* $H_{1\ldots k/k+1\ldots n}(x_1, \ldots, x_k/x_{k+1}, \ldots, x_n)$ *defined by*

$$H_{1\ldots k/k+1\ldots n}(x_1, \ldots, x_k/x_{k+1}, \ldots, x_n) = \frac{\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} d_{1\ldots n}(u_1, \ldots, u_k, x_{k+1}, \ldots, x_n) dF_1(u_1) \ldots dF_k(u_k)}{d_{k+1\ldots n}(x_{k+1}, \ldots, x_n)} \quad (16)$$

when $d_{k+1\ldots n}(x_{k+1}, \ldots, x_n) > 0$ *and otherwise by, say,*

$$H_{1\ldots k/k+1\ldots n}(x_1, \ldots, x_k/x_{k+1}, \ldots, x_n) = F_1(x_1) \ldots F_k(x_k) \quad (17)$$

is a regular conditional distribution of $X_1, \ldots, X_k$ *given* $X_{k+1} = x_{k+1}, \ldots, X_n = x_n$ *and belongs to the family* M_1 .

Proof. It is quite clear from its definition that $H_{1\ldots k/k+1\ldots n}(x_1, \ldots, x_n/x_{k+1}, \ldots, x_n)$ is Borel measurable in $x_{k+1}, \ldots, x_n$ for each fixed $x_1, \ldots, x_k$, and a distribution in $x_1, \ldots, x_k$ for each fixed $x_{k+1}, \ldots, x_n$. Thus we need only prove that (15) is satisfied.

Fix $x_1, \ldots, x_n$. For brevity we let $A = (-\infty, x_1] \times \ldots \times (-\infty, x_k]$ and $B = (-\infty, x_{k+1}] \times \ldots \times (-\infty, x_n]$ and we omit the variables throughout. Notice that if $E = \{d_{k+1\ldots n} > 0\} \subset R^{n-k}$, then on its complement E' we have $d_{k+1\ldots n} = 0$ and thus $H_{k+1\ldots n}(E') = \int_{E'} d_{k+1\ldots n} dF_{k+1} \ldots dF_n = 0$. It follows that $H_{1\ldots n}(R^k \times E') = H_{k+1\ldots n}(E') = 0$ and since $A \times (B \cap E')$ is a subset of $R^k \times E'$,

$$H_{1\ldots n}\{A \times (B \cap E')\} = 0.$$

Now (15) is obtained as follows:

$$\begin{aligned}
\int_B H_{1\ldots k/k+1\ldots n}\, dH_{k+1\ldots n} &= \int_{B\cap E} H_{1\ldots k/k+1\ldots n}\, dH_{k+1\ldots n} \\
&= \int_{B\cap E} \frac{\int_A d_{1\ldots n} dF_1 \ldots dF_k}{d_{k+1\ldots n}}\, dH_{k+1\ldots n} \\
&= \int_{B\cap E} \left(\int_A d_{1\ldots n}\, dF_1 \ldots dF_k \right) dF_{k+1} \ldots dF_n \\
&= \int_{A\times(B\cap E)} d_{1\ldots n}\, dF_1 \ldots dF_n \\
&= H_{1\ldots n}\{A \times (B\cap E)\} = H_{1\ldots n}(A\times B).
\end{aligned}$$

That $H_{1\ldots k/k+1\ldots n}$ belongs to M_1, i.e., is of the form (6), is now clear from (14), (16) and (17) and from

$$\int_{-\infty}^{x} [1 - 2F(u)]dF(u) = F(x)[1 - F(x)].$$

For instance the term $\alpha_{1n}[1 - 2F_1(x_1)][1 - 2F_n(x_n)]$ in (14) will give rise via the integration in (16) to the term $\alpha_{1n}F_1(x_1)[1 - F_1(x_1)][1 - 2F_n(x_n)]$, and thus $\alpha_{1n}[1 - 2F_n(x_n)]$ will be one term in the expression representing the value of the coefficient β_1 in (6). □

Since values $(x_{k+1}, \ldots, x_n)$ with $d_{k+1\ldots n}(x_{k+1}, \ldots, x_n) = 0$ are taken by $(X_{k+1}, \ldots, X_n)$ with probability zero, the expression of $H_{1\ldots k/k+1\ldots n}$ given by (16) is the one of interest and it is only for such $(x_{k+1}, \ldots, x_n)$'s that expressions will be written out in the sequel.

The coefficients β in the representation (6) of $H_{1\ldots k/k+1\ldots n}(x_1, \ldots, x_k/x_{k+1}, \ldots, x_n)$ depend of course on $x_{k+1}, \ldots, x_n$. Using (14) and (15) we can express the coefficients β in terms of the (constant) coefficients α and of $[1 - 2F_{k+1}(x_{k+1})], \ldots, [1 - 2F_n(x_n)]$. For instance for $k = 1$, $n = 2$ we have

$$H_{1/2}(x_1/x_2) = F_1(x_1)\{1 + \alpha_{12}[1 - 2F_2(x_2)][1 - F_1(x_1)]\}$$

i.e., $\beta(x_2) = \alpha_{12}[1 - 2F_2(x_2)]$. For $k = n - 1$ the coefficients β in the representation of $H_{1\ldots n-1/n}$ are given by

$$\beta_i(x_n) = \alpha_{in}[1 - 2F_n(x_n)]$$

$$\beta_{ij}(x_n) = \alpha_{ij} + \alpha_{ijn}[1 - 2F_n(x_n)] \qquad (18)$$

etc.

$$\beta_{1\ldots n-1}(x_n) = \alpha_{1\ldots n-1} + \alpha_{1\ldots n}[1 - 2F_n(x_n)].$$

From the form of the coefficients β it is also clear that

$$H_{1\ldots n-1/n}(x_1,\ldots,x_{n-1}/x_n) = H_{1\ldots n-1}(x_1,\ldots,x_{n-1}) +$$

$$+ F_1(x_1) \ldots F_{n-1}(x_{n-1})[1 - 2F_n(x_n)]\Big\{\sum_{j=1}^{n-1} \alpha_{jn}[1 - F_j(x_j)]$$

$$+ \sum_{j_1<}^{n-2} \sum_{j_2}^{n-1} \alpha_{j_1 j_2 n}[1 - F_{j_1}(x_{j_1})][1 - F_{j_2}(x_{j_2})] + \ldots$$

$$+ \alpha_{1\ldots n-1,n}[1 - F_1(x_1)] \ldots [1 - F_{n-1}(x_{n-1})]\Big\}.$$

Notice that when x_n is the median of F_n then $H_{1\ldots n-1/n} \equiv H_{1\ldots n-1}$. Also, whenever $f(X_1, \ldots, X_{n-1})$ has finite expectation we have

$$E\{f(X_1,\ldots,X_{n-1})/X_n=x_n\} = E\{f(X_1,\ldots,X_{n-1})\}$$

$$+ [1-2F_n(x_n)] \int_{-\infty}^{\infty} \cdots \int f(x_1,\ldots,x_{n-1})h(x_1,\ldots,x_{n-1})dF_1(x_1)\ldots dF_{n-1}(x_{n-1}) \quad (19)$$

where

$$h(x_1,\ldots,x_{n-1}) = \sum_j^{n-1} \alpha_{jn}[1-2F_j(x_j)]+\ldots+\alpha_{1\ldots n-1,n}[1-2F_1(x_1)]\ldots[1-2F_{n-1}(x_{n-1})].$$

In particular (9), (18) and (19) imply that if the means $\mu_i = E(X_i)$ exist then

$$\mu_{i/n}(X_n) = E(X_i/X_n) = E(X_i) - c_i\alpha_{in}[1 - 2F_n(X_n)], \quad i = 1, \ldots, n-1$$

$$E\{(X_1-\mu_1) \ldots (X_{n-1}-\mu_{n-1}) / X_n\} = E\{(X_1-\mu_1) \ldots (X_{n-1}-\mu_{n-1})\}$$

$$+ (-1)^{n-1} c_1 \ldots c_{n-1} \alpha_{1\ldots n}[1 - 2F_n(X_n)].$$

The expression for $E(X_i/X_n)$ has been calculated by Johnson and Kotz [3]. Also we can easily calculate the conditional covariance

$$\mathrm{Cov}(X_1,X_2/X_3) = \mathrm{Cov}(X_1,X_2) + c_1c_2\alpha_{123}[1 - 2F_3(x_3)] - c_1c_2\alpha_{13}\alpha_{23}[1 - 2F_3(x_3)]^2 .$$

As a further example the coefficients $\beta(x_{n-1},x_n)$ in the representation of $H_{1\ldots n-2/n-1,n}$ $(n \geq 3)$ are given by

$$\beta_i(x_{n-1},x_n) = \frac{\alpha_{i,n-1}[1-2F_{n-1}(x_{n-1})]+\alpha_{i,n}[1-2F_n(x_n)]+\alpha_{i,n-1,n}[1-2F_{n-1}(x_{n-1})][1-2F_n(x_n)]}{1+\alpha_{n-1,n}[1-2F_{n-1}(x_{n-1})][1-2F_n(x_n)]}$$

etc.

$$\beta_{1\ldots n-2}(x_{n-1},x_n) =$$
$$\frac{\alpha_{1\ldots n-2}+\alpha_{1\ldots n-1}[1-2F_{n-1}(x_{n-1})]+\alpha_{1\ldots n-2,n}[1-2F_n(x_n)]+\alpha_{1\ldots n}[1-2F_{n-1}(x_{n-1})][1-2F_n(x_n)]}{1+\alpha_{n-1,n}[1-2F_{n-1}(x_{n-1})][1-2F_n(x_n)]}.$$

Also $H_{1\ldots n-2/n-1,n}$ may be expressed as follows

$$H_{1\ldots n-2/n-1,n} = H_{1\ldots n-2} + \frac{F_1\cdots F_{n-2}}{d_{n-1,n}}\Big\{(1 - 2F_{n-1})K^{(n-1)}_{1\ldots n-2} + (1 - 2F_n)K^{(n)}_{1\ldots n-2}$$
$$+ (1 - 2F_{n-1})(1 - 2F_n)K^{(n-1,n)}_{1\ldots n-2}\Big\}$$

where

$$K^{(m)}_{1\ldots n-2} = \sum_j^{n-2} \alpha_{jm}(1-F_j) + \ldots + \alpha_{1\ldots n-2,m}(1-F_1)\ \ldots\ (1-F_{n-2})$$

for $m = n-1,n$ and similarly for $K^{(n-1,n)}_{1\ldots n-2}$ (replace m in expression by $n-1,n$). Notice that the denominator $d_{n-1,n}$ becomes 1 when X_{n-1} and X_n are independent. In particular

$$H_{1/23}(x_1/x_2,x_3) = F_1(x_1)\{1 + \beta(x_2,x_3)[1 - 2F_1(x_1)]\}$$

where

$$\beta = \frac{\alpha_{12}(1-2F_2)+\alpha_{13}(1-2F_3)+\alpha_{123}(1-2F_2)(1-2F_3)}{1+\alpha_{23}(1-2F_2)(1-2F_3)}$$

and thus whenever $f(X_1)$ has finite expectation we have

$$E\{f(X_1)/X_2,X_3\} = E\{f(X_1)\} + \beta(X_2,X_3)E\{f(X_1)[1 - 2F(X_1)]\}.$$

As a final example we give the expression of

$$H_{1|2\ldots n}(x_1|x_2, \ldots ,x_n) = F_1(x_1)\{1 + \beta(x_2, \ldots ,x_n)[1 - F_1(x_1)]\}$$

where

$$\beta(x_2, \ldots ,x_n) = \frac{h^{(1)}_{2\ldots n}(x_2,\ldots,x_n)}{h_{2\ldots n}(x_2,\ldots,x_n)}$$

$$h^{(1)}_{2\ldots n} = \sum_{j=2}^{n} \alpha_{1j} (1 - 2F_j) + \sum_{2=j_1<j_2}^{n-1}\sum^{n} \alpha_{1j_1j_2} (1 - 2F_{j_1})(1 - 2F_{j_2}) + \ldots + \alpha_{12\ldots n} (1 - 2F_2) \ldots (1 - 2F_n)$$

$$h_{2\ldots n} = 1 + \sum_{2=j_1<j_2}^{n-1}\sum^{n} \alpha_{j_1j_2}(1-2F_{j_1})(1-2F_{j_2}) + \ldots + \alpha_{2\ldots n} (1-2F_2)\ldots(1-2F_n).$$

Again $h_{2\ldots n} \equiv 1$ when $X_2, \ldots ,X_n$ are independent, in which case the expression of the conditional distribution, and hence that of the regression, is greatly simplified.

ACKNOWLEDGMENT

The author wishes to thank Professor Norman L. Johnson for introducing him to the problems considered in this paper.

REFERENCES

[1] MORGENSTERN, D. (1956). Einfache Beispiele Zweidimensionaler Verteilungen. Mitt. für Math. Statistik 8 234-235.

[2] JOHNSON, N.L. AND KOTZ, S. (1975). On some generalized Farlie-Gumbel-Morgenstern distributions. Comm. Statist. 4 415-427.

[3] JOHNSON, N.L. AND KOTZ, S. (1976). Private Communication.