A NOTE ON THE JOINT DISTRIBUTION OF THE DURATION OF A BUSY PERIOD AND THE TOTAL QUEUEING TIME*

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§1. Introduction

Computer simulation of queueing models is becoming the paramount method for their investigation; complexities of the model which present formidable difficulties to any analytical attack can often be dealt with in a trivial way in the computer program. Nonetheless the existence of exact mathematical formulae is useful, even when they refer to rather specific models. For one thing they can indicate, in a rough way, the effect of varying relevant parameters. For another thing, they can provide valuable checks on the correctness of a computer simulation for which, maybe, a subtle undetected flow in the program is yielding results which are wrong, but not so conspiciously so as to arouse suspicions. For a further thing, the exact formulae can possibly provide indication of the sampling errors inherent in a proposed Monte Carlo study of a queueing system.

The present note is part of an attack on a number of analytical problems which arose from work being done by Andrew Seila of the Curriculum of Operations Research at the University of North Carolina, Chapel Hill.

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He is investigating, mainly by computer methods, the estimation of quantiles of the waiting time distribution in various queues; his methods focus on the busy period as the basic sampling unit. One check on the correct operation of his methods is through a study of the joint distribution of the total time lost (Z) by customers in queueing during a busy period and the length (U) of that period. This note tackles this problem for the M/M/1 queue and shows that product moments of Z and U are obtainable in analytic form. They are given later in this note up to order three.

§2 Derivation of a Joint Generating Function

Let a busy period begin by a customer C_0, say, with service-time x, arriving to find the server free. Let U be the length of the busy period thus initiated and let Z be the sum of the queueing-times of all customers who are served in that busy period. Plainly the joint distribution of U and Z depends upon x. Let $\alpha > 0$ and $\beta > 0$ be dummy transform-variables and set

$$(2.1) \qquad M(\alpha,\beta|x) = E\{e^{-\alpha Z-\beta U} \mid x\},$$

where the conditional expectation has the obvious meaning. For ease we shall write $M(\alpha,\beta|x)$ simply as $M(x)$.

The distribution of the time from the arrival of C_0 to the next customer C_1, say, has a p.d.f. of exponential form and this leads us to the integral equation

$$(2.2) \quad M(x) = e^{-\lambda x-\beta x} + \int_0^x \lambda e^{-\lambda u-\beta u-\alpha(x-u)} \int_0^\infty M(x+z-u)\mu e^{-\mu z}dzdu.$$

In this equation the constants λ and μ are the intensities of arrival and service, respectively.

If we provisionally set

$$g(x) = e^{-(\alpha-\mu)x} \int_x^\infty e^{-\mu y} M(y)dy$$

then a straightforward computation yields the fact that the Laplace transform, denoted by the notation:

$$g^0(s) = \int_0^\infty e^{-sx} g(x)dx,$$

and defined for real $s > 0$, is given by

$$g^0(s) = \frac{M^0(\mu)-M^0(s+\mu)}{s+\alpha-\mu} . \tag{2.3}$$

If this result is used in (2.2) we obtain after a little computation

$$M^0(s) = \frac{1}{s+\lambda+\beta} + \frac{1}{s+\lambda+\beta}\left\{\frac{M^0(\mu)-M^0(s+\alpha)}{s+\alpha-\mu}\right\} . \tag{2.4}$$

Since we are principally concerned with the unconditional expectation

$$E\{e^{-\alpha Z-\beta U}\} = \int_0^\infty \mu M(x)e^{-\mu x}\, dx$$
$$= \mu M^0(\mu),$$

it follows that $M^0(\mu)$ is of special interest to us. From (2.4) we obtain by putting $s = \mu$,

$$M^0(\mu) = \frac{1}{\lambda+\mu+\beta} + \frac{\lambda\mu}{\lambda+\mu+\beta}\left\{\frac{M^0(\mu)-M^0(\mu+\alpha)}{\alpha}\right\} .$$

and hence

$$M^0(\mu)\left\{1 - \frac{\lambda\mu}{\alpha(\lambda+\mu+\beta)}\right\} = \frac{1}{\lambda+\mu+\beta} - \frac{\lambda\mu M^0(\mu+\alpha)}{\alpha(\lambda+\mu+\beta)} . \tag{2.5}$$

More generally, if n is any positive integer, then putting $s = \mu + n\alpha$

in (2.4) yields

(2.6) $$M^0(\mu+n\alpha) = \frac{1}{\mu+\lambda+\beta+n\alpha}\left\{1 + \frac{\lambda\mu}{\alpha(n+1)}\,[M^0(\mu) - M^0(\mu+\overline{n+1}\alpha)]\right\}.$$

Let us set $\xi = \lambda\mu/\alpha$, and

(2.7) $$K(\xi) = 1 - \frac{\xi}{\lambda+\mu+\beta} + \frac{\xi^2}{2(\lambda+\mu+\beta)(\lambda+\mu+\beta+\alpha)} - \frac{\xi^2}{3!(\lambda+\mu+\beta)(\lambda+\mu+\beta+\alpha)(\lambda+\mu+\beta+2\alpha)} + \text{etc.}$$

It is not difficult to see that this infinite series is always convergent and that $K(\xi)$ is actually an entire function.

Somewhat tedious and repetitive use of (2.6) in (2.5) will then show that

(2.8) $$M^0(\mu) = -\frac{K'(\xi)}{K(\xi)}.$$

In obtaining this result it is helpful to note that $M^0(n\alpha) \to 0$ as $n \to \infty$.

It is possible to express (2.7) as a certain Bessel function, and, indeed, if we temporarily set

$$\zeta = \lambda\mu/\alpha^2 \quad \text{and} \quad \theta = (\lambda+\mu+\beta)/\alpha$$

then

(2.9) $$(\mu\lambda)^{\frac{1}{2}}M^0(\mu) = \frac{J_\theta(2\sqrt{\zeta})}{J_{\theta-1}(2\sqrt{\zeta})}.$$

This result does not seem particularly tractable. The most we can hope to achieve is an expansion of (2.8) as a double Taylor expansion in powers of α and β from which the joint moments (unconditional) of Z and U can be extracted.

§3 Expansion of Joint Generating Function

From a well-known property of the Bessel function (Watson, 1958) we have that, for any argument y, say,

$$J_{\theta-1}(y) + J_{\theta+1}(y) = \frac{2(\theta-1)}{y} J_\theta(y),$$

from which one can derive

$$\frac{J_\theta(y)}{J_{\theta-1}(y)} = \cfrac{1}{\frac{2\theta}{y} - \frac{J_{\theta+1}(y)}{J_\theta(y)}},$$

and thus obtain a continued-fraction expansion of the ratio of Bessel functions in (2.9). Indeed if we now set

$$a = \frac{\lambda+\mu+\beta}{\sqrt{\lambda\mu}}, \quad b = \frac{\alpha}{\sqrt{\lambda\mu}},$$

we find that

$$\text{(3.1)} \qquad \frac{J_\theta(2\sqrt{\zeta})}{J_{\theta-1}(2\sqrt{\zeta})} = \cfrac{1}{a - \cfrac{1}{(a+b) - \cfrac{1}{(a+2b)-\text{etc.}}}}$$

$$= \frac{1}{a-}\,\frac{1}{(a+b)-}\,\frac{1}{(a+2b)-}\,\frac{1}{(a+3b)-\ldots}.$$

Let us call this continued fraction $C(a,b)$. Then, evidently,

$$\text{(3.2)} \qquad \frac{1}{C(a,b)} = a - C(a+b,b).$$

In particular we see

$$\{C(a,0)\}^2 - a\{C(a,0)\} + 1 = 0. \tag{3.3}$$

This quadratic equation gives us the equivocal result

$$C(a,0) = \frac{\lambda+\mu+\beta}{2\sqrt{\lambda\mu}} \pm \frac{\sqrt{[(\lambda-\mu)^2+\beta^2+2(\lambda+\mu)\beta]}}{2\sqrt{(\lambda\mu)}} . \tag{3.4}$$

However, the substitution $\alpha = 0$ and $\beta = 0$ must make $\mu M^0(\mu) = 1$. Thus, from (2.9), we need

$$(\lambda/\mu)^{\frac{1}{2}} = C(a,0)\Big|_{\beta=0}$$

$$= \frac{\lambda+\mu}{2\sqrt{\lambda\mu}} \pm \frac{\sqrt{(\lambda-\mu)^2}}{2\sqrt{\lambda\mu}}$$

and so we may conclude the plus sign to be correct, and hence

$$C(a,0) = \frac{a+\sqrt{a\;-4}}{2} .$$

This result will be found to agree with the well-known formula (Cox and Smith, 1961) for the transform of the distribution of the duration of a busy period, for it yields from (2.9):

$$E(e^{-\beta Z}) = \frac{(\mu/\lambda)^{\frac{1}{2}}}{2}\left\{\frac{\lambda+\mu+\beta}{\sqrt{\lambda\mu}} + \sqrt{\frac{(\lambda+\mu+\beta)^2}{\lambda\mu} - 4}\right\} \tag{3.5}$$

$$= \frac{1}{2\lambda}\{(\lambda+\mu+\beta) + \sqrt{(\lambda+\mu+\beta)^2-4\lambda\mu} .$$

Let us adopt the notation C for $C(a,b)$ and

$$C_{ij}(a,b) = \frac{\partial^{i+j}}{\partial a^i \partial b^j} C(a,b) ,$$

and be prepared to contract $C_{ij}(a,b)$ even further to C_{ij} . Then

(3.2) gives

$$(3.6)\quad \begin{cases} -\dfrac{C_{10}}{C^2} = 1 - C_{10}(a+b,b) \\ -\dfrac{C_{01}}{C^2} = -C_{10}(a+b,b) - C_{01}(a+b,b) \end{cases}$$

If we set $\gamma = \gamma(a) = C(a,0)$ and

$$\gamma_{ij} = \gamma_{ij}(a) = C_{ij}(a,0),$$

then (3.6) gives

$$(3.7)\quad \gamma_{10} = \frac{\gamma^2}{\gamma^2-1}$$

and

$$(3.8)\quad \gamma_{01} = -\left(\frac{\gamma^2}{\gamma^2-1}\right)^2 .$$

Since the function $\gamma^2/(\gamma^2-1)$ occurs frequently we shall denote it by ϕ. Thus

$$(3.9)\quad \begin{cases} \gamma_{10} = \phi \\ \gamma_{01} = -\phi^2 . \end{cases}$$

If we return to (3.6) and perform further partial differentiations then we find

$$(3.10)\quad \begin{cases} \dfrac{2C_{10}^2}{C^3} - \dfrac{C_{20}}{C^2} = -C_{20}(a+b,b) \\ \dfrac{2C_{10}C_{01}}{C^3} - \dfrac{C_{11}}{C^2} = -C_{20}(a+b,b) - C_{11}(a+b,b) \\ \dfrac{2C_{01}^2}{C^3} - \dfrac{C_{02}}{C^2} = -C_{20}(a+b,b) - 2C_{11}(a+b,b) - C_{02}(a+b,b). \end{cases}$$

Computation then yields the equations:

$$(3.11)\quad \begin{cases} \gamma_{20} = -\frac{2}{\gamma^3}\phi^3 \\ \gamma_{11} = \frac{4}{\gamma^3}\phi 4 \\ \gamma_{02} = \frac{2}{\gamma^3}\phi^4 - \frac{10}{\gamma^3}\phi^5 . \end{cases}$$

It is plain that, at the expense of greater and greater complication, one can continue to obtain equations of higher order. We merely list the following results:

$$(3.12)\quad \begin{cases} \gamma_{30} = \frac{6}{\gamma^4}\phi^4 + \frac{12}{\gamma^6}\phi^5 \\ \gamma_{21} = -\frac{6}{\gamma^4}\phi^5 - \frac{20}{\gamma^6}\phi^6 \\ \gamma_{12} = -\frac{6}{\gamma^4}\phi^5 + \frac{12}{\gamma^4}\phi^6 + \frac{4}{\gamma^6}\phi^6 + \frac{56}{\gamma^6}\phi^7 \\ \gamma_{03} = -\frac{6}{\gamma^4}\phi^3 - \frac{6}{\gamma^4}\phi^4 + \frac{36}{\gamma^4}\phi^5 - \frac{36}{\gamma^4}\phi^6 - \frac{12}{\gamma^6}\phi^5 + \frac{60}{\gamma^6}\phi^6 - \frac{228}{\gamma^6}\phi^7 . \end{cases}$$

Let $\overline{\gamma_{ij}}$ denote the value of γ_{ij} when we set $\beta = 0$. Similarly for $\overline{\gamma}$ and $\overline{\phi}$. Then it is easy to see that $\overline{\gamma} = (\lambda/\mu)^{\frac{1}{2}}$ and $\overline{\phi} = \lambda/(\lambda-\mu)$. (Note that we assume $\mu > \lambda$ for stability of the queue, so $\overline{\phi} < 0$.) Then we have the expansion

$$C(a,b) = \overline{\gamma} + \left(\frac{\beta}{\sqrt{\lambda\mu}}\right)\overline{\gamma}_{10} + \left(\frac{\alpha}{\sqrt{\lambda\mu}}\right)\overline{\gamma}_{01} + \frac{1}{2!}\left(\frac{\beta}{\sqrt{\lambda\mu}}\right)^2\overline{\gamma}_{20} + \left(\frac{\beta}{\sqrt{\lambda\mu}}\right)\left(\frac{\alpha}{\sqrt{\lambda\mu}}\right)\overline{\gamma}_{11} + \text{etc.}$$

and so

$$(3.13)\quad \begin{aligned} \mu M^0(\mu) &= \left(\frac{\mu}{\lambda}\right)^{\frac{1}{2}} C(a,b) \\ &= \frac{C(a,b)}{\overline{\gamma}} \\ &= 1 + \frac{1}{\overline{\gamma}}\left\{\left(\frac{\beta}{\sqrt{\lambda\mu}}\right)\overline{\gamma}_{10} + \left(\frac{\alpha}{\sqrt{\lambda\mu}}\right)\overline{\gamma}_{01} + \text{etc. } \ldots\right\}. \end{aligned}$$

We are now in a position to extract the requisite moments, for we see that

(3.14) $$EU^iZ^j = (-1)^{i+j} \frac{\bar{\gamma}_{ij}}{(\lambda\mu)^{\frac{1}{2}(i+j)}\bar{\gamma}} .$$

Thus the equations (3.9), (3.11), (3.12) will yield product moments up to those of the third order. We list our results in the next section, and spare the reader the detailed calculations.

§4 The Product Moments

The following results are obtained as outlined in the previous sections.

(4.1) $$EU = \frac{1}{\lambda}\left(\frac{\rho}{1-\rho}\right).$$

(4.2) $$EZ = \frac{1}{\lambda}\left(\frac{\rho}{1-\rho}\right)^2 .$$

(4.3) $$\text{Var } U = \frac{1}{\lambda^2}\frac{\rho^2(1+\rho)}{(1-\rho)^3} .$$

(4.4) $$\text{Var } Z = \frac{1}{\lambda^2}\frac{\rho^3(2+7\rho+\rho^2)}{(1-\rho)^5} .$$

(4.5) $$\text{Cov }(U,Z) = \frac{1}{\lambda^2}\frac{\rho^3(3+\rho)}{(1-\rho)^4} .$$

From the last three results we obtain for the correlation coefficient $\rho_{U,Z}$, say,

(4.6) $$\rho_{U,Z} = \frac{(3+\rho)\sqrt{\rho}}{(1+\rho)^{\frac{1}{2}}(2+7\rho+\rho^2)^{\frac{1}{2}}} .$$

Thus, as $\rho \to 1$,

$$\rho_{U,Z} \to \frac{2}{\sqrt{5}} = .894427,$$

and, as $\rho \to 0$,

$$\rho_{U,Z} \sim \frac{3}{\sqrt{2}} \sqrt{\rho} .$$

Actually, the correlation of U and Z is high for moderately low values of ρ, as this table displays: --

Traffic Intensity ρ	$\rho_{U,Z}$
0.1	.5678
0.2	.7044
0.3	.7745
0.4	.8160
0.5	.8427
0.6	.8607
0.7	.8734
0.8	.8825
0.9	.8893
→ 1.0	→ .8944

(The arrows in the bottom line indicate limit values.)

For product moments of order three we find:

$$EU^3 = \frac{6(1+\rho)}{\mu^3(1-\rho)^5}$$

$$EU^2Z = \frac{\rho^2(14+6\rho)}{\mu^3(1-\rho)^6}$$

$$EUZ^2 = -\frac{\rho(10-60\rho+6\rho^2)}{\mu^3(1-\rho)^7}$$

$$EZ^3 = -\frac{(6-18\rho+54\rho^2+174\rho^3+12\rho^4)}{\mu^3\rho(1-\rho)^7} .$$

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